Fixed Point Theory, 12(2011), No. 2, 241-246
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

A FIXED POINT THEOREM FOR GENERALIZED NONEXPANSIVE MULTIVALUED MAPPINGS

A. ABKAR* AND M. ESLAMIAN**

*Department of Mathematics, Imam Khomeini International University
P.O. Box 288, Qazvin 34149, Iran.
E-mail: abkar@ikiu.ac.ir

**Department of Mathematics, Imam Khomeini International University
P.O. Box 288, Qazvin 34149, Iran.
E-mail: mhmdeslamian@gmail.com

Abstract. A mapping T on a normed space satisfies Suzuki condition provided that $1/2\|x-Tx\| \leq \|x-y\|$ implies $\|Tx-Ty\| \leq \|x-y\|$. This notion which generalizes the concept of nonexpansiveness was recently introduced by Suzuki [Fixed point thoerems and convergence theorems for some generelized nonexpansive mappings, J. Math. Anal. Appl. 340(2008) 1088–1095]. We shall first modify Suzuki condition to incorporate the class of multivalued mappings, and then prove a fixed point theorem for multivalued mappings satisfying the modified Suzuki condition. In this way, we generalize a theorem of T.C. Lim [A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach spaces, Bull. Amer. Math. Soc., 80(1974) 1123-1126] proved for nonexpansive multivalued mappings to the class of generalized nonexpansive multivalued mappings in the above sense.
Key Words and Phrases: Fixed point, nonexpansive mapping, multivalued mapping, Suzuki condition, uniformly convex Banach space.
2010 Mathematics Subject Classification: 47H10, 47H09.

1. Introduction

Let X be a Banach space. A mapping $T : X \to X$ is said to be *nonexpansive* provided that

$$\|Tx - Ty\| \leq \|x - y\|, \quad x, y \in X.$$

According to a well-known theorem due to Browder, if E is a bounded closed convex subset of a uniformly convex Banach space X, then every nonexpansive mapping $T : E \to E$ has a fixed point. Recently, Tomonari Suzuki [6] introduced a condition which is weaker than nonexpansiveness; Suzuki called it the condition (C). According to Suzuki, a mapping $T : E \to E$ is said to satisfy condition (C) provided that

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \implies \|Tx - Ty\| \leq \|x - y\|, \quad x, y \in E.$$

Among other results, Suzuki proved that if T satisfies the condition (C) and if E is a weakly compact convex subset of a uniformly convex in every direction Banach space X, then T has a fixed point. Recall that a Banach space X is said to be uniformly convex in every direction provided that for every $\epsilon \in (0, 2]$ and $z \in X$ with $\|z\| = 1$, there exists a positive number δ (depending on ϵ and z) such that for all $x, y \in X$ with $\|x\| \le 1$, $\|y\| \le 1$, and $x - y \in \{tz : t \in [-2, -\epsilon] \cup [\epsilon, 2]\}$ we have $\|x + y\| \le 2(1 - \delta)$.

Let us call every mapping satisfying the Suzuki condition (C), a generalized nonexpansive mapping. It follows from the definition that every nonexpansive mapping satisfies the condition (C). Moreover, the following example shows that there is a generalized nonexpansive mapping which is not a nonexpansive mapping; so that the class of generalized nonexpansive mappings contains properly the class of nonexpansive mappings.

Example 1.1. (see [6], Example 1). Define $T : [0, 3] \to [0, 3]$ by

$$T(x) = \begin{cases} 0, & x \neq 3, \\ 1 & x = 3. \end{cases}$$

Then T is a generalized nonexpansive map, but it is not nonexpansive.

The following existence theorem was established in [6].

Theorem 1.2. *([6], Theorem 5). Let C be a weakly compact convex subset of a Banach space which is uniformly convex in every direction. If T satisfies the condition (C), then T has a fixed point.*

In this paper we will study the case of multivalued mappings. Let $T : X \to 2^X$ be a multivalued mapping. An element $x \in X$ is said to be a fixed point of T provided that $x \in T(x)$. Let $CB(X)$ denote the collection of all nonempty closed bounded subsets of X. A multivalued map $T : X \to CB(X)$ is said to be nonexpansive provided that

$$H(Tx, Ty) \le \|x - y\|, \quad x, y \in X$$

where $H(Tx, Ty)$ denotes the Hausdorff metric on $CB(X)$.

In 1974, T.C. Lim [5] proved the following theorem on the existence of a fixed point theorem for multivalued nonexpansive mappings.

Theorem 1.3. *([5], or [3], p. 221). Let E be a bounded closed convex subset of a uniformly convex Banach space X, and let T be a nonexpansive multivalued mapping from E into the nonempty compact subsets of E endowed with the Hausdorff metric H. Then T has a fixed point.*

In this paper we intend to weaken the nonexpansiveness assumption on the multivalued mapping T. To do this, we shall modify Suzuki's (C) condition in the following manner. The norm of $x - Tx$ is replaced by the distance of x to the set Tx, moreover, the norm of $Tx - Ty$ will be replaced by the Hausdorff distance (metric) between these compact sets. In this way we will find a weaker condition than nonexpansiveness. Motivated by Suzuki, we shall assume that the multivalued mapping $T : E \to CB(E)$ satisfies the following (E) condition:

$$(E) \qquad \frac{1}{2} dist(x, Tx) \le \|x - y\| \implies H(Tx, Ty) \le \|x - y\|, \quad x, y \in E.$$

Here the distance between x and a subset $A \subset E$ is defined by

$$dist(x, A) = \inf\{\|x - y\| : y \in A\}.$$

We shall at times refer to such a mapping as a *generalized multivalued nonexpansive mapping*. In the last section we shall give a nontrivial example of a generalized nonexpansive multivalued mapping which is not a nonexpansive multivalued mapping. This means that the new class of multivalued mappings contains properly the class of nonexpansive multivalued mappings (see Example 3.2 below). The main result of this paper reads as follows:

Theorem 1.4. *Let E be a bounded closed convex subset of a uniformly convex Banach space X, and let T be a multivalued mapping from E into the nonempty compact subsets of E endowed with the Hausdorff metric H and satisfying the above condition (E). Then T has a fixed point.*

It is obvious that every multivalued nonexpansive mapping satisfies the condition (E). Therefore Lim's theorem follows immediately from our theorem.

2. Preliminaries

Let E be a nonempty closed convex subset of a Banach space X. Assume that $\{x_n\}$ is a bounded sequence in X. For each $x \in X$, the asymptotic radius of $\{x_n\}$ at x is defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|.$$

Let

$$r(E, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in E\},$$

and

$$A = A(E, \{x_n\}) = \{x \in E : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to E. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to E. In the case that X is a reflexive Banach space and E is a nonempty closed convex subset of X, the set $A(E, \{x_n\})$ is always a nonempty closed convex subset of E. To see this, observe that by the definition of r, for each $\epsilon > 0$, the set

$$C_\epsilon = \{x \in E : \limsup_{n\to\infty} \|x_n - x\| \le r + \epsilon\}$$

is nonempty. It is not difficult to see that each C_ϵ is closed and convex, hence

$$A = \cap_{\epsilon>0} C_\epsilon$$

is closed and convex. Moreover, it follows from the weak compactness of E that A is nonempty. It is easy to see that if X is uniformly convex and if E is a closed convex subset of X, then A consists of exactly one point.

A bounded sequence $\{x_n\}$ is said to be *regular* with respect to E if for every subsequence $\{x'_n\}$ we have

$$r(E, \{x_n\}) = r(E, \{x'_n\}).$$

The following lemma was proved by Goebel and Lim.

Lemma 2.1. *Let $\{x_n\}$ and E be as above. then there exists a subsequence of $\{x_n\}$ which is regular relative to E.*

Proof. See [1] and [5]. □ The following lemma, proved by Goebel and Kirk [2] plays an important role in the coming discussions. See also the recent result of [4].

Lemma 2.2. *Let $\{z_n\}$ and $\{w_n\}$ be two bounded sequences in a Banach space X, and let $0 < \lambda < 1$. If for every natural number n we have $z_{n+1} = \lambda w_n + (1-\lambda)z_n$ and $\|w_{n+1} - w_n\| \leq \|z_{n+1} - z_n\|$, then $\lim_{n\to\infty} \|w_n - z_n\| = 0$.*

Proof. See [2]. □

3. Main Result

We are now in position to prove the main result of this paper. As we mentioned earlier, this result generalizes Lim's theorem on the existence of fixed points for multivalued mappings.

Theorem 3.1. *Let E be a bounded closed convex subset of a uniformly convex Banach space X, and let T be a multivalued mapping from E into the nonempty compact subsets of E endowed with the Hausdorff metric H and satisfying the condition*

$$(E) \qquad \frac{1}{2}dist(x, Tx) \leq \|x - y\| \implies H(Tx, Ty) \leq \|x - y\|, \quad x, y \in E.$$

Then T has a fixed point.

Proof. Choose $x_0 \in E$ and $y_0 \in T(x_0)$. Define

$$x_1 = \frac{1}{2}(x_0 + y_0).$$

Let $y_1 \in T(x_1)$ is chosen in such a way that

$$\|y_0 - y_1\| = dist(y_0, T(x_1)).$$

Similarly, put $x_2 = \frac{1}{2}(x_1 + y_1)$, again we choose $y_2 \in T(x_2)$ in such a way that

$$\|y_1 - y_2\| = dist(y_1, T(x_2)).$$

In this way we will find a sequence $\{x_n\}$ in E such that $x_{n+1} = \frac{1}{2}(x_n + y_n)$ where $y_n \in T(x_n)$ and

$$\|y_{n-1} - y_n\| = dist(y_{n-1}, T(x_n)).$$

Therefore for every natural number $n \geq 1$ we have

$$\frac{1}{2}\|x_n - y_n\| = \|x_n - x_{n+1}\|$$

from which it follows that

$$\frac{1}{2}dist(x_n, T(x_n)) \leq \frac{1}{2}\|x_n - y_n\| = \|x_n - x_{n+1}\|, \quad n \geq 1.$$

Our assumption now gives

$$H(T(x_n), T(x_{n+1})) \leq \|x_n - x_{n+1}\|, \quad n \geq 1,$$

hence

$$\|y_n - y_{n+1}\| = dist(y_n, T(x_{n+1})) \leq H(T(x_n), T(x_{n+1})) \leq \|x_n - x_{n+1}\|, \quad n \geq 1.$$

We now apply Lemma 2.2 to conclude that $\lim_{n\to\infty} \|x_n - y_n\| = 0$ where $y_n \in T(x_n)$. Moreover, by passing to a subsequence we may assume that $\{x_n\}$ is regular (see

Lemma 2.1). Since X is uniformly convex, $A(E,\{x_n\})$ is singleton, say $\{w\}$. Let $r = r(E,\{x_n\})$. For each $n \geq 1$, we choose $z_n \in T(w)$ such that

$$\|y_n - z_n\| = dist(y_n, T(w)).$$

On the other hand, there is a natural number n_0 such that for every $n \geq n_0$ we have $\frac{1}{2}\|x_n - y_n\| \leq \|x_n - w\|$. This implies that

$$\frac{1}{2} dist(x_n, T(x_n)) \leq \|x_n - w\|,$$

and hence from the assumption we obtain

$$H(T(x_n), T(w)) \leq \|x_n - w\|, \quad n \geq n_0.$$

Therefore

$$\|y_n - z_n\| \leq H(T(x_n), T(w)) \leq \|x_n - w\|, \quad n \geq n_0.$$

Since $T(w)$ is compact, the sequence $\{z_n\}$ has a convergent subsequence $\{z_{n_k}\}$ with $\lim_{k\to\infty} z_{n_k} = v \in T(w)$. Note that

$$\|x_{n_k} - v\| \leq \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - v\|$$

and for $n_k \geq n_0$ we have $\|y_{n_k} - z_{n_k}\| \leq \|x_{n_k} - w\|$. These entail

$$\limsup_{k\to\infty} \|x_{n_k} - v\| \leq \limsup_{k\to\infty} \|x_{n_k} - w\| \leq r.$$

Since $\{x_n\}$ is regular, this shows that $w = v \in T(w)$. $\square$

We now provide an example of a generalized nonexpansive multivalued mapping satisfying the condition

$$\frac{1}{2} dist(x, Tx) \leq \|x - y\| \implies H(Tx, Ty) \leq \|x - y\|, \quad x, y \in E$$

which is not a nonexpansive multivalued mapping.

Example 3.2. Let $X = \mathbb{R}$ be equipped with the usual metric $d(x,y) = |x - y|$, and let $E = [0,5]$. We define

$$T(x) = \begin{cases} [0, \frac{x}{5}], & x \neq 5 \\ \{2\} & x = 5. \end{cases}$$

Let $x, y \in [0,5)$. It is easy to see that

$$H(Tx, Ty) = \left|\frac{x-y}{5}\right| \leq \|x - y\|.$$

If $x \in [0, \frac{15}{4}]$ and $y = 5$, then $H(Tx, Ty) = 2 - \frac{x}{5}$, while $\|x - y\| = 5 - x$. It is clear that $2 - \frac{x}{5} \leq 5 - x$ if and only if $x \leq \frac{15}{4}$. Therefore $H(Tx, Ty) \leq \|x - y\|$.

In case that $x \in (\frac{15}{4}, 5)$, and $y = 5$, we have $dist(x, Tx) = \frac{4x}{5}$. Therefore

$$\frac{1}{2} dist(x, Tx) = \frac{4x}{10} > \frac{4}{10}\frac{15}{4} = \frac{3}{2} > \|x - y\|.$$

Moreover

$$\frac{1}{2} dist(y, Ty) = \frac{3}{2} > \|x - y\|.$$

These inequalities show that the implication (E) mentioned in Theorem 3.1 holds true.

Now we shall see that the mapping T is not nonexpansive. To see this we take $x = 4$ and $y = 5$. Then we have

$$H(Tx, Ty) = 2 - \frac{4}{5} > 1 = \|x - y\|.$$

Finally we mention that 0 is the fixed point of T. Needless to say that Lim's theorem does not guarantee the existence of a fixed point for the mapping T, simply because T is not nonexpansive. $\square$

As a result we obtain Lim's theorem as a corollary.

Corollary 3.3. *(Lim's Theorem). Let E be a bounded closed convex subset of a uniformly convex Banach space X, and let T be a nonexpansive multivalued mapping from E into the nonempty compact subsets of E endowed with the Hausdorff metric H. Then T has a fixed point.*

References

[1] K. Goebel, *On a fixed point theorem for multivalued nonexpansive mappings*, Ann. Univ. Mariae Curie-Sklodowska, **29**(1975), 70-72.

[2] K. Goebel, W.A. Kirk, *Iteration processes for nonexpansive mappings*, Contemp. Math., **21**(1983), 115-123.

[3] M.A. Khamsi, W.A. Kirk, *An Introduction to Metric Spaces and Fixed Point Theory*, John Wiley, New York, 2001.

[4] W.A. Kirk, *Approximate fixed points of nonexpansive maps*, Fixed Point Theory, **10**(2009), no.2, 275-288.

[5] T.C. Lim, *A fixed point theorem for multivalued nonexpansive mappings in a uniformly convex Banach space*, Bull. Amer. Math. Soc., **80**(1974), 1123-1126.

[6] T. Suzuki, *Fixed point theorems and convergence theorems for some generelized nonexpansive mappings*, J. Math. Anal. Appl., **340**(2008), 1088-1095.

Received: August 12, 2009; Accepted: June 24, 2010.