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FIXED POINT THEOREMS ON ADMISSIBLE MULTIRETRACTS APPLICABLE TO DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is to present new fixed point theorems which can be used in the theory of dynamical systems. The main novelty consists especially in the fact that non-metric versions of these theorems are formulated. In the metric case, we give some simple applications to differential equations.

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1. Introduction

In [24], [25], the authors generalized, in the metric case, the concept of absolute neighbourhood retracts to absolute neighbourhood multiretracts. Here, this notion will be furthermore extended, for non-metric spaces, to the one of admissible multiretracts.

The main aim of this paper is to formulate Lefschetz-type and Nielsen-type asymptotic fixed point theorems on these very general Hausdorff topological spaces.

As explained e.g. in [19, Chapter 15.5], the asymptotic fixed point theory concerns theorems in which the existence of fixed points of a map is established from assumptions imposed on its iterates. Usually, only a certain amount of compactness is needed in this way, for maps under consideration. The typical example is the Browder fixed point theorem [11] and its various generalizations in [2]–[4], [5]–[7], [8], [9], [10], [12], [13], [14], [15], [16], [17], [18], [19], [22], [23], [24].

Since discrete dynamical systems are generated by the iterates of maps, the asymptotic fixed point theorems can be regarded in this light as a crucial part of their theory, because stationary regimes play always an essential role among dynamics.

On the other hand, the nontrivial application of our new fixed point theorems to differential equations (i.e. practically only in the metric case) seems to be rather delicate. In the last part of our paper, we indicate possible applications in the form of two theorems and several remarks.

2. Some auxiliary definitions

In this paper, all spaces are assumed to be Hausdorff topological spaces and all mappings under consideration are continuous.

Let $f\colon X \to X$ be a map. Then the sequence

$$O(x) := \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$$

is called the *orbit* of x under f.

A compact set $A \subset X$ is called a *window* for f if, for every $x \in X$, we have:

$$O(x) \cap A \neq \emptyset,$$

and it is called an *attractor* for f if

$$\overline{O(x)} \cap A \neq \emptyset,$$

where $\overline{O(x)}$ denotes the closure of $O(x)$ in X.

(1.1) *Evidently, if A is a window for f, then A is also an attractor for f.*

The following example demonstrates that the converse to (1.1) is not true.

(1.2) Example. Let (X, d) be a complete metric space and $f\colon X \to X$ be a contraction. Letting $A := \{x_0\}$, where $x_0 \in X$ is a unique fixed point of f, A is an attractor for f, but it is not a window for f.

Recall that a map $f\colon X \to X$ is *compact* if there exists a compact set $K \subset X$ such that $f(X) \subset K$. Similarly, f is *locally compact* if, for every point $x \in X$, there exists an open neighbourhood V_x of x in X and a compact set $K_x \subset X$ such that $f(V_x) \subset K_x$. The set

$$C_f := \bigcap_{n=1}^{\infty} f^n(X)$$

is called the *core* of f.

Let E be a topological vector space over the field of real numbers $\mathbb{R}$. We say that E is a *Klee admissible space* (cf. [2]) if, for any compact $K \subset E$ and for any open neighbourhood V of $0 \in E$, there exists a map:

$$\pi_V\colon K \to E$$

such that the following conditions are satisfied:

(1.3) $\pi_V(x) \in x + V$, for any $x \in K$,

(1.4) there exists a natural number $n = n_K$ such that $\pi_V(K) \subset E^n$, where E^n is an n-dimensional subspace of E.

Roughly speaking, a space E is Klee admissible if compact mappings into E can be approximated by compact finite dimensional mappings.

(1.5) Open Problem. *Is it true that any topological vector space is Klee admissible?*

Let Top_2 be the category of pairs of Hausdorff topological spaces and continuous mappings of such pairs. By a pair (X, A) in Top_2, we understand a Hausdorff space X and its subset A. A pair $(X, \emptyset)$ will be shortly denoted by X. By a map $f\colon (X, A) \to (Y, B)$, we understand a continuous map from X to Y such that $f(A) \subset B$. We shall use the following notations: if $f\colon (X, A) \to (Y, B)$ is a map of pairs, then by $f_X : X \to Y$ and $f_A\colon A \to B$ we shall understand the respective induced mappings. Let us also denote by Vect_G the category of graded vector spaces over the field of rational numbers $\mathbb{Q}$ and linear maps of degree zero between such spaces. By $H\colon \mathrm{Top}_2 \to \mathrm{Vect}_G$, we denote the Čech homology functor with compact carriers and coefficients in $\mathbb{Q}$.

Thus, for any pair (X, A), we have

$$H(X, A) = \{H_q(X, A)\}_{q\geq 0},$$

a graded vector space in Vect_G and, for any map $f\colon (X, A) \to (Y, B)$, we have the induced linear map

$$f_* := \{f_{*q}\}\colon H(X, A) \to H(Y, B),$$

where $f_{*q}\colon H_q(X, A) \to H_q(Y, B)$ is a linear map from the q-dimensional homology $H_q(X, A)$ of the pair (X, A) into the q-dimensional homology $H_q(Y, B)$ of the pair (Y, B).

For further properties of H, we recommend the monograph [2].

A non-empty space X is called *acyclic* if:

(1.6) $H_q(X) = 0$, for every $q \geq 1$, and

(1.7) $H_0(X) \approx \mathbb{Q}$.

(1.8) Definition. A map $p\colon \Gamma \to X$ is called a *Vietoris map* if the following conditions are safisfied:

(1.8.1) p is onto and closed,

(1.8.2) for every $x \in X$, the set $p^{-1}(x)$ is compact and acyclic.

(1.9) Vietoris Theorem. ([15]) *If $p\colon \Gamma \to X$ is a Vietoris map, then the induced linear map $p_*\colon H(\Gamma) \xrightarrow{\sim} H(X)$ is an isomorphism, i.e., for every $q \geq 0$, the linear map*

$$p_{*q}\colon H_q(\Gamma) \xrightarrow{\sim} H_q(X)$$

is an isomorphism.

For further properties of Vietoris mappings, see e.g. [20].

The following notions introduced in definitions will play here a crucial role. At first, by $\varphi\colon X \multimap Y$, we shall denote a multivalued map, i.e. a map which assigns to every point $x \in X$ a compact nonempty set $\varphi(x) \subset Y$.

(1.10) Definition. (cf. [2], [15]) A multivalued map $\varphi\colon X \multimap Y$ is called *admissible* if there exists a diagram

$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y$$

in which p is a Vietoris map, q is continuous and we have:

$$\varphi(x) = q(p^{-1}(x)), \quad \text{for every } x \in X.$$

Note that the class of admissible mappings is quite large. In particular, it contains compositions of acyclic mappings.

(1.11) Definition. (cf. [24]) A map $r\colon X \to Y$ is said to be a *multiretraction map* (*mr-map*) if there exists an admissible map $\varphi\colon Y \multimap X$ such that $r \circ \varphi = \mathrm{Id}_Y$.

(1.12) Definition. (cf. [24]) A Hausdorff topological space X is called an *admissible multiretract* ($X \in \mathrm{AMR}$) if there exists an open subset U of some space E which is admissible in the sense of the Klee such that X is a multiretract of U, i.e. there is an mr-map $r\colon U \to X$.

Observe that if X is a retract of $U \subset E$, then $X \in \mathrm{AMR}$, where E is a Klee admissible space. Moreover, if there exists a Vietoris map $p\colon X \to U$, then $X \in \mathrm{AMR}$. Note that the class of AMR-spaces is obviously larger than of ANR-spaces.

For some nontrivial examples and more details concerning metric AMR-spaces, we recommend [24].

3. Maps with only a certain amount of compactness

In this section, we shall consider noncompact mappings for which the Lefschetz and, in particular, the Schauder fixed point theorems are true.

(2.1) Definition. ([12], cf. also [13], [2] and [21]) A map $f\colon X \to X$ is called a *compact absorbing contraction* ($f \in \mathrm{CAC}(X)$) if there exists an open set $U \subset X$ such that:

(2.1.1) $f(U) \subset U$,
(2.1.2) $\overline{f(U)}$ is a compact subset of U,
(2.1.3) for every $x \in X$, there exists a natural number n_x such that $f^{n_x}(x) \in U$.

We let

$\mathrm{K}(X) := \{f : X \to X \mid f$ is compact$\}$,

$\mathrm{EC}(X) := \{f\colon X \to X \mid f$ is locally compact and there exists a natural number n such that $f^n \in \mathrm{K}(X)\}$,

$\mathrm{CW}(X) := \{f\colon X \to X \mid f$ is locally compact and there exists a compact window A for $f\}$,

$\mathrm{ASC}(X) := \{f\colon X \to X \mid f$ is locally compact, every orbit $O(x)$ is relatively compact and the core C_f of f is nonempty and relatively compact$\}$,

$\mathrm{CA}(X) := \{f\colon X \to X \mid f$ is locally compact and there exists a compact attractor A for $f\}$.

Note that if X is a locally compact Hausdorff space, then every map $f\colon X \to X$ is locally compact. Let us also note that in Definition (2.1) we do not assume that f is locally compact.

It is well known that, for locally compact Hausdorff spaces, it holds (see [15], [16], [21]):

$$\mathrm{K}(X) \subset \mathrm{EC}(X) \subset \mathrm{CW}(X) \subset \mathrm{ASC}(X) \subset \mathrm{CA}(X) \subset \mathrm{CAC}(X). \tag{2.2}$$

Moreover, each of the above inclusions is proper. The inclusion $\mathrm{CW}(X) \subset \mathrm{ASC}(X)$ is obvious.

Now, we are still going to generalize the class of compact absorbing contractions. For a map $f\colon X \to X$, by $\Lambda(f)$, we shall denote the *Lefschetz number* of f.

We say that f is a *Lefschetz map* if the Lefschetz number $\Lambda(f)$ of f is well defined and $\Lambda(f) \neq \emptyset$ implies that $\mathrm{Fix}(f) := \{x \in X \mid f(x) = x\} \neq \emptyset$.

For a map $f\colon (X,A) \to (X,A)$, by $f_X\colon X \to X$ and $f_A\colon A \to A$, we denote the respective mappings induced by f, i.e., $f_X(x) = f(x)$, for every $x \in X$, and $f_A(x) = f(x)$, for every $x \in A$.

(2.3) Definition. (cf. [18]) Let $f, h\colon (X,A) \to (X,A)$ be two mappings. We say that $f_X\colon X \to X$ is a *generalized compact absorbing contraction* with respect to h (written $f_X \in \mathrm{GCAC}(X)$) if the following conditions are satisfied:

(2.3.1) $f_A\colon A \to A$ is a Lefschetz map,

(2.3.2) for every compact $K \subset X$, there exists $n = n_K$ such that $f^n(h(K)) \subset A$ (or $h(f^n(K)) \subset A$ and $f(h^{-1}(A)) \subset h^{-1}(A)$),

(2.3.3) $h_*\colon H(X,A) \to H(X,A)$ is an epimorphism (or $h_*\colon H(X,A) \to H(X,A)$ is a monomorphism).

(2.4) Remark. Observe that if $X \in \mathrm{AMR}$, A is an open subset of X and $h = \mathrm{Id}_{(X,A)}$, then the class of GCAC-mappings reduces to the class of CAC-mappings.

It will be also useful to consider the following class of mappings.

(2.5) Definition. (cf. [18]) Let $f\colon (X,A) \to (X,A)$ be a map. We say that $f_X\colon X \to X$ is *acyclically compact absorbing contraction* (written $f_X \in \mathrm{ACAC}(X)$) if the following conditions are satisfied:

(2.5.1) $f_A\colon A \to A$ is a compact map,

(2.5.2) there exists an acyclic set $K \subset X$ such that $\overline{f_A(A)} \subset K$ and $(K \cap A) \in \mathrm{AMR}$,

(2.5.3) there exists an $n = n_K$ such that $f^n(K) \subset A$.

4. Lefschetz theorems for generalized CAC-maps on AMR-spaces

At first, we shall formulate the Lefschetz-type fixed point theorems for CAC-mappings on AMR-spaces. We are able to obtain the following result:

(3.1) Theorem. *Let $X \in \mathrm{AMR}$ and $f \in \mathrm{CAC}(X)$. Then f is a Lefschetz map.*

(3.2) Remark.

(3.2.1) For a metric AMR-space X, Theorem (3.1) was formulated as a single-valued version of Theorem 5.3 in [24].

(3.2.2) In the above case, we have to prove (3.1) for compact mappings (cf. [24] and [21]), and then we can proceed quite analogously as in the proof of Theorem 4.4 in [18].

The most general result in this field is the following:

(3.3) Theorem. *Let X be a Hausdorff topological space and $f_X \in \mathrm{GCAC}(X)$. Then f_X is a Lefschetz map.*

Sketch of the proof. According to Definition (2.3), we have a map $f\colon (X,A) \to (X,A)$ such that the conditions (2.3.1)–(2.3.3) are satisfied. So, we have three maps:

$$f\colon (X,A) \to (X,A), \quad f_X\colon X \to X, \quad f_A\colon A \to A$$

such that f_A is a Lefschetz map. Now, from the assumptions (2.3.2) and (2.3.3), we can deduce that the Lefschetz number $\Lambda(f) = 0$ (cf. the proof of (4.4) in [18]).

On the other hand, it is known (see [2] and [16]) that

$$0 = \Lambda(f) = \Lambda(f_X) - \Lambda(f_A).$$

Assuming $\Lambda(f_X) \neq 0$, we obtain $\Lambda(f_A) \neq 0$, and since f_A is a Lefschetz map, we get that $\mathrm{Fix}(f_A) \neq \emptyset$. Thus, $\mathrm{Fix}(f_X) \neq \emptyset$, and the proof is complete. □

Note that from Theorem (3.3), we can deduce far reaching generalizations of the Schauder fixed point theorem. Recall that if X is an acyclic space then, for every $f\colon X \to X$, we have $\Lambda(f) = 1$ (cf. e.g. [2], [15]).

Hence, we get:

(3.4) Corollary. *If X is still an acyclic space and $f_X \in \mathrm{GCAC}(X)$, then*

$$\mathrm{Fix}(f_X) \neq \emptyset.$$

Similarly, in view of the above inclusion hierarchy (2.2), Theorem (3.1) immediately implies:

(3.5) Corollary. *Let X be an acyclic* AMR*-space. If $f \in \mathrm{K}(X)$ or $f \in \mathrm{EC}(X)$or $f \in \mathrm{CW}(X)$ or $f \in \mathrm{ASC}(X)$ or $f \in \mathrm{CA}(X)$ or $f \in \mathrm{CAC}(X)$, then* $\mathrm{Fix}(f) \neq \emptyset$.

Finally, using the same techniques as in the proof of Theorem 4.5 in [18], we are able to proof the following:

(3.6) Theorem. *If $f_X \in \mathrm{ACAC}(X)$, then* $\mathrm{Fix}(f_X) \neq \emptyset$.

Let us also note that all the results obtained in this section represent very general asymptotic fixed point theorems on a large class of topological spaces.

5. Nielsen theorem for CAC-maps on AMR-spaces

In this section, by X we shall denote an AMR-space and by $f\colon X \to X$ a CAC-mapping. We shall assume the following:

(4.1) Assumption. We additionally suppose that X admits a universal covering $P_X\colon \tilde{X} \to X$.

Using the above Assumption (4.1), there exists a lift $\tilde{f}\colon \tilde{X} \to \tilde{X}$ of $f\colon X \to X$, i.e. that the following diagram is commutative:

$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "P_x"'] & \tilde{X} \arrow[d, "P_x"] \\ X \arrow[r, "f"] & X \end{tikzcd}$$

We let

$$\Theta_X := \{\alpha\colon \tilde{X} \to \tilde{X} \mid P_X \circ \alpha = P_X\}.$$

We fix points $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ and a loop $\omega\colon [0,1] \to X$ based at x_0. Let $\tilde{\omega}$ denote the unique lift of ω starting from $\tilde{x}_0$. We subordinate to $[\omega] \in \pi_1(X, x_0)$ the unique transformation from Θ_X sending $\tilde{\omega}(0)$ to $\tilde{\omega}(1)$, where $\pi_1(X, x_0)$ is the fundamental group of X based at x_0. Then the homomorphism $\tilde{f}!\colon \Theta_X \to \Theta_X$ corresponds to the induced endomorphism $f_{\#}\colon \pi_1(X, x_0) \to \pi_1(X, f(x_0))$.

Now, it is easy to see that the set $\mathrm{Fix}(f) := \{x \in X \mid x = f(x)\}$ is compact ($f \in \mathrm{CAC}(X)$). The above lifts split the set $\mathrm{Fix}(f)$ into finite Nielsen classes (for more details, see [5], [6]).

We still need the following assumption which, in particular, implies (4.1):

(4.2) Assumption. X is paracompact, connected, locally contractible and $\pi_1(X)$ is a finitely generated abelian fundamental group of X.

Assumption (4.2) also implies that there exists a normal subgroup $H \subset \Theta_X$ of a finite index satisfying $\tilde{f}!(H) \subset H$.

We say that f is *H-admissible* if $f \in \mathrm{CAC}(X)$, $X \in \mathrm{AMR}$ and (4.2) is satisfied. In the rest of this section, we will consider only H-admissible mappings.

(4.3) Definition. A Nielsen class of $\mathrm{Fix}(f)$ is *essential* if the related Lefschetz number is nontrivial, i.e. if $\Lambda(\alpha \circ \tilde{f}) \neq 0$, where $\alpha \in H$.

The above definition does not depend on the choice of $\alpha \in H$ (see Lemma 5.5 in [5]).

(4.4) Definition. We define the *Nielsen number* $N(f)$ of f (modulo H) as the number of essential Nielsen classes of $\mathrm{Fix}(f)$.

Now, from the homotopy invariance of the Lefschetz number, it immediately follows:

(4.5) Theorem. *Assume that $f, g\colon X \to X$ are two homotopic H-admissible mappings. Then we have:*

(4.5.1) *$N(f) = N(g)$,*

(4.5.2) *f and g have at least $N(f)$ fixed points.*

6. Simple application in the metric case

For a simple example of the application of the above fixed point theory, let us consider the system of differential equations

$$x' = F(t, x), \quad F(t, x) \equiv F(t + \tau, x),\ \tau > 0, \tag{5.1}$$

where $F\colon [0, \tau] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, i.e. $F(\cdot, x)$ is measurable, for every $x \in \mathbb{R}^n$, and $F(t, \cdot)$ is continuous, for almost all $t \in [0, \tau]$. Assuming, furthermore, that system (5.1) satisfies a uniqueness condition and that the (Carathéodory) locally absolutely continuous solutions $x(\cdot)$ of (5.1) exist on the whole line, we can define the *Poincaré translation operator* $T_\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ along the trajectories of system (5.1) as

$$T_\tau(x_0) := \{x(\tau) \mid x(\cdot) \text{ is a solution of (5.1) with } x(0) = x_0\}, \tag{5.2}$$

where $x_0 \in \mathbb{R}^n$ and $\tau > 0$ is a given number.

This operator is well known (see e.g. [1], [4]) to be completely continuous and locally compact and to have the property $T_\tau^k(x_0) = T_{k\tau}(x_0)$, where T_τ^k denotes the kth iterate of T_τ defined in (5.2).

Suppose, for a moment, that the system (5.1) is *dissipative* in the sense of N. Levinson, i.e. there exists a constant $D > 0$ such that

$$\limsup_{t \to \infty} |x(t)| < D \tag{5.3}$$

holds, for every solution $x(\cdot)$ of (5.1), i.e.

$$\limsup_{t \to \infty} \left| T_\tau^k(x_0) \right| < D, \tag{5.4}$$

for every $x_0 \in \mathbb{R}^n$, where T_τ is defined in (5.2).

As already pointed out e.g. in [1], [10], we can easily check that $T_\tau \in \mathrm{CA}(\mathbb{R}^n)$, and so that $T_\tau\colon \mathbb{R}^n \to \mathbb{R}^n$ is H-admissible.

Thus, as a trivial consequence of Corollary (3.5), T_τ has a fixed point determining a τ-periodic solution $x(\cdot)$ of (5.1) with $|x(0)| < D$.

Assuming still that an AMR-space $A \subset \mathbb{R}^n$ is a compact subinvariant set w.r.t. T_τ such that $\{x \in \mathbb{R}^n \mid |x| < D\} \subset A$, Theorem (3.6) can also be applied, when taking $X = K := \mathbb{R}^n$. Nevertheless, such an application would not have much meaning, because under the additional assumption above only $|x(0)| < D$ can be proved again, for a τ-periodic solution $x(\cdot)$ of (5.1).

On the other hand, assuming still that $T_\tau|_A \colon A \to A$ holds, where $A \subset \mathbb{R}^n$ is a compact, acyclic AMR-space (observe that, in view of (5.3) and the subinvariance of A, it must be $A \cap \{x \in \mathbb{R}^n \mid |x| < D\} \neq \emptyset$), and subsequently $T_\tau|_A \in \mathrm{K}(A)$, Corollary (3.5) implies the existence of a τ-periodic solution $x(\cdot)$ of (5.1), this time with $x(0) \in A \cap \{x \in \mathbb{R}^n \mid |x| < D\}$.

(5.5) Remark. If $A \subset \mathbb{R}^n$ is as above, then (5.1) need not be dissipative. On the other hand if, for a compact $A \subset \mathbb{R}^n$,

$$\lim_{t\to\infty} \varrho(x(t), A) = 0$$

holds, for all solutions $x(\cdot)$ of (5.1) with $x(0) \in \mathbb{R}^n$, where ϱ stands for the set distance, then we can always assume that (5.3) is satisfied, because $A \subset \{x \in \mathbb{R}^n \mid |x| < D\}$ holds, for a sufficiently large $D > 0$.

For a not necessarily dissipative system (5.1), we can give the following theorems:

(5.6) Theorem. *Assume that $F \colon [0,\tau] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that a uniqueness condition is satisfied. Assume, furthermore, that $A \subset \mathbb{R}^n$ is a locally compact, acyclic AMR-space and $A_0 \subset A$ its compact subset such that $x(\tau) \in A$ and*

$$\lim_{t\to\infty} \varrho(x(t), A_0) = 0$$

hold, for all solutions $x(\cdot)$ of (5.1) with $x(0) \in A$, where ϱ stands for the set distance. Then the system (5.1) admits a τ-periodic solution $x(\cdot)$ such that $x(0) \in A_0$.

Sketch of the proof. One can readily check that $T_\tau|_A \colon A \to A$ belongs to the $\mathrm{CA}(A)$-class, where A is as above. Therefore, the conclusion follows by means of (3.5). □

(5.7) Remark. Let F be as in Theorem (5.6). Assume that $x(\lambda\tau) \in A$ holds, for each $\lambda \in (0,1]$ and all solutions $x(\cdot)$ of (5.1) with $x(0) \in A$, where $A \subset \mathbb{R}^n$ is a compact AMR-space. Then, by means of (3.1), we have that (cf. e.g. [2])

$$\Lambda(T_\tau|_A) = \Lambda(\mathrm{Id}_A) = \chi(A),$$

where $\chi(A)$ denotes the Euler–Poincaré characteristic of A. Hence, (5.1) also admits a τ-periodic solution, provided $\chi(A) \neq 0$. Observe that A need not be acyclic.

(5.8) Theorem. *Assume that $F \colon [0,\tau] \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function such that a uniqueness condition is satisfied. Assume, furthermore, that a compact AMR-subset $A \subset \mathbb{R}^n$ exists such that $x(\lambda\tau) \in A$ holds, for each $\lambda \in (0,1]$ and every solution $x(\cdot)$ of (5.1) with $x(0) \in A$. Let h be a continuous self-map of A, i.e. $h \colon A \to A$. If A additionally satisfies (4.2), but need not be acyclic, then the system (5.1) has at least $N(h)$ solutions $x(\cdot)$ such that $x(0) = h(x(\tau))$.*

Sketch of the proof. One can readily check that, for each $\lambda \in [0,1]$, $T_{\lambda\tau}|_A \colon A \to A$ as well as the composition $h \circ T_{\lambda\tau}|_A \colon A \to A$ belong to the $\mathrm{K}(A)$-class and with (4.2) they are H-admissible maps. Moreover, $T_\tau|_A$ is admissibly homotopic to the identity on A (see e.g. [1], [2]), and subsequently $N(h \circ T_\tau|_A) = N(h)$ holds. The application of Theorem (4.5) completes the proof. □

(5.9) Remark. Theorem (5.8) provides the lower estimate of the number of solutions $x(\cdot)$ of (5.1) such that $x(0) = h(x(\tau))$. The computation of the Nielsen number $N(h)$ can, however, be a difficult task. For $h = \mathrm{Id}_A$, we already know that at least one τ-periodic solution of (5.1) exists, provided A is still acyclic or such that $\chi(A) \neq 0$. On the other hand, it is hopeless to expect that $N(\mathrm{Id}_A) > 1$, for some A.

(5.10) Remark. The subinvariance of A w.r.t. $T_{\lambda\tau}|_A$, $\lambda \in (0,1]$, in (5.6)–(5.8) can be guaranteed by means of Liapunov-like bounding functions. For more details, see e.g. [2], [4].

7. Concluding remark

The results presented above can be obviously generalized in many directions. In particular,

(i) admissible multivalued maps,
(ii) periodic points and orbits,
(iii) further relative versions

can be taken into account with this respect.

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