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# FIXED POINT THEOREMS FOR CONDENSING MULTIVALUED MAPPINGS UNDER WEAK TOPOLOGY FEATURES

RAVI P. AGARWAL*, DONAL O'REGAN** AND MOHAMED-AZIZ TAOUDI***

*Department of Mathematical Sciences, Florida Institute of Technology
150 West University Boulevard, Melbourne Florida 32901, USA
E-mail: agarwal@fit.edu
and
KFUPM Chair Professor, Mathematics and Statistics Department
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia.

**Department of Mathematics, National University of Ireland
Galway, Ireland.
E-mail: donal.oregan@nuigalway.ie

***Université Cadi Ayyad, Laboratoire de Mathématiques
et de Dynamique de Populations, Marrakech, Maroc
and
Centre Universitaire Kalaa des Sraghna
B.P. 263, Kalaa des Sraghna, Maroc.
E-mail: taoudi@ucam.ac.ma

**Abstract.** We present new fixed point theorems for weakly condensing multivalued maps with weakly sequentially closed graph. Our fixed point results are obtained under Sadovskii's, Leray-Schauder's and Furi-Pera's type conditions.

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## 1. Introduction

In this paper, we present new fixed-point results for weakly condensing multivalued maps with weakly sequentially closed graph on a Banach space. Our results in particular extend those of Arino et al. [4], Agarwal and O'Regan [3] and O'Regan [14, 15]. For the remainder of this section we gather some notations and preliminary facts. Let $X$ be a Banach space, let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of $X$ and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of $X$. Also, let $B_r$ denote the closed ball centered at 0 with radius $r$.

In our considerations the following definition will play an important role.

**Definition 1.1.** [5] *A function* $\psi\colon \mathcal{B}(X) \to \mathbb{R}_+$ *is said to be a measure of weak noncompactness if it satisfies the following conditions :*

(1) *The family* $\ker(\psi) = \{M \in \mathcal{B}(X) : \psi(M) = 0\}$ *is nonempty and* $\ker(\psi)$ *is contained in the set of relatively weakly compact sets of* $X$.

(2) $M_1 \subseteq M_2 \Rightarrow \psi(M_1) \leq \psi(M_2)$.

(3) $\psi(\overline{co}(M)) = \psi(M)$, *where* $\overline{co}(M)$ *is the closed convex hull of* $M$.

(4) $\psi(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\psi(M_1) + (1-\lambda)\psi(M_2)$ *for* $\lambda \in [0,1]$.

(5) *If* $(M_n)_{n\geq 1}$ *is a sequence of nonempty weakly closed subsets of* $X$ *with* $M_1$ *bounded and* $M_1 \supseteq M_2 \supseteq \ldots \supseteq M_n \supseteq \ldots$ *such that* $\lim_{n\to\infty} \psi(M_n) = 0$, *then* $M_\infty := \bigcap_{n=1}^{\infty} M_n$ *is nonempty.*

The family $\ker\psi$ described in (1) is said to be the kernel of the measure of weak noncompactness $\psi$. Note that the intersection set $M_\infty$ from (5) belongs to $\ker\psi$ since $\psi(M_\infty) \leq \psi(M_n)$ for every $n$ and $\lim_{n\to\infty}\psi(M_n) = 0$. Also, it can be easily verified that the measure $\psi$ satisfies

$$\psi(\overline{M^w}) = \psi(M) \tag{1.1}$$

where $\overline{M^w}$ is the weak closure of $M$.

A measure of weak noncompactness $\psi$ is said to be *regular* if

$$\psi(M) = 0 \text{ if and only if } M \text{ is relatively weakly compact.} \tag{1.2}$$

*subadditive* if

$$\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2), \tag{1.3}$$

*homogeneous* if

$$\psi(\lambda M) = |\lambda|\psi(M), \quad \lambda \in \mathbb{R}, \tag{1.4}$$

*set additive* if

$$\psi(M_1 \cup M_2) = max(\psi(M_1), \psi(M_2)). \tag{1.5}$$

The first important example of a measure of weak noncompactness has been defined by De Blasi [7] as follows :

$$w(M) = \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\}, \tag{1.6}$$

for each $M \in \mathcal{B}(X)$.

Notice that $w(.)$ is regular, homogeneous, subadditive and set additive (see [7]).

In what follows we shall recall some classical definitions and results regarding multivalued mappings. Let $X$ and $Y$ be topological spaces. A multivalued map $F\colon X \to 2^Y$ is a point to set function if for each $x \in X$, $F(x)$ is a nonempty subset of $Y$. For a subset $M$ of $X$ we write $F(M) = \cup_{x\in M} F(x)$ and $F^{-1}(M) = \{x \in X : F(x) \cap M \neq \emptyset\}$. The *graph* of $F$ is the set $Gr(F) = \{(x,y) \in X \times Y : y \in F(x)\}$. We say that $F$ is *upper semicontinuous* (u.s.c. for short) at $x \in X$ if for every neighborhood $V$ of $F(x)$ there exists a neighborhood $U$ of $x$ with $F(U) \subseteq V$ (equivalently, $F : X \to 2^Y$ is u.s.c. if for any net $\{x_\alpha\}$ in $X$ and any closed set $B$ in $Y$ with $x_\alpha \to x_0 \in X$ and $F(x_\alpha) \cap B \neq \emptyset$ for all $\alpha$, we have $F(x_0) \cap B \neq \emptyset$). We say that $F\colon X \to 2^Y$ is upper semicontinuous if it is upper semicontinuous at every $x \in X$ The function $F$ is lower semicontinuous (l.s.c.) if the set $F^{-1}(B)$ is open for any open set $B$ in $Y$ . If $F$ is l.s.c. and u.s.c., then $F$ is continuous.

If $Y$ is compact, and the images $F(x)$ are closed, then $F$ is upper semicontinuous if and only if $F$ has a closed graph. In this case, if $Y$ is compact, we have that $F$ is upper semicontinuous if $x_n \to x$, $y_n \to y$, and $y_n \in F(x_n)$, together imply that $y \in F(x)$. When $X$ is a Banach space we say that $F\colon X \to 2^X$ is weakly upper semicontinuous if $F$ is upper semicontinuous in $X$ endowed with the weak topology. Also, $F\colon X \to 2^X$ is said to have weakly sequentially closed graph if the graph of $F$ is sequentially closed w.r.t. the weak topology of $X$.

**Definition 1.2.** *Let $X$ be a Banach space and let $\psi$ be a measure of weak noncompactness on $X$. A multivalued mapping $B\colon D(B) \subseteq X \to 2^X$ is said to be $\psi$-condensing if it maps bounded sets into bounded sets and $\psi(B(S)) < \psi(S)$ whenever $S$ is a bounded subset of $D(B)$ such that $\psi(S) > 0$.*

We recall the following extension of the Arino-Gautier-Penot fixed point theorem for multivalued mappings. For a proof we refer to [15, Theorem 2.2.].

**Theorem 1.1.** *Let $X$ be a metrizable locally convex linear topological space and let $C$ be a weakly compact, convex subset of $X$. Suppose $F\colon C \to C(C)$ has a weakly sequentially closed graph. Then $F$ has a fixed point; here $C(C)$ denotes the family of nonempty, closed, convex subsets of $C$.*

## 2. Fixed point theorems

We start with the following Sadovskii type fixed point theorem for multivalued mappings with weakly sequentially closed graph.

**Theorem 2.1.** *Let $X$ be a Banach space, $\psi$ a regular set additive measure of weak noncompactness on $X$ and $C$ a nonempty closed convex subset of $X$. Suppose $F\colon C \to C(C)$ is $\psi$-condensing, $F(C)$ is bounded and $F$ has a weakly sequentially closed graph. Then $F$ has a fixed point.*

**Proof.** Choose a point $x_0 \in C$ and let

$$\mathcal{F} = \{A \subseteq C,\ \overline{co}(A) = A,\ x_0 \in A \,\text{and}\ F(x) \in C(A) \,\text{for all}\, x \in A\}.$$

The set $\mathcal{F}$ is nonempty since $C \in \mathcal{F}$. Set

$$M = \bigcap_{A \in \mathcal{F}} A$$

and

$$K = \overline{co}(F(M) \cup \{x_0\}).$$

Clearly $M$ is a closed convex subset of $C$ and $F(x) \in C(M)$ for all $x \in M$. Thus $M \in \mathcal{F}$. This implies $K \subseteq M$. Hence $F(K) \subseteq F(M) \subseteq K$. Consequently $K \in \mathcal{F}$. Hence $M \subseteq K$. As a result $K = M$. Using the properties of the measure of weak noncompactness we get

$$\psi(M) = \psi(K) = \psi(\overline{co}(F(M) \cup \{x_0\})) = \psi(FM),$$

which yields that $M$ is weakly compact. Since $F\colon M \to C(M)$, then the result follows from Theorem 1.1. □

**Remark 2.1.** Theorem 2.1 is the mutivalued version of [11, Theorem 12] and [13, Theorem 2]. It is also an extension of [14, Theorem 2.2] and [15, Theorem 2.3].

Our next result is the following fixed point theorem of Leray-Schauder type.

**Theorem 2.2.** *Let $X$ be a Banach space and $\psi$ a regular set additive measure of weak noncompactness on $X$. Let $Q$ and $C$ be closed, convex subsets of $X$ with $Q \subseteq C$. In addition, let $U$ be a weakly open subset of $Q$ with $0 \in U$. Suppose $F\colon \overline{U^w} \to C(C)$ has a weakly sequentially closed graph , $F(\overline{U^w})$ is bounded and $F$ is a $\psi$-condensing map; here $C(C)$ denotes the family of nonempty, closed, convex subsets of $C$. Also assume $U$ is weakly open in $C$. Then either*

$$F \text{ has a fixed point,} \tag{2.7}$$

*or*

$$\text{there is a point } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda F u; \tag{2.8}$$

*here $\partial_Q U$ is the weak boundary of $U$ in $Q$.*

**Proof.** Suppose (2.8) does not occur and $F$ does not have a fixed point on $\partial_Q U$ (otherwise we are finished since (2.7) occurs). Let

$$M = \{x \in \overline{U^w} : x \in \lambda F x \text{ for some } \lambda \in [0,1]\}.$$

The set $M$ is nonempty since $0 \in U$. Also $M$ is weakly sequentially closed. Indeed let $(x_n)$ be sequence of $M$ which converges weakly to some $x \in \overline{U^w}$ and let $(\lambda_n)$ be a sequence of $[0,1]$ satisfying $x_n \in \lambda_n F x_n$. Then for each $n$ there is a $z_n \in F x_n$ with $x_n = \lambda_n z_n$. By passing to a subsequence if necessary, we may assume that $(\lambda_n)$ converges to some $\lambda \in [0,1]$ and $\lambda_n \neq 0$ for all $n$. This implies that the sequence $(z_n)$ converges to some $z \in \overline{U^w}$ with $x = \lambda z$. Since $F$ has a weakly sequentially closed graph then $z \in F(x)$. Hence $x \in \lambda F x$ and therefore $x \in M$. Thus $M$ is weakly sequentially closed. We now claim that $M$ is relatively weakly compact. Suppose $\psi(M) > 0$. Since $M \subseteq co(F(M) \cup \{0\})$ then

$$\psi(M) \leq \psi(co(F(M) \cup \{0\})) = \psi(F(M)) < \psi(M),$$

which is a contradiction. Hence $\psi(M) = 0$ and therefore $\overline{M^w}$ is weakly compact. This proves our claim. Now let $x \in \overline{M^w}$. Since $\overline{M^w}$ is weakly compact then there is a sequence $(x_n)$ in $M$ which converges weakly to $x$. Since $M$ is weakly sequentially closed we have $x \in M$. Thus $\overline{M^w} = M$. Hence $M$ is weakly closed and therefore weakly compact. From our assumptions we have $M \cap \partial_Q U = \emptyset$. Since $X$ endowed with the weak topology is a locally convex space then there exists a weakly continuous mapping $\rho\colon \overline{U^w} \to [0,1]$ with $\rho(M) = 1$ and $\rho(\partial_Q U) = 0$ (see [9]). Let

$$J(x) = \begin{cases} \rho(x)F(x), & x \in \overline{U^w}, \\ 0, & x \in C \setminus \overline{U^w}. \end{cases}$$

Clearly $J: C \to C(C)$ has a weakly sequentially closed graph since $F$ has a sequentially closed graph. Moreover, for any $S \subseteq C$ we have

$$J(S) \subseteq co(F(S \cap U) \cup \{0\}).$$

This implies that

$$\psi(J(S)) \leq \psi(co(F(S \cap U) \cup \{0\})) = \psi(F(S \cap U)) \leq \psi(F(S)) < \psi(S),$$

if $\psi(S) > 0$. Thus $J: C \to C(C)$ is $\psi$-condensing. By Theorem 2.1 there exists $x \in C$ such that $x \in J(x)$. Now $x \in U$ since $0 \in U$. Consequently $x \in \rho(x)F(x)$ and so $x \in M$. This implies $\rho(x) = 1$ and so $x \in F(x)$. □

**Remark 2.2.** Theorem 2.2 extends [15, Theorem 2.4], [15, Theorem 2.5] and [14, Theorem 2.3]. In Theorem 2.2 above notice $\partial_Q U = \partial_C U$ (see [2] for a discussion). We note that the condition $U$ is weakly open in $C$ was omitted in Theorem 2.6 in [1] and in Theorem 2.1 (and the other results in Section 2) in [16].

Now we present a fixed point theorem of Furi-Pera type for weakly compact multivalued mappings with weakly sequentially closed graph.

**Theorem 2.3.** *Let $X$ be a reflexive separable Banach space, $C$ a closed bounded convex subset of $X$, and $Q$ a closed convex subset of $C$ with $0 \in Q$. Also, assume $F: Q \to C(C)$ has a weakly sequentially closed graph. In addition, assume that the following condition is satisfied:*

*if $\{(x_j, \lambda_j)\}_{j=1}^{\infty}$ is a sequence in $Q \times [0,1]$ with $x_j \rightharpoonup x \in \partial Q$, $\lambda_j \to \lambda$ and $x \in \lambda F(x), 0 \leq \lambda < 1$, then $\lambda_j F(x_j) \subseteq Q$ for $j$ sufficiently large; here $\partial Q$ is the weak boundary of $Q$ relative to $C$.*

*Then $F$ has a fixed point in $Q$.*

**Proof.** Since $X'$ is separable (note $X$ reflexive and separable implies $X'$ separable) and $Q$ is weakly compact (note that a closed convex bounded subset of a reflexive Banach space is weakly compact) we know from [12] that there exists a weakly continuous retraction $r: X \to Q$. Consider

$$B = \{x \in X \, : \, x \in Fr(x)\}.$$

We first show that $B \neq \emptyset$. To see this, consider $rF: Q \to C(Q)$. Clearly $rF$ has a weakly sequentially closed graph, since $F$ has a weakly sequentially closed graph and $r$ is weakly continuous. Also $rF(Q)$ is relatively weakly compact since $F(Q)$ is relatively weakly compact and $r$ is weakly continuous. Applying Theorem 2.1 we infer that there exists $y \in Q$ with $y \in rF(y)$. Let $z \in F(y)$ such that $y = r(z)$. Then $z \in F(y) = Fr(z)$. Thus $z \in B$ and $B \neq \emptyset$. In addition $B$ is weakly sequentially closed, since $Fr$ has a weakly sequentially closed graph. Moreover, since $B \subseteq Fr(B) \subseteq F(Q)$ then $B$ is relatively weakly compact. Now let $x \in \overline{B^w}$. Since $\overline{B^w}$ is weakly compact then there is a sequence $(x_n)$ of elements of $B$ which converges weakly to some $x$. Since $B$ is weakly sequentially closed then $x \in B$. Thus, $\overline{B^w} = B$. This implies that $B$ is weakly compact. We now show that $B \cap Q \neq \emptyset$. Suppose $B \cap Q = \emptyset$. Now since $X$ is separable we know from [8, 17] (note $C$ is weakly compact) that the weak topology on $C$ is metrizable, let $d^*$ denote the metric. With respect to $(C, d^*)$ note $Q$ is closed, $B$ is compact, $B \cap Q = \emptyset$ so there exists $\epsilon > 0$ with

$$d^*(B, Q) = \inf\{d^*(x, y) : x \in B, y \in Q\} > \epsilon.$$

For $i \in \{1, 2 \ldots\}$, let $U_i = \{x \in C \,:\, d^*(x, Q) < \frac{\epsilon}{i}\}$. For each $i \in \{1, 2 \ldots\}$ fixed, $U_i$ is open with respect to $d^*$ and so $U_i$ is weakly open in $C$. Also

$$\overline{U_i^w} = \overline{U_i^{d^*}} = \{x \in C \,:\, d^*(x, Q) \le \frac{\epsilon}{i}\} \quad \text{and} \quad \partial U_i = \{x \in C \,:\, d^*(x, Q) = \frac{\epsilon}{i}\}.$$

Keeping in mind that $\overline{U_i^w} \cap B = \emptyset$, Theorem 2.2 (with $F = Fr$, $U = U_i$, $Q = C$) guarantees that there exists $y_i \in \partial Q$ and $\lambda_i \in (0, 1)$ with $y_i \in \lambda_i Fr(y_i)$. Note since $y_i \in \partial U_i$ that $\lambda_i Fr(y_i) \not\subseteq Q$. We now consider

$$D = \{x \in X \,:\, x \in \lambda Fr(x), \text{ for some } \lambda \in [0, 1]\}.$$

The same reasoning as above implies that $D$ is weakly compact. Then, up to a subsequence, we may assume that $\lambda_i \to \lambda^* \in [0, 1]$ and $y_i \rightharpoonup y^* \in \partial Q$. Since $F$ has a weakly sequentially closed graph then $y^* \in \lambda^* Fr(y^*)$. Note $\lambda^* \ne 1$ since $B \cap Q = \emptyset$. From the assumption in the statement of Theorem 2.3 it follows that $\lambda_i Fr(y_i) \subseteq Q$ for $j$ sufficiently large, which is a contradiction. Thus $B \cap Q \ne \emptyset$, so there exists $x \in Q$ with $x \in Fr(x)$, i.e., $x \in Fx$. □

**Remark 2.3.** Theorem 2.3 extends [15, Theorem 2.6] and [14, Theorem 2.4]. We note that one of the conditions in Theorem 2.10 in [1] is stated incorrectly and that the proof there has to be adjusted slightly (see the proof of Theorem 2.3 above). We also refer the reader to [3].

The following lemma was proved in [1]. We give here the proof for the convenience of the reader.

**Lemma 2.1.** *Let $X$ be a Banach space and $B \colon X \to X$ a $k$-Lipschitzian map, that is*

$$\forall x, y \in X, \quad \|Bx - By\| \le k\|x - y\|.$$

*In addition, suppose that $B$ is weakly sequentially continuous. Then for each bounded subset $S$ of $X$ we have*

$$w(BS) \le kw(S);$$

*here, $w$ is the De Blasi measure of weak noncompactness.*

**Proof.** Let $S$ be a bounded subset of $X$ and $r > w(S)$. There exist $0 \le r_0 < r$ and a weakly compact subset $K$ of $X$ such that $S \subseteq K + B_{r_0}$. Now we show that

$$BS \subseteq BK + B_{kr_0} \subseteq \overline{BK^w} + B_{kr_0}. \tag{2.9}$$

To see this let $x \in S$. Then there is a $y \in K$ such that $\|x - y\| \le r_0$. Since $B$ is $k$-Lipschizian then $\|Bx - By\| \le k\|x - y\| \le kr_0$. This proves (2.9). Further, since $B$ is weakly sequentially continuous, then the Eberlein-Šmulian theorem [8, p. 430] implies that $\overline{BK^w}$ is weakly compact. Consequently

$$w(BS) \le kr_0 \le kr. \tag{2.10}$$

Letting $r \to w(S)$ we get

$$w(BS) \leq kw(S). \tag{2.11}$$

□

Now we are is a position to prove our next result.

**Theorem 2.4.** *Let $Q$ and $C$ be closed convex subsets of a Banach space $X$ with $Q \subseteq C$. In addition, let $U$ be a weakly open subset of $Q$ with $0 \in U$, $A\colon \overline{U^w} \to C(X)$ and $B\colon X \to X$ are two mappings satisfying :*

(i) *$A(\overline{U^w})$ is relatively weakly compact and $A$ has a weakly sequentially closed graph,*
(ii) *$B$ is a weakly sequentially continuous nonexpansive map,*
(iii) *if $(x_n)$ is a sequence of $M$ such that $((I-B)x_n)$ is weakly convergent then the sequence $(x_n)$ has a convergent subsequence,*
(iv) *$Ax + Bx \subseteq C$ for all $x \in \overline{U^w}$.*

*Also assume $U$ is weakly open in $C$. Then either*

$$A + B \text{ has a fixed point,} \tag{2.12}$$

*or*

$$\text{there is } u \in \partial_Q U \text{ and } \lambda \in (0,1) \text{ with } u \in \lambda(A+B)u; \tag{2.13}$$

*here $\partial_Q U$ is the weak boundary of $U$ in $Q$.*

**Proof.** Let $\mu \in (0,1)$ and $F_\mu := \mu A + \mu B$. Keeping in mind that $A$ has closed convex values and using assumption (iv) we infer that $F_\mu(x) \subseteq C$ for all $x \in \overline{U^w}$, since $0 \in U$. Now we show that $F_\mu\colon \overline{U^w} \to C(C)$ is $w$-condensing; here $w$ is the De Blasi measure of weak noncompactness. To see this let $S$ be a bounded subset of $\overline{U^w}$. Using the homogeneity and the subadditivity of the De Blasi measure of weak noncompactness we obtain

$$w(F_\mu(S)) \leq w(\mu AS + \mu BS) \leq \mu w(AS) + \mu w(BS). \tag{2.14}$$

Taking into account that $A$ is weakly compact and using Lemma 2.1 we deduce that

$$w(F_\mu(S)) \leq \mu w(S) < w(S), \tag{2.15}$$

whenever $w(S) > 0$. This proves that $F_\mu$ is $w$-condensing. Next suppose (2.13) does not occur. If there exists a $u \in \partial_Q U$ and $\lambda \in (0,1)$ with $u \in \lambda F_\mu u$, then $u \in \lambda\mu Au + \lambda\mu Bu$ which is a impossible since $\lambda\mu \in (0,1)$. By Theorem 2.2 there exists $x_\mu \in \overline{U^w}$ such that $x_\mu \in F_\mu(x_\mu)$. Now choose a sequence $\{\mu_n\}$ in $(0,1)$ such that $\mu_n \to 1$ and consider the corresponding sequence $\{x_n\}$ of elements of $\overline{U^w}$ satisfying

$$x_n \in F_{\mu_n}(x_n) = \mu_n Ax_n + \mu_n Bx_n. \tag{2.16}$$

Hence

$$x_n - \mu_n Bx_n \in \mu_n Ax_n. \tag{2.17}$$

Using the fact that $A(\overline{U^w})$ is weakly compact and passing eventually to a subsequence, we may assume that $\{x_n - \mu_n Bx_n\}$ converges weakly to some $y \in \overline{U^w}$. Since $\{x_n\}$ is a sequence in $\overline{U^w}$ then it is norm bounded and so is $\{Bx_n\}$. Consequently

$$\|(x_n - Bx_n) - (x_n - \mu_n Bx_n)\| = (1-\mu_n)\|Bx_n\| \to 0. \tag{2.18}$$

As a result

$$x_n - Bx_n \rightharpoonup y. \tag{2.19}$$

By hypothesis (iii) the sequence $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges weakly to some $x \in \overline{U^w}$. The weak sequential continuity of $B$ implies $y = x - Bx$. Now since $A$ has a weakly sequentially closed graph, the use of (2.17) gives $x - Bx \in Ax$, i.e., $x \in Bx + Ax$. □

**Remark 2.4.** We note that the condition $U$ is weakly open in $C$ was omitted in Theorem 2.9 in [1].

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