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FIXED POINT THEOREMS OF EXPANSIVE TYPE MAPPINGS IN MODULAR FUNCTION SPACES

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Abstract. In this paper, the existence of fixed point of expansive type mappings is studied in modular function spaces.

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1. Introduction

The theory of modular spaces was initiated by Nakano [11] in connection with the theory of order spaces which was further generalized by Musielak and Orlicz [9] (see also [10]). The fixed point theory for nonlinear mappings is an important subject of nonlinear functional analysis and is widely applied to nonlinear integral equations and differential equations. The study of this theory in the context of modular function spaces was initiated by Khamsi [5] (see also [1] and [4]). Kumam [7] obtained some fixed point theorems for nonexpansive mappings in arbitrary modular spaces. Recently, Kutabi and Latif [8] studied fixed points of multivalued maps in modular function spaces. The objective of this paper is to prove some fixed point theorems for expansive type mappings in modular function space. Due to this, some basic facts and notations about modular spaces are recalled from [6].

Definition 1.1. *Let X be an arbitrary vector space. A functional $\rho : X \to [0, \infty]$ is called a modular if for any arbitrary x, y in X*

(m_1) $\rho(x) = 0$ *if and only if* $x = 0$.
(m_2) $\rho(\alpha x) = \rho(x)$ *for every scalar* α *with* $|\alpha| = 1$.
(m_3) $\rho(\alpha x + \beta y) \le \rho(x) + \rho(y)$ *if* $\alpha + \beta = 1, \alpha \ge 0, \beta \ge 0$.

If (m_3) is replaced by $\rho(\alpha x + \beta y) \le \alpha\rho(x) + \beta\rho(y)$ if $\alpha + \beta = 1, \alpha \ge 0, \beta \ge 0$ then ρ is called convex modular.

The vector space X_ρ given by $X_\rho = \{x \in X;\ \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$ is called a modular space. Generally, the modular ρ is not sub-additive and therefore does not behave as a norm or a distance. One can associate to a modular an $F-$norm.
The modular space X_ρ can be equipped with an $F-$norm defined by

$$\|x\|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq \alpha\}.$$

When ρ is convex modular, then $\|x\|_\rho = \inf\{\alpha > 0; \rho(\frac{x}{\alpha}) \leq 1\}$ defines a norm on the modular space X_ρ which is called the Luxemburg norm.
Define the $\rho-$ball, $B_\rho(x, r)$, centered at $x \in X_\rho$ with radius r as

$$B_\rho(x, r) = \{h \in X\rho; \rho(x - h) \leq r\}.$$

A point $x \in X_\rho$ is called a fixed point of $T : X_\rho \to X_\rho$ if $T(x) = x$.

Definition 1.2. *A function modular is said to satisfy the Δ_2-type condition, if there exists $K > 0$ such that for any $x \in X_\rho$, we have $\rho(2x) \leq K\rho(x)$.*

Definition 1.3. *Let X_ρ be a modular space. The sequence $\{x_n\} \subset X_\rho$ is called:*

- (t_1) *$\rho-$convergent to $x \in X_\rho$, if $\rho(x_n - x) \to 0$ as $n \to \infty$.*
- (t_2) *$\rho-$Cauchy, if $\rho(x_n - x_m) \to 0$ as n and $m \to \infty$.*

Note that, $\rho-$convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, this will happen if and only if ρ satisfies the Δ_2-condition. We know that [1] the norm and modular convergence are also the same when we deal with the Δ_2-type condition. In the sequel, suppose the modular function ρ is convex and satisfies the Δ_2-type condition. We also state the following definition and results given in [2] (see also, [3]).

Definition 1.4. *The growth function w_ρ of a function modular ρ is defined as:*

$$w_\rho(t) = \sup\left\{\frac{\rho(tx)}{\rho(x)},\ x \in X_\rho \backslash \{0\}\right\} \text{ for all } 0 \leq t < \infty.$$

Observe that $w_\rho(t) \leq 1$ for all $t \in [0, 1]$.

Lemma 1.5. *The growth function ω has the following properties:*

- (g_1) *$\omega(t) < \infty$, for each $t \in [0, \infty)$.*
- (g_2) *$\omega : [0, \infty) \to [0, \infty)$ is a convex, strictly increasing function. So, it is continuous.*
- (g_3) *$\omega(\alpha\beta) \leq \omega(\alpha)\omega(\beta)$; for all $\alpha, \beta \in [0, \infty)$.*
- (g_4) *$\omega^{-1}(\alpha)\omega^{-1}(\beta) \leq \omega^{-1}(\alpha\beta)$; for all $\alpha\beta \in [0, \infty)$, where ω^{-1} is the inverse function of ω.*

The following lemma shows that the growth function can be used to give an upper bound for $\|x\|_\rho$ for each $x \in X_\rho$.

Lemma 1.6. *Let ρ be a convex modular function satisfying the Δ_2-type condition. Then*

$$\|x\|_\rho \leq \frac{1}{\omega^{-1}(\frac{1}{\rho(x)})}, \text{ whenever } x \in X_\rho.$$

2. Fixed point for expansive type mappings

In this section, some fixed point theorems for expansive type mappings are proved as follows:

Theorem 2.1. *Let X_ρ be a modular function space. If a surjective mapping $T : X_\rho \to X_\rho$ satisfies*

$$\rho(Tx - Ty) \geq a_1\rho(x - y) + a_2\rho(x - Tx) + a_3\rho(y - Ty)$$

for all $x, y \in X_\rho$ with $x \neq y$, where $a_1, a_2, a_3 \geq 0$, $a_2 < 1$ and $a_1 + a_2 + a_3 > 1$. Then T has a fixed point in X_ρ.

Proof. Suppose x_0 is an arbitrary point of X_ρ. Since T is surjective, there is $x_1 \in X_\rho$ such that $x_0 = Tx_1$. Also,there exists $x_2 \in X\rho$ such that $x_1 = Tx_2$. Continuing this process, having chosen x_n in X_ρ, we can choose x_{n+1} in X_ρ such that $x_n = Tx_{n+1}$. Assume that for any $n = 0, 1, \cdots, x_{n-1} \neq x_n$. Otherwise, if there is n_0 such that $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$, then x_{n_0-1} becomes a fixed point of T and the result is proved. Now

$$\begin{aligned}\rho(x_{n-1} - x_n) = {} & \rho(Tx_n - Tx_{n+1}) \\ \geq {} & a_1\rho(x_n - x_{n+1}) + a_2\rho(x_n - Tx_n) + a_3\rho(x_{n+1} - Tx_{n+1}) \\ = {} & a_1\rho(x_n - x_{n+1}) + a_2\rho(x_n - x_{n-1}) + a_3\rho(x_{n+1} - x_n),\end{aligned}$$

which further implies that $(1 - a_2)\rho(x_{n-1} - x_n) \geq (a_1 + a_3)\rho(x_n - x_{n+1})$. Thus $\rho(x_n - x_{n+1}) \leq h\rho(x_{n-1} - x_n)$, where $h = \dfrac{1 - a_2}{a_1 + a_3} < 1$. By continuing this process we get that $\rho(x_n - x_{n+1}) \leq h^n\rho(x_0 - x_1)$. Hence

$$\frac{1}{h^n\rho(x_0 - x_1)} \leq \frac{1}{\rho(x_n - x_{n+1})}.$$

Since ρ is a convex function modular satisfying the Δ_2-type condition we have

$$\|x_n - x_{n+1}\|_\rho \leq \frac{1}{w^{-1}(\dfrac{1}{\rho(x_n - x_{n+1})})}.$$

Since

$$w^{-1}(\frac{1}{h^n\rho(x_0 - x_1)}) \leq w^{-1}(\frac{1}{\rho(x_n - x_{n+1})})$$

from (g_3) we obtain

$$w^{-1}((\frac{1}{h})^n)w^{-1}(\frac{1}{\rho(x_0 - x_1)}) \leq w^{-1}(\frac{1}{\rho(x_n - x_{n+1})}).$$

Therefore

$$\|x_n - x_{n+1}\|_\rho \leq \frac{1}{w^{-1}(\dfrac{1}{h})^n w^{-1}(\dfrac{1}{\rho(x_0 - x_1)})}.$$

Since $w(1) = 1$ and $h < 1$ then $1 < w^{-1}(\frac{1}{h})$. Hence $\{x_n\}$ is a Cauchy sequence in $(X_\rho, \|.\|_\rho)$. Since $(X_\rho, \|.\|_\rho)$ is complete so there exists $x \in X_\rho$ such that $\|x_n - x\|_\rho \to 0$. Moreover, Δ_2-type condition implies the equivalence of norm and modular convergence. Therefore $\rho(x_n - x) \to 0$ as $n \to \infty$. Since T is surjective on X_ρ, there exists j in X_ρ such that $x = T(j)$. Without loss of generality, for any n we assume that $x_n \neq x$. Thus

$$\begin{aligned}\rho(x_n - x) = & \ \rho(Tx_{n+1} - Tj) \\ \geq & \ a_1\rho(x_{n+1} - j) + a_2\rho(x_{n+1} - x_n) + a_3\rho(j - Tj) \\ = & \ a_1\rho(x_{n+1} - j) + a_2\rho(x_{n+1} - x_n) + a_3\rho(j - x).\end{aligned}$$

By taking limit as $n \to \infty$, we have $0 \geq (a_1 + a_3)\rho(j - x)$, which implies that $\rho(j - x) = 0$. Hence $j = x = Tj$ and j is a fixed point of T in X_ρ. □

Theorem 2.2. *Let X_ρ be a modular function space and suppose that mapping $T : X_\rho \to X_\rho$ is surjective and satisfies*

$$\rho(T^p x - T^q y) \geq h\rho(x - y), \text{ for all } x, y \in X_\rho$$

where p, q are the integers and h is constant with $h > 1$. Then T has a unique fixed point.

Proof. Suppose x_0 is an arbitrary point of X_ρ. Since T is surjective, there is $x_1 \in X_\rho$ such that $x_0 = Tx_1$. Also there exists $x_2 \in X_\rho$ such that $x_1 = Tx_2$. Continuing this process, having chosen x_n in X_ρ,we can choose x_{n+1} in X_ρ such that $x_n = Tx_{n+1}$. Take

$$\begin{aligned}y_0 = & \ x_0, \\ y_{2n-1} = & \ x_{(n-1)(p+q)+q} = T^p(x_{n(p+q)}) \text{ and} \\ y_{2n} = & \ x_{n(p+q)} = T^q(x_{n(p+q)+q})\end{aligned}$$

Note that, $\{y_n\}$ is a Cauchy sequence in X_ρ because

$$\begin{aligned}\rho(y_{2n-1} - y_{2n}) = & \ \rho(T^p(x_{n(p+q)}) - T^q(x_{n(p+q)+q})) \\ \geq & \ h\rho(x_{n(p+q)} - x_{n(p+q)+q}) \\ = & \ h\rho(y_{2n} - y_{2n+1})\end{aligned}$$

which gives that $\rho(y_{2n} - y_{2n+1}) \leq \frac{1}{h}\rho(y_{2n-1} - y_{2n})$. Similarly, $\rho(y_{2n-1} - y_{2n}) \leq \frac{1}{h}\rho(y_{2n-2} - y_{2n-1})$. Thus $\rho(y_{n+2} - y_{n+1}) \leq \frac{1}{h}\rho(y_{n+1} - y_n)$. Since $h > 1$ we have that $\{y_n\}$ is a contractive sequence. Following the similar arguments to those given in Theorem 2.1, we conclude that $\{y_n\}$ is a Cauchy sequence in X_p. Since X_ρ is complete there, is $y \in X_\rho$ such that $y_n \to y$ as $n \to \infty$. As, T is surjective on X_ρ, one obtains $j \in X_\rho$ such that $y = T^p j$. Moreover

$$\begin{aligned}\rho(y - y_{2n}) = & \ \rho(T^p j - T^q(x_{n(p+q)+q})) \\ \geq & \ h\rho(j - x_{n(p+q)+q})) = h\rho(j - y_{2n+1}).\end{aligned}$$

Letting $n \to \infty$, we have $0 \geq h\rho(j-y)$. Therefore $j = y = T^p(j)$. Also, $j = y = T^q(j)$. To prove uniqueness, assume that $w \in X_\rho$ is a common fixed point of T^p and T^q then

$$\rho(y - w) = \rho(T^p y - T^q w) \geq h\rho(y - w)$$

which implies that $y = w$. Finally, from $Ty = T(T^p y) = T^p(Ty)$, and $Ty = T(T^q y) = T^q(Ty)$, we conclude that y is a unique fixed point of T. □

Theorem 2.3. *Let X_ρ be a modular function space. If a surjective mapping $T : X_\rho \to X_\rho$ satisfies*

$$\rho(Tx - Ty) \geq a_1[\rho(x - Tx) + \rho(y - Ty)] + a_2\rho(x - y)) + a_3\rho(x - Ty)$$

for all $x, y \in X_\rho$ with $x \neq y$, where $a_1, a_2, a_3 \geq 0$, $a_1 + a_2 + a_3 > 1$ and $2a_1 + a_2 > 1$. Then T has a fixed point in X_ρ.

Proof. Suppose x_0 is an arbitrary point of X_ρ. Since T is surjective, there is $x_1 \in X_\rho$ such that $x_0 = Tx_1$. Moreover, there exists $x_2 \in X\rho$ such that $x_1 = Tx_2$. Continuing this process, having chosen x_n in X_ρ we can choose x_{n+1} in X_ρ such that $x_n = Tx_{n+1}$. Assume, $x_{n-1} \neq x_n$, for all $n = 0, 1, \cdots$. Otherwise, if there is n_0 such that $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$, then x_{n_0-1} is a fixed point of T and the result is proved. We have

$$\begin{aligned}\rho(x_{n-1} - x_n) = &\ \rho(Tx_n - Tx_{n+1}) \\ \geq &\ a_1[\rho(x_n - x_{n-1}) + \rho(x_{n+1} - x_n)] + a_2\rho(x_n - x_{n+1}) \\ &\ +a_3\rho(x_n - x_n)\end{aligned}$$

Thus

$$\rho(x_{n+1} - x_n) \leq h\rho(x_{n-1} - x_n), \text{ where } h = \frac{1 - a_1}{a_1 + a_2} < 1.$$

Continuing this process we obtain that $\rho(x_{n+1} - x_n) \leq h^n\rho(x_0 - x_1)$. Following the similar arguments to those given in Theorem 2.1, it can be shown that $\{x_n\}$ is a Cauchy sequence in $(X_\rho, \|.\|_\rho)$. Since $(X_\rho, \|.\|_\rho)$ is complete, there exists $x \in X_\rho$ such that $\|x_n - x\|_\rho \to 0$. Also Δ_2-condition implies the equivalence of norm and modular convergence. Therefore $\rho(x_n - x) \to 0$ as $n \to \infty$. Since T is surjective on X_ρ, there exists j in X_ρ such that $x = T(j)$. Without loss of generality, we assume that $x_n \neq x$, for any n. Thus

$$\begin{aligned}\rho(x_n - x) = &\ \rho(Tx_{n+1} - Tj) \\ \geq &\ a_1[\rho(x_{n+1} - x_n) + \rho(j - x)] + a_2\rho(x_{n+1} - j) \\ &\ +a_3\rho(x_{n+1} - x).\end{aligned}$$

By taking limit as $n \to \infty$, we obtain $0 \geq (a_1 + a_2)\rho(j - x)$, which implies that $\rho(j - x) = 0$. Hence $j = x = Tj$ and j is a fixed point of T in X_ρ. □

Theorem 2.4. *Let X_ρ be a modular function space. If a surjective mapping $T : X_\rho \to X_\rho$ satisfies*

$$\rho(Tx - Ty) \geq a_1\rho(x - Tx) + a_2 \max\{\rho(y - Ty) + \rho(x - y), \rho(x - Ty)\}$$

for all $x, y \in X_\rho$ with $x \neq y$, where $a_1, a_2 \geq 0$ and $a_1 + 2a_2 > 1$. Then T has a fixed point in X_ρ.

Proof. Suppose x_0 is an arbitrary point of X_ρ. Since T is surjective, there is $x_1 \in X_\rho$ such that $x_0 = Tx_1$. Moreover, there exists $x_2 \in X_\rho$ such that $x_1 = Tx_2$. Continuing this process, having chosen x_n in X_ρ, we can choose x_{n+1} in X_ρ such that $x_n = Tx_{n+1}$. Assume, $x_{n-1} \neq x_n$, for all $n = 0, 1, \cdots$. Otherwise, if there is n_0 such that $x_{n_0-1} = x_{n_0} = Tx_{n_0-1}$, then x_{n_0-1} becomes a fixed point of T and the result is proved.

$$\begin{aligned}\rho(x_{n-1} - x_n) = &\ \rho(Tx_n - Tx_{n+1}) \\ \geq &\ a_1\rho(x_n - x_{n-1}) + a_2 \max\{2\rho(x_{n+1} - x_n), \rho(x_n - x_n)\}.\end{aligned}$$

Thus

$$\rho(x_{n-1} - x_n) \geq a_1\rho(x_n - x_{n-1}) + 2a_2\rho(x_{n+1} - x_n),$$

which implies that

$$\rho(x_{n+1} - x_n) \leq \frac{1 - a_1}{2a_2}\rho(x_{n-1} - x_n).$$

Thus

$$\rho(x_{n+1} - x_n) \leq h\rho(x_{n-1} - x_n), \text{ where } h = \frac{1 - a_1}{2a_2} < 1.$$

Continuing this process we obtain that $\rho(x_{n+1} - x_n) \leq h^n \rho(x_0 - x_1)$. Following the similar arguments to those given in Theorem 2.1, it can be shown that $\{x_n\}$ is a Cauchy sequence in $(X_\rho, \|.\|_\rho)$. Since $(X_\rho, \|.\|_\rho)$ is complete so there exists $x \in X_\rho$ such that $\|x_n - x\|_\rho \to 0$. Moreover, Δ_2-condition implies the equivalence of norm and modular convergence. Therefore $\rho(x_n - x) \to 0$ as $n \to \infty$. Since T is surjective on X_ρ, there exists j in X_ρ such that $x = T(j)$. Without loss of generality, we assume that $x_n \neq x$, for any n. Thus

$$\begin{aligned}\rho(x_n - x) &= \rho(Tx_{n+1} - Tj) \\ &\geq a_1\rho(x_{n+1} - Tx_{n+1}) + a_2 \max\{\rho(j - Tj) + \rho(x_{n+1} - j), \rho(x_{n+1} - Tj)\} \\ &\geq a_1(x_{n+1} - x_n) + a_2 \max\{\rho(j - x) + \rho(x_{n+1} - j), \rho(x_{n+1} - x)\},\end{aligned}$$

where by taking limit as $n \to \infty$, we obtain $0 \geq 2a_2\rho(j - x)$, which implies that $\rho(j - x) = 0$. Hence $j = x = Tj$. □

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