JORDAN *−HOMOMORPHISMS BETWEEN UNITAL 
C∗−ALGEBRAS: A FIXED POINT APPROACH

M. ESHAGHI GORDJI

Department of Mathematics, Semnan University, P. O. Box 35195-363, Semnan, Iran; 
Research Group of Nonlinear Analysis and Applications (RGNAA), Semnan, Iran; 
Center of Excellence in Nonlinear Analysis and Applications (CENAA), Semnan University, Iran. 
E-mail: madjid.eshaghi@gmail.com

Abstract. Let A, B be two unital C∗−algebras. By using fixed pint methods, we prove that:
a) Every almost unital almost linear mapping h : A −→ B which satisfies 
h(2nuy + 2nyu) = h(2nu)h(y) + h(y)h(2nu) for all u ∈ U(A), all y ∈ A, and all n = 0, 1, 2, ..., is a Jordan homomor-
phism.
b) For a unital C∗−algebra A of real rank zero, every almost unital almost linear continuous 
mapping h : A −→ B is a Jordan homomorphism when 
h(2nuy + 2nyu) = h(2nu)h(y) + h(y)h(2nu) holds for all u ∈ I(A), all y ∈ A, and all n = 0, 1, 2, ...

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1. Introduction

B.E. Johnson (Theorem 7.2 of [3]) investigated almost algebra *−homomorphisms 
between Banach *−algebras: Suppose that U and B are Banach *−algebras which 
satisfy the conditions of (Theorem 3.1 of [3]). Then for each positive ε and K there is 
a positive δ such that if T ∈ L(U, B) with ∥T∥ < K, ∥T′∥ < δ and ∥T(x)∗ − T(x)∥ < 
δ∥x∥(x ∈ U), then there is a *−homomorphism T′ : U −→ B with ∥T − T′∥ < ε. 
Here L(U, B) is the space of bounded linear maps from U into B, and T′(x, y) = 

We recall a fundamental result in fixed point theory (see [6, 7, 8] for more details).

Theorem 1.1. (The alternative of fixed point [2]). Suppose that we are given a 
complete generalized metric space (Ω, d) and a strictly contractive mapping T : Ω −→ Ω 
with Lipschitz constant L. Then for each given x ∈ Ω, either 
d(Tmx, Tm+1x) = ∞ for all m ≥ 0, 
or other exists a natural number m0 such that 
d(Tmx, Tm+1x) < ∞ for all m ≥ m0; 
the sequence {Tmx} is convergent to a fixed point y* of T; 
y* is the unique fixed point of T in the set Λ = {y ∈ Ω : d(Tm0x, y) < ∞}; 
\d(y, y*) ≤ \frac{1}{1−L}d(y, Ty) for all y ∈ Λ.

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Throughout this paper, let $A$ be a unital $C^*$-algebra with unit $e$, and $B$ a unital $C^*$-algebra. Note that a linear mapping $h : A \to B$ is a Jordan $*$-homomorphism if $h$ is $*$-preserving and
\[
h(ab + ba) = h(a)h(b) + h(b)h(a)
\]
for all $a, b \in A$. Let $U(A)$ be the set of unitary elements in $A$, $A_{sa} := \{x \in A| x = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa}||v|| = 1, v \in Inv(A)\}$. Recently, C. Park, D.-H. Boo and J.-S. An [5] investigated almost homomorphisms between unital $C^*$-algebras. In this paper, by using the fixed point methods, we prove that every almost unital almost linear mapping $h : A \longrightarrow B$ is a Jordan homomorphism when $h(2^nuy + 2^nuy) = h(2^n)y + h(y)h(2^n)$ holds for all $u \in U(A)$, all $y \in A$, and all $n = 0, 1, 2, \ldots$, and that for a unital $C^*$-algebra $A$ of real rank zero (see [1]), every almost unital almost linear continuous mapping $h : A \longrightarrow B$ is a Jordan homomorphism when $h(2^nuy + 2^nuy) = h(2^n)y + h(y)h(2^n)$ holds for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n = 0, 1, 2, \ldots$.

Note that a unital $C^*$-algebra is of real rank zero, if the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [1]). We denote the algebraic center of algebra $A$ by $Z(A)$.

2. Main results

We start our work with the following theorem which investigate almost Jordan $*$-homomorphisms between unital $C^*$-algebras.

**Theorem 2.1.** Let $f : A \to B$ be a mapping such that $f(0) = 0$ and that
\[
f(2^nuy + 2^nuy) = f(2^n)uf(y) + f(y)f(2^n)
\]
for all $u \in U(A), y \in A, n \in \mathbb{Z}_+$. There exists a function $\phi : A^3 \to [0, \infty)$ such that:
(i) \[
\|\mu f(x + y) + \mu f(x - y) - f(\mu x) + f(u^*) - f(u)\| \leq \phi(x, y, u)
\]
for all $\mu \in \mathbb{T}$ and all $x, y, u \in (U(A) \cup \{0\})$.

(ii) There exists an $L < 1$ such that
\[
\phi(x, y, u) \leq 2L\phi(\frac{x}{2}, \frac{y}{2}, \frac{u}{2})
\]
for all $x, y, u \in A$.

If $\lim_{n \to \infty} f(2^n) \in U(B) \cap Z(B)$ then the mapping $f : A \to B$ is a Jordan $*$-homomorphism.

**Proof.** It follows from $\phi(x, y, a) \leq 2L\phi(\frac{x}{2}, \frac{y}{2}, \frac{a}{2})$ that
\[
\lim_{j \to \infty} 2^{-j}\phi(2^jx, 2^jy, 2^ja) = 0
\]
for all $x, y, a \in A$.

Put $\mu = 1, y = u = 0$ in (2.2) to obtain
\[
\|2f(\frac{x}{2}) - f(x)\| \leq \phi(x, 0, 0)
\]
for all $x \in A$. Hence,
\[ \| \frac{1}{2} f(2x) - f(x) \| \leq \frac{1}{2} \phi(2x, 0, 0) \leq L \phi(x, 0, 0) \] (2.5)
for all $x \in A$.

Consider the set $X := \{ g : A \to B \}$ and introduce the generalized metric on $X$:
\[ d(h, g) := \inf \{ C \in \mathbb{R}^+ : \| g(x) - h(x) \| \leq C \phi(x, 0, 0) \forall x \in A \}. \]
It is easy to show that $(X, d)$ is complete. Now we define the linear mapping $J : X \to X$ by
\[ J(h)(x) = \frac{1}{2} h(2x) \]
for all $x \in A$. By Theorem 3.1 of [2],
\[ d(J(g), J(h)) \leq L d(g, h) \]
for all $g, h \in X$.
It follows from (2.5) that
\[ d(f, J(f)) \leq L. \]

By Theorem 1.1, $J$ has a unique fixed point in the set $X_1 := \{ h \in X : d(f, h) < \infty \}$. Let $H$ be the fixed point of $J$. $H$ is the unique mapping with
\[ H(2x) = 2H(x) \]
for all $x \in A$ satisfying there exists $C \in (0, \infty)$ such that
\[ \| H(x) - f(x) \| \leq C \phi(x, 0, 0) \]
for all $x \in A$. On the other hand we have $\lim_n d(J^n(f), D) = 0$. It follows that
\[ \lim_n \frac{1}{2^n} f(2^n x) = H(x) \] (2.6)
for all $x \in A$. It follows from (2.2), (2.3) and (2.6) that
\[
\begin{align*}
\| H(\frac{x+y}{2}) + H(\frac{x-y}{2}) &- H(x) \| \\
&= \lim_n \frac{1}{2^n} \| f(2^{n-1}(x+y)) + f(2^{n-1}(x-y)) - f(2^n x) \| \\
&\leq \lim_n \frac{1}{2^n} \phi(2^n x, 2^n y, 0) \\
&= 0
\end{align*}
\]
for all $x, y \in A$. So
\[ H(\frac{x+y}{2}) + H(\frac{x-y}{2}) = H(x) \]
for all $x, y \in A$. Put $z = \frac{x+y}{2}, t = \frac{x-y}{2}$ in above equation, we get
\[ H(z) + H(t) = H(z + t) \] (2.7)
for all $z, t \in A$. Hence, $H$ is Cauchy additive. By putting $y = x, u = 0$ in (2.2), we have
\[ \| \mu f(\frac{2x}{2}) - f(\mu x) \| \leq \phi(x, x, 0) \]
for all \( x \in A \). It follows that
\[
\|H(2\mu x) - 2\mu H(x)\| = \lim_{n} \frac{1}{2^n} \|f(2\mu 2^n x) - 2\mu f(2^n x)\| \leq \lim_{n} \frac{1}{2^n} \phi(2^n x, 2^n x, 0) = 0
\]
for all \( \mu \in \mathbb{T} \), and all \( x \in A \). One can show that the mapping \( H : A \rightarrow B \) is \( \mathbb{C} \)-linear. On the other hand by using (2.2), we have
\[
\|H(u^*) - (H(u))^*\| = \lim_{n} \frac{1}{2^n} \|f(2^n u^*) - \frac{1}{2^n} (f(2^n u))^*\|
\]
\[
\leq \lim_{n} \frac{1}{2^n} \phi(0, 0, 2^n u) \leq \lim_{n} \frac{1}{2^n} \phi(0, 0, 2^n u)
\]
\[
= 0
\]
\[(2.8)\]
for all \( u \in U(A) \). Now, let \( x \in A \). By Theorem 4.1.7 of [4], \( x \) is a finite linear combination of unitary elements, i.e., \( x = \sum_{j=1}^{n} c_{j} u_{j} \) \((c_{j} \in \mathbb{C}, u_{j} \in U(A))\). Since \( H \) is \( \mathbb{C} \)-linear, it follows from (2.8) that
\[
H(x^*) - H(x)^* = H(\sum_{j=1}^{n} c_{j} u_{j}^*) - H(\sum_{j=1}^{n} c_{j} u_{j})^* = 0.
\]
Hence, \( H \) is \( \ast \)-preserving. Now, we show that \( H \) is a Jordan homomorphism. To this end, let \( u \in U(A), y \in A \). Then by linearity of \( H \) and (2.1), we have
\[
H(uy + yu) = \lim_{n} \frac{f(2^n uy + 2^n yu)}{2^n} = \lim_{n} \frac{f(2^n u) f(y) + f(y) f(2^n u)}{2^n}
\]
\[
= H(u)f(y) + f(y)H(u)
\]
\[(2.9)\]
for all \( u \in U(A), y \in A \). Since \( H \) is additive, then by (2.9), we have
\[
2^n H(uy + yu) = H(u(2^n y) + (2^n y)u) = H(u)f(2^n y) + f(2^n y)H(u)
\]
for all \( u \in U(A) \) and all \( y \in A \). Hence,
\[
H(uy + yu) = \lim_{n} [H(u) \frac{f(2^n y)}{2^n} + \frac{f(2^n y)}{2^n} H(u)] = H(u)H(y) + H(y)H(u)
\]
\[(2.10)\]
for all \( u \in U(A) \) and all \( y \in A \).

Now, let \( x \in A \). Then there are \( n \in \mathbb{N}, c_{j} \in \mathbb{C}, u_{j} \in U(A), 1 \leq j \leq n \), such that \( x = \sum_{j=1}^{n} c_{j} u_{j} \), it follows from (2.10) that
\[
H(xy + yx) = H(\sum_{j=1}^{n} c_{j} u_{j} y + \sum_{j=1}^{n} c_{j} y u_{j}) = \sum_{j=1}^{n} c_{j} H(u_{j} y + y u_{j})
\]
\[
= \sum_{j=1}^{n} c_{j} (H(u_{j} y) + H(y u_{j})) = \sum_{j=1}^{n} c_{j} (H(u_{j})H(y) + H(y)H(u_{j}))
\]
\[
= H(\sum_{j=1}^{n} c_{j} u_{j}) H(y) + H(y) H(\sum_{j=1}^{n} c_{j} u_{j})
\]
\[
= H(x) H(y) + H(y) H(x)
\]
for all \( y \in A \). This means that \( H \) is a Jordan homomorphism.
On the other hand, we have
\[ H(e) = \lim_n \frac{f(2^n e)}{2^n} \in U(B) \cap Z(B). \]

Hence, it follows from (2.9) and (2.10) that
\[ \frac{2H(e)H(y) = H(e)H(y) + H(y)H(e) = H(\mu e + ey)}{e} = H(e)f(y) + f(y)H(e) = 2H(e)f(y) \]

for all \( y \in A \). Since \( H(e) \) is invertible, then \( H(y) = f(y) \) for all \( y \in A \). This completes the proof of theorem. \( \square \)

**Corollary 2.2.** Let \( p \in (0, 1), \theta \in [1, \infty) \) be real numbers. Let \( f : A \to B \) be a mapping such that \( f(0) = 0 \). Let
\[ f(2^n uy + 2^n yu) = f(2^n u)f(y) + f(y)f(2^n u) \]
for all \( u \in U(A), y \in A, n \in Z_+ \).

(i) \[ \|\mu f(x, y) + \mu f(x, -y) - f(\mu x) + f(\mu f(x) - f(u)^* - f(u)*\| \leq \theta(\|x\|^p + |y|^p + \|u\|^p) \]

for all \( \mu \in \mathbb{T} \) and all \( x, y \in A, u \in \{U(A) \cup \{0\}\} \).

If \( \lim_n f(2^n e) \in U(B) \cap Z(B) \) then the mapping \( f : A \to B \) is a Jordan \( * \)-homomorphism.

**Proof.** The conclusion follows from Theorem 2.1, by putting \( \phi(x, y, u) := \theta(\|x\|^p + |y|^p + \|u\|^p) \) all \( x, y, u \in A \) and \( L = 2^p - 1 \). \( \square \)

**Theorem 2.3.** Let \( A \) be a \( C^* \)-algebra of real rank zero. Let \( f : A \to B \) be a mapping such that \( f(0) = 0 \) and
\[ f(2^n uy + 2^n yu) = f(2^n u)f(y) + f(y)f(2^n u) \quad (2.11) \]
for all \( u \in I_1(A_{sa}), y \in A, n \in Z_+ \). There exists a function \( \phi : A^3 \to [0, \infty) \) such that

(i) \[ \|\mu f(x, y) + \mu f(x, -y) - f(\mu x) + f(\mu f(x) - f(u)^* - f(u)*\| \leq \phi(x, y, u) \quad (2.12) \]

for all \( \mu \in \mathbb{T} \) and all \( x, y \in A, u \in \{I_1(A_{sa}) \cup \{0\}\} \).

(ii) There exists an \( L < 1 \) such that \( \phi(x, y, u) \leq 2L\phi \left( \frac{x}{2}, \frac{y}{2}; \frac{u}{2} \right) \) for all \( x, y, u \in A \).

If \( \lim_n f(2^n e) \in U(B) \cap Z(B) \) then the mapping \( f : A \to B \) is a Jordan \( * \)-homomorphism.

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique \( \mathbb{C} \)-linear mapping \( H : A \to B \) satisfying (2.6). It follows from (2.11) that
\[ H(uy + yu) = \lim_n \frac{f(2^n uy + 2^n yu)}{2^n} = \lim_n \frac{f(2^n u)}{2^n} f(y) + f(y) f(2^n u) \]
\[ = H(u)f(y) + f(y)H(u) \quad (2.13) \]

for all \( u \in I_1(A_{sa}) \), and all \( y \in A \). By additivity of \( H \) and (2.13), we obtain that
\[ 2^n H(uy + yu) = H(u(2^n y) + (2^n y)u) = H(u)f(2^n y) + f(2^n y)H(u) \]
for all \( u \in I_1(A_{sa}) \) and all \( y \in A \). Hence,
\[
H(u y + y u) = \lim_{n} \left[ H(u) \frac{f(2^n y)}{2^n} + \frac{f(2^n y)}{2^n} H(u) \right] = H(u) H(y) + H(y) H(u) \tag{2.14}
\]
for all \( u \in I_1(A_{sa}) \) and all \( y \in A \). By the assumption, we have
\[
H(e) = \lim_{n} \frac{f(2^n e)}{2^n} \in U(B) \cap Z(B).
\]

Similar to the proof of Theorem 2.1, it follows from (2.13) and (2.14) that \( H = f \) on \( A \). So \( H \) is continuous.

Putting \( x = y = 0 \) in (2.12), we have
\[
\|H(u^*) - (H(u))^*\| = \lim_{n} \left\| \frac{1}{2^n} f(2^n u^*) - \frac{1}{2^n} (f(2^n u))^* \right\|
\leq \lim_{n} \frac{1}{2^n} \|f(0, 0, 2^n u)\| \leq \lim_{n} \frac{1}{2^n} \|f(0, 0, 2^n u)\| = 0 \tag{2.15}
\]
for all \( u \in I_1(A_{sa}) \). Since \( A \) is real rank zero, it is easy to show that \( I_1(A_{sa}) \) is dense in \( \{x \in A_{sa} : \|x\| = 1\} \). Let \( v \in \{x \in A_{sa} : \|x\| = 1\} \). Then there exists a sequence \( \{z_n\} \) in \( I_1(A_{sa}) \) such that \( \lim_n z_n = v \). Since \( H \) is continuous, it follows from (2.15) that
\[
H(u^*) = H(\lim_n z_n^*) = \lim_n H(z_n^*) = H(\lim_n z_n)^* = H(v)^*. \tag{2.16}
\]

Also, it follows from (2.14) that
\[
H(v y + y v) = H(\lim_n (z_n y + y z_n)) = \lim_n H(z_n y + y z_n)
= \lim_n H(z_n) H(y) + \lim_n H(y) H(z_n)
= H(\lim_n z_n) H(y) + H(y) H(\lim_n z_n)
= H(v) H(y) + H(y) H(v) \tag{2.17}
\]
Now, let \( x \in A \). Then we have \( x = x_1 + i x_2 \), where \( x_1 := \frac{x + x^*}{2i} \) and \( x_2 := \frac{x - x^*}{2i} \) are self-adjoint.

First, consider the case that \( x_1 \neq 0, x_2 \neq 0 \). Since \( H \) is \( C \)-linear, then it follows from (2.16) that
\[
f(x^*) = H(x^*) = H((x_1 + i x_2)^*) = H(\|x_1\| \frac{x_1^*}{\|x_1\|} + H(i \|x_2\| \frac{x_2^*}{\|x_2\|})
= \|x_1\| H(\frac{x_1^*}{\|x_1\|}) - i \|x_2\| H(\frac{x_2^*}{\|x_2\|}) = \|x_1\| H(\frac{x_1^*}{\|x_1\|}) + H(i \|x_2\| \frac{x_2^*}{\|x_2\|})
= H(\|x_1\| \frac{x_1^*}{\|x_1\|}) + H(i \|x_2\| \frac{x_2^*}{\|x_2\|}) = [H(x_1) + H(i x_2)]^*
= H(x)^* = f(x)^*.
\]
So, it follows from (2.17) that
\[
f(xy + yx) = H(xy + yx) = H(x_1 y + i x_2 y + y x_1 + y (i x_2))
\]
Also, we have

\[ H(A, u) = 0 \]

for all \( y \in A \).

Now, consider the case that \( x_1 \neq 0, x_2 = 0 \). Then it follows from (2.16) that

\[ f(x^*) = H(x^*) = H((x_1)^*) = H(\|x_1\| \frac{x_1^*}{\|x_1\|}) = \|x_1\| H(\frac{x_1^*}{\|x_1\|}) = \|x_1\| H(\frac{x_1}{\|x_1\|})^* \]

Also, we have

\[ f(xy + yx) = H(xy + yx) = H(x_1 y + y(x_1)) = H(\|x_1\| \frac{x_1}{\|x_1\|} y + y(\|x_1\| \frac{x_1}{\|x_1\|})) \]

for all \( y \in A \).

Finally, consider the case that \( x_1 = 0, x_2 \neq 0 \). Then it follows from (2.16) that

\[ f(x^*) = H(x^*) = H((ix_2)^*) = H(i\|x_2\| \frac{x_2^*}{\|x_2\|}) = -i\|x_2\| H(\frac{x_2^*}{\|x_2\|}) = -i\|x_2\| H(\frac{x_2}{\|x_2\|})^* \]

Similarly we can show that

\[ f(xy + yx) = H(xy + yx) = H(x)(y) + H(y)H(x) = f(x)f(y) + f(y)f(x) \]

for all \( y \in A \). Hence, \( f \) is a Jordan \( \ast \)-homomorphism. □

**Corollary 2.4.** Let \( p \in (0, 1), \theta \in [1, \infty) \) be real numbers. Let \( f : A \to B \) be a mapping such that \( f(0) = 0 \) and

\[ f(2^n uy + 2^n yu) = f(2^n u)f(y) + f(y)f(2^n u) \]

for all \( u \in I_1(A_{sa}), y \in A, n \in \mathbb{Z}_+ \). Let

\[
\|\mu f(\frac{x + y}{2}) + \mu f(\frac{x - y}{2}) - f(\mu x) + f(u^*) - f(u)^*\| \leq \theta(\|x\|^p + \|y\|^p + \|u\|^p)
\]

for all \( \mu \in \mathbb{T} \) and all \( x, y \in A, u \in (I_1(A_{sa}) \cup \{0\}) \).
If \( \lim_n \frac{f(2^n x)}{2^n} \in U(B) \cap Z(B) \) then the mapping \( f : A \to B \) is a Jordan *-homomorphism.

**Proof.** The conclusion follows from Theorem 2.3, by putting \( \phi(x, y, u) := \theta(\|x\|^p + \|y\|^p + \|u\|^p) \) all \( x, y, u \in A \) and \( L = 2^{p-1} \). \( \square \)

**References**


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