Fixed Point Theory, 12(2011), No. 2, 349-354
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

SOME REMARKS IN THE STUDY OF IMPULSIVE DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH DELAY

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Abstract. Some remarks in the study of impulsive differential equations and inclusions with delay are given.
Key Words and Phrases: Impulsive differential equation with delay, impulsive differential inclusion with delay, fixed point.
2010 Mathematics Subject Classification: 34A37, 34A60, 47H10, 54C60, 54C65.

1. Introduction

Recently many works treated the existence problem for impulsive differential equations and inclusions with delay and impulsive neutral functional differential equations and inclusions with delay by the methods of fixed-point theory. We only mention the works of some authors ([1], [2], [5], [6], [9]) in the case of finite delay and ([3], [4], [7]) in the case of infinite delay. Let us mention , however, that the treatment of these has been made in the spirit of non impulsive case, more exactly, the norm or seminorm considered for the phase space in the non-impulsive case have been retained for the impulsive one. We will see that this choice of norm or seminorm for the phase space in the impulsive case is not always a good choice because it creates a problems of substance in the study of such problems.

The paper is organized as follows. In Section 2 we give necessary preliminaries from the fields of measurable functions and multivalued maps. In Section 3 we are interested in the nonlinear singlevalued and multivalued parts appearing respectively in such equations and inclusions with finite delay. We give an example through which we can get relevant conclusions. In Section 4 we focus on impulsives equations and inclusions with infinite delay and two examples are given.

2. Preliminaries

If (X, Σ, μ) is a measure space and $\mathbb{Y}$ is a Banach space over a field $\mathbb{K}$ (usually the real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$), then $f : X \to \mathbb{Y}$ is said to be:
(i) weakly measurable if, for every continuous linear functional $g : \mathbb{Y} \to K$, the function $g \circ f : X \to \mathbb{K} : x \longrightarrow g(f(x))$ is a measurable function with respect to Σ

and the usual Borel σ-algebra on $\mathbb{K}$;
(ii) almost surely separably valued (or essentially separably valued) if there exists a subset $N \subset X$ with $\mu(N) = 0$ such that $f(X \setminus N) \subset \mathbb{Y}$ is separable;
(iii) (strongly) measurable if there is a sequence $(f_n)_n$ of measurable step functions for which $f_n(s) \to f(s)$ in $\mathbb{Y}$, for a.e. $s \in X$. For more details see for example [8].

Theorem 2.1 *(Pettis). A function $f : X \to \mathbb{Y}$ is measurable if and only if f is almost surely separably valued and f is weakly measurable.*

Theorem 2.2 *(Bochner). The function $f : X \to \mathbb{Y}$ is integrable if and only if f is measurable and $s \to \|f(s)\|$ is integrable.*

A single valued map $h : [0, T] \times \Pi \to \mathbb{Y}$, where $(\Pi, \|.\|_\Pi)$ is a seminormed space, is called L^1-Carathéodory if satisfies :

$h1)$ the map $h : t \to F(t, u)$ is measurable for each $u \in \Pi$;
$h2)$ the map $h : u \to F(t, u)$ is continuous for a.e. $t \in [0, T]$;
$h3)$ there exists a function $\beta \in L^1([0, T], \mathbb{R}^+)$ such that, for all $u \in \Pi$

$$\|h(t, u)\|_{\mathbb{Y}} \leq \beta(t)\,(1 + \|u\|_\Pi), a.e.\ t \in [0, T].$$

A multimap $H : [0, T] \times \Pi \to Kv(\mathbb{Y})$, where $Kv(\mathbb{Y})$ denotes the class of all nonempty compact convex subsets of Y, is called L^1-upper-Carathéodory if satisfies :

$H1)$ the multimap $H : t \to F(t, u)$ has a strongly measurable selector for each $u \in \Pi$;
$H2)$ the multimap $H : u \to F(t, u)$ is upper semicontinuous for a.e. $t \in [0, T]$;
$H3)$ there exists a function $\beta \in L^1([0, T], \mathbb{R}^+)$ such that, for all $u \in \Pi$

$$\|H(t, u)\|_{\mathbb{Y}} \leq \beta(t)(1 + \|u\|_\Pi), a.e.\ t \in [0, T].$$

Remark 2.3 *The condition $H1)$ is fulfilled if $H(., u)$ is strongly measurable for every $u \in \Pi$.*

Throughout this work E denotes a Banach space with norm $\|.\|$.

3. Finite delay case

Let $F : [0, T] \times \Lambda \to Kv(E)$,a multivalued map, where Λ stands for the space formed by all functions $\xi : [-r, 0] \to E$ such that ξ is continuous everywhere except for a finite number of points s at which $\xi(s)$ and the right limit $\xi(s^+)$ exist and $\xi(s) = \xi(s^-)$ endowed with the norm $\|\xi\|_\Lambda = \sup_{\theta \in [-r,0]} \|\xi(\theta)\|$.
Let denote by $PC([0, T], E)$ the space consisting of functions $x : [0, T] \to E$ such that $x(.)$ is continuous everywhere except for some t_k at which $x(t_k^-)$ and $x(t_k^+), k = 1, ...m$, exist and $x(t_k^-) = x(t_k)$, endowed with the norm

$$\|x\|_{PC} = \sup_{t \in [0,T]} \|x(t)\| \,;$$

and by $PC_\Lambda([-r, T], E)$ the space of functions $x : [-r, T] \to E$ such that $x_{|\,[-r,0]} \in \Lambda$ and $x|_{[0,T]} \in PC([0, T], E)$, endowed with the norm

$$\|x\|_{PC_\Lambda} = \sup_{t \in [-r,T]} \|x(t)\| \,.$$

For any function $x : [-r, T] \longrightarrow E$ and for every $t \in [-r, T]$, x_t represents the function from $[-r, 0]$ into E defined by $x_t(\theta) = x(t + \theta), \; -r \leq \theta \leq 0$.
The aim of this section is to give an example for a function $\varphi \in PC_\Lambda([-r, T], \mathbb{R})$ such that the function $t \to \varphi_t$ is not measurable. This fact will have some consequences.

Example 3.1 Set $E = \mathbb{R}$ and let define the function φ by

$$\varphi(t) = \begin{cases} 0\,, & -r \leq t \leq t_1 \\ 1\,, & t_1 < t \leq T, \end{cases}$$

where $0 < t_1 < T$. Let N be an arbitrary subset of $[0, T]$ such that $\mu(N) = 0$, where μ denotes a Lebesgue measure on $[0, T]$. Set $W = \{\varphi_t, \, t \in [t_1, T] \setminus N\}$. It is clear that for $\varphi_t \in W$, we have

$$\varphi_t(\theta) = \begin{cases} 0\,, & -r \leq \theta \leq t_1 - t \\ 1\,, & t_1 - t < \theta \leq 0, \end{cases}$$

and $\varphi_t \neq \varphi_s$ if $t \neq s$. Obviously $\mu\left([t_1, T] \setminus N\right) = T - t_1 > 0$. Consequently, the set $[t_1, T] \setminus N$ is not denumerable. Thus W is not denumerable. Now, observe that for all $\varphi_t, \, \varphi_s \in W, \, s \neq t$, we have

$$\|\varphi_t - \varphi_s\|_\Lambda = \sup_{\theta \in [-r,0]} |\varphi_t(\theta) - \varphi_s(\theta)| = 1.$$

As a consequence, since W is not denumerable it can not be separable. Thus the range of the function $t \longrightarrow \varphi_t, \, t \in [t_1, T] \setminus N$ is not separable. Since $\varphi_t \equiv 0$ for every $t \in [0, t_1]$, we have

$$\{\varphi_t, \, t \in [0, T] \setminus N\} = \{\varphi_t, \, t \in [t_1, T] \setminus N\} \cup \{\psi\}\,.$$

where $\psi(\theta) = 0$ for every $\theta \in [-r, 0]$. Thus the range of the function $t \to \varphi_t, \, t \in [0, T] \setminus N$ is not separable. Since N is chosen arbitrarily, the function $t \to \varphi_t, \, t \in [0, T]$ is not almost surely separably valued. Hence by Theorem 2.1 it is not measurable. Now consider the multivalued $F : [0, T] \times \Lambda \to Kv(\mathbb{R})$ defined by

$$F(t, u) = \{g(u)\}\,, \text{ where } g : \Lambda \longrightarrow \mathbb{R} \text{ is an arbitrary continuous function.}$$

It is clear that F is Upper-Carathéodory (i.e. F statisfies conditions similar to $H1$) and $H2$)) and for $x \in PC([-r, T], \mathbb{R})$

$$F(t, x_t) = \{g(x_t)\}$$

We know that the function $t \longrightarrow \varphi_t$ is not measurable, hence the function $t \longrightarrow g(\varphi_t)$ is not necessarily measurable. Otherwise, taking in account that the function $t \longrightarrow \varphi_t$ is not measurable, we have to believe that for all continuous function $g : \Lambda \longrightarrow E$ the function $t \longrightarrow g(\varphi_t)$ is measurable which is absurd.

Remark 3.2 *Now, we know that for $x \in PC_\Lambda([-r, T], E)$, the function $t \to x_t$ need not to be measurable. It results that, in the case when F is L^1-Carathéodory single valued map, the function $t \longrightarrow F(t, x_t)$ need not to be measurable for all $x \in PC_\Lambda([-r, T], E)$. Even, if one suppose that $t \longrightarrow F(t, x_t)$ is measurable for all $x \in PC_\Lambda([-r, T], E)$, it is not clear why the use of the condition $(h3)$ after integration will be always true. Otherwise, we have to believe that for all $x \in PC_\Lambda([0, T], \mathbb{R})$, the function $t \to \|x_t\|$ is Lebesgue measurable.*

Consequences. In general, under condition that $F : [0,T] \times \Pi \to Kv(E)$ is L^1-Upper Carathéodory:
(1) the multimap $t \longrightarrow F(t, x_t)$ need not to have a measurable selection for all $x \in PC_\Lambda([-r,T],E)$, as a consequence the superposition operator sel_F generated by F assigning to every function $x \in PC_\Lambda([-r,T],E)$ the set of all strongly measurable selections i.e.;

$$sel_F(x) = \{f : f \text{ is strongly measurable and } f(t) \in F(t,x_t), a.e. t \in [0,T]\},$$

need not to be well defined.
(2) It is not true that for all $x \in PC_\Lambda([-r,T],E)$, the function $t \to x_t$ is Bochner integrable.

4. Infinite delay case

We will see that all our remarks given above in the case of finite delay can be true in the case of infinite delay. Also, in order to obtain general remarks, we want to understand why this can happen.
For any function $x :]-\infty, T] \longrightarrow E$ and for every $t \in [0,T]$, x_t represents the function from $]-\infty, 0]$ into E defined by $x_t(\theta) = x(t+\theta)$, $-\infty < \theta \leq 0$.
Let $G : [0,T] \times \mathbb{B} \to Kv(E)$, a multivalued map, where $\mathbb{B}$ is a linear space of functions mapping $]-\infty,0]$ to E endowed with the seminorm $\|.\|_{\mathbb{B}}$ and verifying some axioms. By $PC_{\mathbb{B}}(]-\infty,T],E)$ we denote the space of functions $x :]-\infty,T] \to E$ such that $x_0 \in \mathbb{B}$ and $x|_{[0,T]} \in PC([0,T],E)$ (for the definition of $PC([0,T],E)$ see section 3), endowed with the seminorm

$$\|x\|_{PC_{\mathbb{B}}} = \|x_0\|_{\mathbb{B}} + \left\|x|_{[0,T]}\right\|_{PC}.$$

Set $E = \mathbb{R}$ and let define the fuction ψ in the following way:

$$\psi(t) = \begin{cases} 0, & -\infty < t \leq t_1 \\ 1, & t_1 < t \leq T, \end{cases}$$

Notice that $\psi_0 \equiv 0$ and hence ψ_0 belongs to any abstract space $\mathbb{B}$. Since $\psi|_{[0,T]} \in PC([0,T],\mathbb{R})$ it results that $\psi \in PC_{\mathbb{B}}(]-\infty,T],\mathbb{R})$.
Now we give two examples of phase spaces which have been used, for example, in ([4], [7]) and we prove that the function $t \to \psi_t$ is not measurable. As a consequence all our remarks given in section 3 are valid for the choice of these two phase spaces in the study of impulsive differential equations and inclusions with infinite delay.

Example 4.1 Let $r > 0$ and let $g :]-\infty,-r] \longrightarrow \mathbb{R}$ be a positive function such that $g(.)$ is Lebesgue integrable on $]-\infty,-r]$ and that there exists a non negative and locally bounded function γ on $]-\infty,0]$ such that $g(\xi+\theta) \leq \gamma(\xi)g(\theta)$, for all $\xi \leq 0$ and $\theta \in]-\infty,-r] \setminus N_\xi$, where $N_\xi \subset]-\infty,-r]$ is a set with Lebesgue measure zero.
Set $\mathbb{B}$ be the space of all functions $\zeta :]-\infty,0] \longrightarrow \mathbb{R}$ such that $\zeta\,|_{[-r,0]} \in \Lambda$ (for the definition of Λ see section 3), ζ is Lebesgue measurable on $]-\infty,-r]$ and $g(.)\,|\zeta(\,.)|^p$ is Lebesgue integrable in $]-\infty,-r]$. The seminorm $\|.\|_{\mathbb{B}}$ in $\mathbb{B}$ is defined by

$$\|\zeta\|_{\mathbb{B}} = \left(\int_{-\infty}^{-r} g(\theta)\,|\zeta(\theta)|^p\,d\theta\right)^{1/p} + \sup_{\theta\in[-r,0]} |\zeta(\theta)|\,.$$

We want to prove that the function $t \to \psi_t$ is not measurable.
Let N be an arbitrary subset of $[0,T]$ such that $\mu(N)=0$, where μ denotes a Lebesgue measure on $[0,T]$. Set $\widetilde{W} = \{\psi_t,\, t \in [t_1, \min\{t_1+r, T\}] \setminus N\}$. It is clear that for $\psi_t \in \widetilde{W}$,

$$\psi_t(\theta) = \begin{cases} 0\,, & -\infty < \theta \le t_1 - t \\ 1\,, & t_1 - t < \theta \le 0, \end{cases}$$

and $\psi_t \neq \psi_s$ if $t \neq s$. Since μ is countably additive and monotone, we have

$$\mu\left([t_1, \min\{t_1+r, T\}] \setminus N\right) = \mu\left([t_1, \min\{t_1+r, T\}]\right) = \min\{r,\, T-t_1\} > 0$$

Consequently the set $[t_1, \min\{t_1+r, T\}] \setminus N$ is not denumerable. Thus $\widetilde{W}$ is not denumerable.

Now, observe that for all $\psi_t,\, \psi_s \in \widetilde{W},\, s \neq t$, we have $t_1 - t \ge -r$ and $t_1 - s \ge -r$. Thus

$$\left(\int_{-\infty}^{-r} g(\theta)\, |\psi_t(\theta) - \psi_s(\theta)|^p \, d\,\theta \right)^{1/p} = 0.$$

Then

$$\|\psi_t - \psi_s\|_{\mathbb{B}} = \sup_{\theta \in [-r,0]} |\psi_t(\theta) - \psi_s(\theta)| = 1.$$

As a consequence, since $\widetilde{W}$ is not denumerable it can not be separable. Thus the range of the function $t \longrightarrow \psi_t,\, t \in [t_1, \min\{t_1+r, T\}] \setminus N$ is not separable. using the same reasoning as in Example 3.1 one can show that the function $t \to \varphi_t,\, t \in [0,,T]$ is not almost surely separably valued and hence not measurable.

Example 4.2 Let $D = \{\phi :]-\infty, 0] \to \mathbb{R},\, \phi$ is continuous everywhere exept for finite number of points $\bar{t}$ at which $\phi(\bar{t}^-)$ and $\phi(\bar{t}^+)$ exist $\phi(\bar{t}^-) = \phi(\bar{t})\}$. Let γ a positive real constant. Set

$$\mathbb{B}_\gamma = \left\{ y \in D : \lim_{\theta \to -\infty} e^{\gamma\theta} y(\theta) \text{ exists in } \mathbb{R} \right\}$$

and the norm of $\mathbb{B}_\gamma$ be given by $\|y\|_{\mathbb{B}_\gamma} = \sup_{-\infty<\theta\le 0} e^{\gamma\theta}\, |y(\theta)|$. Let N be an arbitrary subset of $[0,T]$ such that $\mu(N)=0$, where μ denotes a Lebesgue measure on $[0,T]$. Set $\hat{W} = \{\psi_t,\, t \in [t_1, T] \setminus N\}$. It is clear that for $\psi_t \in \hat{W}$,

$$\psi_t(\theta) = \begin{cases} 0\,, & -\infty < \theta \le t_1 - t \\ 1\,, & t_1 - t < \theta \le 0 \end{cases}$$

Now, observe that for all $\psi_t,\, \psi_s \in \hat{W},\, s > t$, we have

$$\|\psi_t - \psi_s\|_{\mathbb{B}} = \sup_{\theta \in)-\infty, 0]} e^{\gamma\theta}\, |\psi_t(\theta) - \psi_s(\theta)| = \sup_{\theta \in]t_1-s, t_1-t]} e^{\gamma\theta} = e^{\gamma(t_1-t)} \ge e^{\gamma(t_1-T)}.$$

Using the same reasoning as in Example 3.1 one can show that the function $t \to \varphi_t,\, t \in [0,T]$ is not almost surely separably valued and hence not measurable.

Remark 4.3 *Taking $\mathbb{B}$ one of phase spaces considered in Examples 4.1 and Example 4.2, and instead of $PC_\Lambda([-r,T], E)$ the space $PC_{\mathbb{B}}(]-\infty, T), E)$, Remark 3.2 remains true.*

Conclusions. 1) In general, in the study of differential equations and inclusions with finite delay where the multivalued part is defined via F which is L^1-Carathéodory, the phase space is Λ and the solutions are sought in the space $PC_\Lambda([-r,T],E)$ (see section 3), we find that the choice of the uniform convergence norm for the space Λ is not a good choice because it creates a problems of substance. As we have seen, in this case:
(i) for $x \in PC_\Lambda([0,T],E)$ the function $t \longrightarrow x_t$ is not necessarily measurable and hence not necessarily Bochner integrable.
(ii) The superposition operator is not well defined.

2) For the choice of some concrete phase spaces, this situation is the same in the study of differential equations and inclusions with infinite delay. Example 4.1 and Example 4.2 give two examples of phase spaces for which the study can not be achieved. What really has been done in Examples 4.1 and Example 4.2 can be made for other concrete phase spaces whose norms or seminorms are written, partly or entirely, through the uniform convergence norm taken in some interval $[\alpha,\beta] \subset]-\infty,0]$ or taken all the interval $]-\infty,0]$. This can be explained by the fact that for a given $x \in PC_\Lambda([-r,T],E)$ or $x \in PC_{\mathbb{B}}(]-\infty,T],E)$ the functions $x_t(.)$, $t \in [0,T]$ are impulsive with dependent pulses of t and the sup norm works very poorly when the pulses are variable, it already can not keep the uniforme convegence.

Acknowledgements 1. I thank Professor Górniewicz for encouragements. The author wishes to thank the referee for his (her) corrections and remarks.

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Received: November 10, 2009; Accepted: December 2, 2010.