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ULAM STABILITIES OF A FIRST ORDER ITERATIVE FUNCTIONAL-DIFFERENTIAL EQUATION

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Abstract. In this paper we deal with special types of data dependence, the so-called Ulam stabilities, regarding the first order iterative functional-differential equation

$$x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[m]}(t)),\ t \in [a, b],\ a, b \in \mathbb{R},\ a < b,$$

with $x \in C^1([a, b], [a, b])$. Here $x^{[m]}$ denotes the m^{th} iterate of the function x, $(m \geq 0)$ i.e.

$$x^{[m]} := \underbrace{x \circ x \circ \dots \circ x}_{m}.$$

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1. Introduction

Let $a, b \in \mathbb{R}$ $(a < b)$, and $f : [a, b]^{m+1} \to \mathbb{R}$, $\varphi : [a, b] \to \mathbb{R}_+$ be two continuous functions. Our aim is to investigate the Ulam–Hyers stability and the generalized Ulam–Hyers–Rassias stability of the first order iterative functional-differential equation

$$x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \dots, x^{[m]}(t)), \quad t \in [a, b], \tag{1.1}$$

$x \in C^1([a, b], [a, b])$.

For this purpose, consider first the following associated differential inequalities:

$$|y'(t) - f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t))| \leq \varepsilon, \quad t \in [a, b], \tag{1.2}$$

$$|y'(t) - f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t))| \leq \varphi(t), \quad t \in [a, b], \tag{1.3}$$

$$|y'(t) - f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t))| \leq \varepsilon\varphi(t), \quad t \in [a, b]. \tag{1.4}$$

We make the following remarks.

Remark 1.1. *A function $y \in C^1([a, b], [a, b])$ is a solution of the inequality* (1.2) *(also called an ε-solution of the equation* (1.1)*) if and only if there exists a function $g \in C([a, b], \mathbb{R})$, which depends on y, such that*

(1) $|g(t)| \leq \varepsilon, \quad \forall t \in [a, b]$;
(2) $y'(t) = f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t)) + g(t), \quad \forall t \in [a, b]$.

Remark 1.2. *A function $y \in C^1([a,b],[a,b])$ is a solution of the inequality* (1.3) *if and only if there exists a function $h \in C([a,b],\mathbb{R})$, which depends on y, such that*

(1) $|h(t)| \leq \varphi(t), \quad \forall t \in [a,b]$;
(2) $y'(t) = f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t)) + h(t), \quad \forall t \in [a,b]$.

Remark 1.3. *A function $y \in C^1([a,b],[a,b])$ is a solution of the inequality* (1.4) *if and only if there exists a function $k \in C([a,b],\mathbb{R})$, which depends on y, such that*

(1) $|k(t)| \leq \varepsilon\varphi(t), \quad \forall t \in [a,b]$;
(2) $y'(t) = f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t)) + k(t), \quad \forall t \in [a,b]$.

Following [5] and [7] we present the following notions (see also [3], [4],[6],[8], [1]):

Definition 1.1. *The equation* (1.1) *is* Ulam–Hyers stable *if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a,b],[a,b])$ of* (1.2) *there exists a solution $x \in C^1([a,b],[a,b])$ of the equation* (1.1) *with*

$$\|y - x\| \leq c_f \varepsilon.$$

Definition 1.2. *The equation* (1.1) *is* generalized Ulam–Hyers stable *if there exists a real valued function $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$ such that for each solution $y \in C^1([a,b],[a,b])$ of* (1.2) *there exists a solution $x \in C^1([a,b],[a,b])$ of* (1.1) *with*

$$\|y - x\| \leq \theta_f(\varepsilon).$$

Definition 1.3. *The equation* (1.1) *is* Ulam–Hyers–Rassias stable with respect to φ *if there exists a real number $c_{f,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a,b],[a,b])$ of* (1.4) *there exists a solution $x \in C^1([a,b],[a,b])$ of* (1.1) *with*

$$\|y - x\| \leq c_{f,\varphi} \varepsilon \varphi(t).$$

Definition 1.4. *The equation* (1.1) *is* generalized Ulam–Hyers–Rassias stable with respect to φ *if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $y \in C^1([a,b],[a,b])$ of* (1.3) *there exists a solution $x \in C^1([a,b],[a,b])$ of* (1.1) *with*

$$\|y - x\| \leq c_{f,\varphi} \varphi(t).$$

In the present paper we shall give some results in terms of Ulam–Hyers stability and generalized Ulam–Hyers–Rassias stability of the equation (1.1). For this we will need the following result:

Lemma 1.1 (data dependence, [2]). *Let (X,d) be a complete metric space and $A, B : X \to X$ be to operators. We suppose that:*

i) *the operator A is a contraction, i.e. there exists $L_A \in [0,1)$ with*

$$d(A(x), A(y)) \leq L_A d(x,y), \quad \forall x, y, \in X;$$

ii) $F_B \neq \emptyset$;
iii) *there exists η such that $d(A(x), B(x)) \leq \eta, \quad$ for all $x \in X$.*

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1 - L_A}.$$

2. Ulam–Hyers stability

Consider the equation (1.1) and let $L > 0$ be a real number. Denote by $C_L([a,b],[a,b])$ the space of those continuous functions, defined on $[a,b]$ taking values on $[a,b]$, which have the same Lipschitz constant, $L > 0$. We endow the set $C_L([a,b],[a,b]) \subset C([a,b],\mathbb{R})$ with the Chebyshev metric $d_C(x,y) := \|x-y\|_C = \max_{t\in[a,b]} |x(t)-y(t)|$, for all $x,y \in C_L([a,b],[a,b])$. In what follows we will work in this metric space.

We have our first result:

Theorem 2.1. *Consider the iterative functional-differential equation* (1.1) *with the Cauchy condition*

$$x(a) = x_0. \tag{2.1}$$

Suppose that the following conditions are satisfied:

(i) $f \in C([a,b]^{m+1},\mathbb{R})$;

(ii) *there exists* $L_f > 0$, *such that*

$$|f(t,u_1,u_2,\ldots,u_m) - f(t,v_1,v_2,\ldots v_m)| \le L_f \sum_{i=1}^{m} |u_i - v_i|,$$

for all $t, u_i, v_i \in [a,b]$, $i = \overline{1,m}$;

(iii) *for* $M_f := \max\limits_{[t_0-a,t_0+a]^{m+1}} |f(t,u_1,u_2,\ldots,u_m)|$ *we have*

$$M_f \le \min\left\{\frac{b-x_0}{b-a}, \frac{x_0-a}{b-a}, L\right\};$$

(iv) $L_f(b-a)\displaystyle\sum_{i=1}^{m} iL^{m-i} < 1$.

Then

(1) *the Cauchy problem* (1.1) + (2.1) *has a unique solution in* $C_L([a,b],[a,b])$;

(2) *Let* $M_f \le L$ *and*

$$X_1 := \{y \in C_L([a,b],[a,b]) \mid M_f \le \min\left\{\frac{b-y(a)}{b-a}, \frac{y(a)-a}{b-a}, L\right\}\}.$$

Then the equation (1.1) *is Ulam–Hyers stable in* X_1.

Proof.

(1) In this first part we consider the operator

$$A : C_L([a,b],[a,b]) \to C_L([a,b],[a,b]),$$

given by

$$A(x)(t) := x_0 + \int_a^t f(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,\mathrm{d}s,\ t \in [a,b]. \tag{2.2}$$

This operator is well-defined, since for all $x \in C_L([a,b],[a,b])$ we have $A(x) \in C_L([a,b],[a,b])$. Indeed, let $x \in C([a,b],[a,b])$ with

$$|x(t_1) - x(t_2)| \le L|t_1 - t_2|,\ \forall t_1, t_2 \in [a,b].$$

Then, from the definition of the operator A and due to the assumption (iii), for all $t \in [a,b]$ we get $A(x)(t) \le x_0 + M_f(b-a) \le b$ and $A(x)(t) \ge x_0 - M_f(b-a) \ge a$. Since the mapping $t \mapsto f(t, y^{[1]}(t), y^{[2]}(t), \dots, y^{[m]}(t))$ is continuous, the latter affirmations justifies that $A(x) \in C([a,b],[a,b])$. From (iii) we have that $M_f \le L$ and so we can obtain that

$$\begin{aligned}|A(x)(t_1) - A(x)(t_2)| = \left| \int_{t_1}^{t_2} f(s, y^{[1]}(s), y^{[2]}(s), \dots, y^{[m]}(s))\,\mathrm{d}s \right| \le \\ \le M_f|t_1 - t_2| \le L|t_1 - t_2|, \quad \forall t_1, t_2 \in [a,b].\end{aligned}$$

In this manner we get that $A(x) \in C_L([a,b],[a,b])$.

We will prove now that the operator A is a contraction. Indeed, we have:

$$\begin{aligned}&|A(x)(t) - A(y)(t)| = \\ =&\left| \int_a^t \left[f(s, x(s), x^{[2]}(s), \dots, x^{[m]}(s)) - f(s, y(s), y^{[2]}(s), \dots, y^{[m]}(s)) \right] \mathrm{d}s \right| \le \\ \le& L_f|t-a| \sum_{i=1}^m |x^{[i]}(s) - y^{[i]}(s)|.\end{aligned}$$

By induction we can prove that, for all $x, y \in C_L([a,b],[a,b])$ the following successive estimations hold:

$$\begin{aligned}|x^{[1]}(s) - y^{[1]}(s)| &\le \|x-y\|_C; \\ |x^{[2]}(s) - y^{[2]}(s)| &= |x(x(s)) - x(y(s)) + x(y(s)) - y(y(s))| \le \\ &\le L|x(s) - y(s)| + |x(y(s)) - y(y(s))| \le \\ &\le (1+L)\|x-y\|_C; \\ |x^{[3]}(s) - y^{[3]}(s)| &= |x(x^{[2]})(s) - x(y^{[2]}(s)) + x(y^{[2]}(s)) - y(y^{[2]})(s)| \le \\ &\le L|x^{[2]}(s) - y^{[2]}(s)| + \|x-y\|_C \le \\ &\le (1+L+L^2)\|x-y\|_C; \\ &\vdots \\ |x^{[m]}(s) - y^{[m]}(s)| &\le (1+L+L^2+\dots+L^{m-1})\|x-y\|_C.\end{aligned}$$

Summing up the corresponding relations, we receive

$$\sum_{i=1}^m |x^{[i]}(s) - y^{[i]}(s)| \le \|x-y\|_C \sum_{i=1}^m iL^{m-i}.$$

Therefore, we obtain

$$|A(x)(t) - A(y)(t)| \le L_f(b-a)\|x-y\|_C \sum_{i=1}^m iL^{m-i}, \quad \forall t \in [a,b],$$

and finally,

$$\|A(x) - A(y)\|_C \le \left[L_f(b-a) \sum_{i=1}^m iL^{m-i} \right] \|x-y\|_C.$$

Due to (iv) we have that $L_f(b-a)\sum_{i=1}^{m} iL^{m-i} < 1$ and, therefore, the operator A is a contraction. From Banach's fixed point theorem A has a unique fixed point. Consequently the Cauchy problem (1.1)+(2.1) has a unique solution in $C_L([a,b],[a,b])$.

(2) Since $y \in C_L([a,b],[a,b])$ is a solution of the differential inequality (1.2), it satisfies the relation

$$|y'(t) - f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t))| \le \varepsilon, \quad t \in [a,b].$$

Then, y is a solution of the integral inequality

$$\left| y(t) - y(a) - \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s \right| \le (t-a)\varepsilon, \forall t \in [a,b].$$

Indeed, by Remark 1.1 we have that

$$y'(t) = f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t)) + g(t),$$

with

$$|g(t)| \le \varepsilon, \quad \forall t \in [a,b].$$

Accordingly, we obtain

$$y(t) = y(a) + \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s + \int_a^t g(s)\mathrm{d}s, \quad t \in [a,b], \tag{2.3}$$

which implies that

$$\left| y(t) - y(a) - \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s \right| \le \left| \int_a^t g(s)\mathrm{d}s \right| \le$$
$$\le \int_a^t |g(s)|\mathrm{d}s \le \varepsilon(t-a).$$

Moreover, we denote by $x \in C_L([a,b],[a,b])$ the unique solution of the Cauchy problem

$$x'(t) = f(t, x^{[1]}(t), x^{[2]}(t), \ldots, x^{[m]}(t)),\ t \in [a,b], \tag{2.4}$$
$$x(a) = y(a), \tag{2.5}$$

which solution is ensured by the previous proven part. We mention that the function y in the initial value condition (2.5) is the solution of the inequality (1.2).

Then we clearly have,

$$x(t) = y(a) + \int_a^t f(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,\mathrm{d}s, \quad t \in [a,b]. \tag{2.6}$$

We introduce the operators

$$A : C_L([a,b],[a,b]) \to C_L([a,b],[a,b]),$$
$$A(x)(t) := \text{the right hand side of the equation (2.6)},$$

and

$$B : C_L([a,b],[a,b]) \to C_L([a,b],[a,b]),$$
$$B(y)(t) := \text{the right hand side of the equation (2.3)}.$$

These are well-defined. Observe that from (2.6) and (2.3) we have

$$\begin{aligned}|A(x)(t)-B(x)(t)| &= \left|y(a)+\int_a^t f(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[m]}(s))\,\mathrm{d}s - x(a)-\right.\\ &\left.-\int_a^t f(s,x^{[1]}(s),x^{[2]}(s),\dots,x^{[m]}(s))\,\mathrm{d}s-\int_a^t g(s)\mathrm{d}s\right| =\\ &=\left|\int_a^t g(s)\mathrm{d}s\right|\le \varepsilon(b-a).\end{aligned}$$

We mention that the operator A is a contraction with the constant

$$L_A := L_f(b-a)\sum_{i=1}^{n} iL^{m-i}.$$

Denote by x_A^* its unique fixed point and let y_B^* denote a fixed point of the operator B. Then due to Lemma 1.1 we have:

$$\|x_A^*-y_B^*\|_C \le \frac{\varepsilon(b-a)}{1-L_f(b-a)\sum_{i=1}^{n} iL^{m-i}}.$$

Now the obtained relation means that the equation (1.1) is Ulam–Hyers stable. Note that $c_f = \dfrac{b-a}{1-L_f(b-a)\sum_{i=1}^{n} iL^{m-i}}$. □

3. Example

Consider the first order iterative functional-differential equation

$$x'(t)=ax(x(t))+bx(t)+c, \quad t\in[-h,h],\ h>0, \tag{3.1}$$

with the initial value condition

$$x(0)=x_0 \tag{3.2}$$

and the inequality

$$|y'(t)-ay(y(t))-by(t)-c|<\varepsilon, \quad t\in[-h,h]. \tag{3.3}$$

Theorem 3.1. *Let $L>0$. Suppose that, concerning the initial value problem* (3.1) + (3.2)*, the following assumptions hold:*

i. $h\big(|a|+|b|\big)+|c|\le \min\left\{\dfrac{h-x_0}{2h},\dfrac{h+x_0}{2h},L\right\}$;

ii. $2h\max\{|a|,|b|\}(L+2)<1$.

Then

(1) *the Cauchy problem* (3.1)+(3.2) *has a unique solution in* $C_L([-h,h],[-h,h])$;

(2) *if* $y\in C_L([-h,h],[-h,h])$ *is a solution of the inequality* (3.3) *so that*

$$h\big(|a|+|b|\big)+|c|\le \min\left\{\frac{h-y(0)}{2h},\frac{h+y(0)}{2h},L\right\},$$

then the equation (3.1) *is Ulam–Hyers stable.*

Proof. Notice that in this case we have

$$f(t,u,v) = av + bu + c,$$
$$L_f = \max\{|a|, |b|\},$$
$$M_f = h(|a| + |b|) + |c|.$$

Using Theorem 2.1, the proof follows. □

4. Generalized Ulam–Hyers–Rassias stability

We present now another stability result.

Theorem 4.1. *Let $\varphi : [a,b] \to \mathbb{R}_+$ be an increasing continuous operator. Consider the iterative functional–differential equation* (1.1)*. Suppose that we are in the conditions of Theorem 1.1 and, additionally, there exists $\lambda_\varphi > 0$, such that*

$$\int_a^t \varphi(s)\,ds \le \lambda_\varphi \varphi(t), \quad \forall t \in [a,b].$$

Then the equation (1.1) *is generalized Ulam-Hyers-Rassias stable on the set* $C_L([a,b],[a,b])$.

Proof. Let $y \in C_L([a,b],[a,b])$ be a solution of the differential inequality (1.3), i.e. satisfying

$$|y'(t) - f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t))| \le \varphi(t), \quad t \in [a,b].$$

Then, y is a solution of the integral inequality

$$\left| y(t) - y(a) - \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s \right| \le \lambda_\varphi \varphi(t), \quad \forall t \in [a,b].$$

Indeed, we have the following

$$y'(t) = f(t, y^{[1]}(t), y^{[2]}(t), \ldots, y^{[m]}(t)) + h(t), \text{ with } |h(t)| \le \varphi(t), \quad \forall t \in [a,b].$$

Consequently, we obtain

$$y(t) = y(a) + \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s + \int_a^t h(s)\mathrm{d}s, \quad t \in [a,b], \tag{4.1}$$

implying

$$\left| y(t) - y(a) - \int_a^t f(s, y^{[1]}(s), y^{[2]}(s), \ldots, y^{[m]}(s))\,\mathrm{d}s \right| \le \left| \int_a^t h(s)\mathrm{d}s \right| \le$$
$$\le \int_a^t |h(s)|\mathrm{d}s \le \int_a^t \varphi(s)\mathrm{d}s \le \lambda_\varphi \varphi(t).$$

Henceforward we take $x \in C_L([a,b],[a,b])$, denoting the unique solution of the Cauchy problem (2.4)+(2.5).

Then, the following relation holds:

$$x(t) = y(a) + \int_a^t f(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,\mathrm{d}s, \quad t \in [a,b]. \tag{4.2}$$

Similarly to the proof of Theorem 1.1 we define the operators

$A^1 : C_L([a,b],[a,b]) \to C_L([a,b],[a,b]),\ A^1(x)(t) :=$ the right hand side of (4.1),

$B^1 : C_L([a,b],[a,b]) \to C_L([a,b],[a,b]),\ B^1(y)(t) :=$ the right hand side of (4.2).

Then, we have

$$|A^1(x)(t) - B^1(x)(t)| = \Big| y(a) + \int_a^t f(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,\mathrm{d}s - x(a) -$$

$$- \int_a^t f(s, x^{[1]}(s), x^{[2]}(s), \ldots, x^{[m]}(s))\,\mathrm{d}s - \int_a^t h(s)\mathrm{d}s \Big| =$$

$$= \left| \int_a^t h(s)\mathrm{d}s \right| \le \int_a^t |h(s)|\mathrm{d}s \le \lambda_\varphi \varphi(t)$$

Denote by $x^*_{A^1}$ the unique fixed point of the contraction A^1 and let $y^*_{B^1}$ denote a fixed point of the operator B^1. From Lemma 1.1 we obtain

$$\|x^*_{A^1} - y^*_{B^1}\|_C \le \frac{\lambda_\varphi \varphi(t)}{1 - L_f(b-a)\sum_{i=1}^n iL^{m-i}},$$

that is, the equation (1.1) is generalized Ulam–Hyers–Rassias stable with

$$c_{f,\varphi} = \frac{\lambda_\varphi}{1 - L_f(b-a)\sum_{i=1}^n iL^{m-i}}. \qquad \square$$

References

[1] L. Cădariu, *Stabilitatea Ulam-Hyers-Bourgin pentru ecuaţii funcţionale*, Editura Universităţii de Vest, Timişoara, 2007.

[2] E. Egri, *On First and Second Order Iterative Functional-Differential Equations and Systems*, Presa Universitară Clujeană, Cluj-Napoca, 2008.

[3] S.-M. Jung, *Hyers-Ulam stability of linear differential equations of first order, III,* J. Math. Anal. Appl., **311**(2005), 139-146.

[4] S.-M. Jung and Th.M. Rassias, *Generalized Hyers-Ulam stability of Riccati differential equation,* Math. Ineq. Appl., **11**(2008), No. 4, 777-782.

[5] I.A. Rus, *Ulam stability of ordinary differential equations,* Studia Univ. Babeş-Bolyai Math., **54**(2009), No. 4, 125-133.

[6] I.A. Rus, *Gronwall lemmas: Ten open problems,* Sci. Math. Japon., **70**(2009), No. 2, 221-228.

[7] I.A. Rus, *Remarks on Ulam stability of the operatorial equations,* Fixed Point Theory, **10**(2009), No. 2, 305-320.

[8] S.-E. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$*, Bull. Korean Math. Soc. **39**(2002), No. 2, 309-315.

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