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AN ITERATIVE METHOD FOR GENERALIZED VARIATIONAL INEQUALITIES WITH APPLICATIONS

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Abstract. In this paper, we introduce a composite iterative process for an inverse-strongly accretive mapping and a nonexpansive mapping. We study the convergence analysis of the iterative algorithm. Strong convergence theorems are established in a real Banach space.
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1. Introduction and Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H and P_C the metric projection of H onto C. Let $A : C \to H$ be a nonlinear mapping. Recall that the mapping A is said to be monotone if $\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C$.

Recall that the classical variational inequality problem, denoted by $VI(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

For given $z \in H$ and $u \in C$, we see that

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C \Leftrightarrow u = P_C z.$$

Recall also that A is said to be α-inverse-strongly monotone if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant. Iterative methods have been considered for the variational inequality (1.1) recently; see [7-10,13-16,20].

For finding solutions of the variational inequality (1.1) for an inverse-strongly monotone mapping, Iiduka, Takahashi and Toyoda [11] proved the following theorem.

Theorem ITM. *Let C be a nonempty closed convex subset of a real Hilbert space H and A an α-inverse strongly monotone operator of C into H with $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows: $x_1 = x \in C$ and*

$$x_{n+1} = P_C(\alpha_n x_n + (1-\alpha_n)P_C(x_n - \lambda_n Ax_n))$$

for every $n = 1, 2, \ldots$, where P_C is the metric projection from H onto C, $\{\alpha_n\}$ is a sequence in $[-1,1]$, and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\{\alpha_n\} \in [a,b]$ for some a, b with $-1 < a < b < 1$ and $\{\lambda_n\} \in [c,d]$ for some c, d with $0 < c < d < 2(1+a)\alpha$, then $\{x_n\}$ converges weakly to some element of $VI(C,A)$.

Recently, Aoyama, Iiduka and Takahashi [1] introduced a Banach version of the variational inequality (1.1). Before we proceed further, we first give some basic concepts in real Banach spaces.

Let C be a nonempty closed convex subset of a Banach space E. Let E^* be the dual space of E and $\langle \cdot, \cdot \rangle$ the pairing between E and E^*. For $q > 1$, the generalized duality mapping $J_q : E \to 2^{E^*}$ is defined by $J_q(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$, for all $x \in E$. In particular, $J = J_2$ is called the normalized duality mapping. It is known that $J_q(x) = \|x\|^{q-2}J(x)$ for all $x \in E$. Further, we have the following properties of the generalized duality mapping J_q:

(a) $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$;
(b) $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0, \infty)$;
(c) $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let $U = \{x \in E : \|x\| = 1\}$. E is said to uniformly convex if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \text{implies} \quad \left\|\frac{x+y}{2}\right\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit is attained uniformly for $x \in U$. The norm of E is said to be Fréchet differentiable, if for each $x \in U$, the limit is attained uniformly for $y \in U$. The norm of E is said to be uniformly Fréchet differentiable, if the limit is attained uniformly for $x, y \in U$. It is well-known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

The modulus of smoothness of E is defined by

$$\rho(t) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| \leq t\Big\}.$$

A Banach space E is said to be uniformly smooth if $\lim_{t\to 0} \frac{\rho(t)}{t} = 0$. Let $q > 1$. A Banach space E is said to be q-uniformly smooth if there exists a fixed constant $c > 0$ such that $\rho(t) \leq ct^q$. It is well-known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \leq 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$.

We also remark that

(a) All Hilbert spaces, L^p (or l^p) spaces ($p \geq 2$) and the Sobolev spaces W_m^p ($p \geq 2$) are 2-uniformly smooth, while L^p (or l^p) and W_m^p spaces ($1 < p \leq 2$) are p-uniformly smooth.

(b) L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

From now on, we always assume that E is 2-uniformly smooth and uniformly convex. We denote by $F(S)$ the set of fixed points of the nonlinear mapping S. Let C be a nonempty closed convex subset of E. Recall that the mapping $S : C \to C$ is said to be nonexpansive if $\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C$. Recall also that an operator A of C into E is said to be accretive if $\langle Ax - Ay, J(x - y)\rangle \geq 0, \quad \forall x, y \in C$. Moreover, A is said to be α-inverse-strongly accretive if there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, J(x - y)\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C$.

Let D be a subset of C and Q be a mapping of C into D. Then Q is said to be sunny if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping Q of C into itself is called a retraction if $Q^2 = Q$. If a mapping Q of C into itself is a retraction, then $Qz = z$ for all $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1.1 ([18]). *Let E be a smooth Banach space and C a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive;*

(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in E$;

(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in E, y \in C$.

Proposition 1.2 ([12]). *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and S a nonexpansive mapping of C into itself with $F(S) \neq \emptyset$. Then the set $F(S)$ is a sunny nonexpansive retract of C.*

For the class of nonexpansive mappings, one classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([2],[18]). More precisely, take $t \in (0, 1)$ and define a contraction $S_t : C \to C$ by

$$S_t x = tu + (1 - t)Sx, \quad \forall x \in C,$$

where $u \in C$ is a fixed point. Banach's contraction mapping principle guarantees that S_t has a unique fixed point x_t in C. That is, $x_t = tu + (1 - t)Sx_t$. It is unclear, in general, what the behavior of x_t is as $t \to 0$, even if S has a fixed point. However, in the case of S having a fixed point, Browder [2] proved that if E is a Hilbert space, then x_t converges strongly to a fixed point of S. Reich [18] extended Broweder's result to the setting of Banach spaces and proved that if E is a uniformly smooth Banach space, then x_t converges strongly to a fixed point of S and the limit defines the (unique) sunny nonexpansive retraction from C onto $F(S)$. Reich [18] showed that if E is uniformly smooth and if D is the fixed point set of a nonexpansive mapping from C into itself, then there exists a unique sunny nonexpansive retraction from C onto D and it can be constructed as follows.

Proposition 1.3. *Let E be a uniformly smooth Banach space and $S : C \to C$ a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $t \in (0, 1)$, the unique fixed point $x_t \in C$ of the contraction $C \ni x \mapsto tu + (1 - t)Sx$ converges strongly as $t \to 0$ to a fixed point of S. Define $Q : C \to D$ by $Qu = s - \lim_{t\to 0} x_t$. Then Q is the unique sunny nonexpansive retract from C onto D; that is, Q satisfies the property $\langle u - Qu, J(y - Qu)\rangle \leq 0, \quad \forall u \in C,$ and $\forall y \in D$.*

In 2006, Aoyama, Iiduka and Takahashi [1] introduced a Banach version of the variational inequality (1.1). That is, find a point $u \in C$ such that

$$\langle Au, J(v - u)\rangle \geq 0, \quad \forall v \in C, \tag{1.2}$$

where A is an accretive operator. Next, we use $BVI(C, A)$ to denote the set of solutions of the generalized variational inequality (1.2). In Hilbert spaces, the generalized variational inequality is reduced to the classical variational inequality (1.1).

For the generalized variational inequality (1.2), Aoyama, Iiduka and Takahashi [1] obtained the following weak convergence theorem.

Theorem AIT. *Let E be a uniformly convex and 2-uniformly smooth Banach space and C a nonempty closed convex subset of E. Let Q_C be a sunny nonexpansive retraction from E onto C, $\alpha > 0$ and A be an α-inverse strongly-accretive operator of C into E with $S(C, A) \neq \emptyset$, where $S(C, A) = \{x^* \in C : \langle Ax^*, J(x-x^*)\rangle \geq 0, \ x \in C\}$. If $\{\lambda_n\}$ and $\{\alpha_n\}$ are chosen such that $\lambda_n \in [a, \frac{\alpha}{K^2}]$ for some $a > 0$ and $\alpha_n \in [b, c]$ for some b, c with $0 < b < c < 1$, then the sequence $\{x_n\}$ defined by the following manners:*

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \quad n \geq 1,$$

converges weakly to some element z of $S(C, A)$, where K is the 2-uniformly smoothness constant of E.

Very recently, Cho, Yao and Zhou [6] further studied the generalized variational (1.2) by considering the following iterative process

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(x_n - \lambda_n Ax_n), \quad n \geq 1,$$

where u is a fixed element in C, A is an α-inverse-accretive operator and Q_C is the sunny nonexpansive retraction from E onto C. They showed that the sequence $\{x_n\}$ generated by above iterative algorithm converges strongly to Qu, where Q is a sunny nonexpansive retraction of C onto $BVI(C, A)$.

In this paper, motivated by the research work going on in this direction, we continue to study the generalized variational inequality (1.2). We introduce and analyze a composite iterative algorithm for finding a common element in the set of solutions of the generalized variational inequality (1.2) for an inverse-strongly accretive mapping and in the set of fixed points of a nonexpansive mapping S. Strong convergence theorems are established in the framework of Banach spaces.

In order to prove our main results, we also need the following lemmas.

Lemma 1.1 ([1]). *Let C be a nonempty closed convex subset of a smooth Banach space E. Let Q_C be a sunny nonexpansive retraction from E onto C and A an accretive operator of C into E. Then, for all $\lambda > 0$, we have that $BVI(C,A) = F(Q_C(I - \lambda A))$.*

Lemma 1.2 ([19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and*

$$\limsup_{n\to\infty} \left(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \right) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.3 ([22]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that $\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n$, where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that:*

(a) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(b) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.4 ([21]). *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 1.5 ([3]). *Let E be a uniformly convex Banach space, C a nonempty closed convex subset of E and $S : C \to C$ a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

Lemma 1.6. ([4]). *Let C be a nonempty closed convex subset of a real strictly convex Banach space E. Let S_1 and S_2 be two nonexpansive mappings such that $F(S_1) \cap F(S_2) \neq \emptyset$. Define $Sx = \delta S_1 x + (1-\delta)S_2 x$, where $\delta \in (0,1)$. Then $S : C \to C$ is a nonexpansive mapping with $F(S) = F(S_1) \cap F(S_2)$.*

2. Main results

Theorem 2.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse-strongly accretive mapping and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} = BVI(C,A) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 = u \in C, \\ y_n = \delta_n x_n + (1-\delta_n)Q_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\mu_n S x_n + (1-\mu_n)y_n], \quad n \geq 1, \end{cases} \tag{2.1}$$

where u is a fixed element in C, $\lambda \in (0, \alpha/K^2]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\mu_n\}$ are sequences in $(0,1)$. Assume that the above control sequences are chosen such that

(a) $\alpha_n + \beta_n + \gamma_n = 1$, $\forall n \geq 1$;
(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(c) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(d) $\lim_{n\to\infty} \mu_n = \mu \in (0,1)$ *and* $\lim_{n\to\infty} \delta_n = \delta \in [0,1)$.

Then the sequence $\{x_n\}$ *defined by* (2.1) *converges strongly to* $q = Q_{\mathcal{F}}u$, *where* $Q_{\mathcal{F}}$ *is a sunny nonexpansive retraction of* C *onto* $\mathcal{F}$.

Proof. First, we show that the mapping $I - \lambda A$ is nonexpansive. Indeed, from the assumption $\lambda \in (0, \alpha/K^2]$ and Lemma 1.4, for all $x, y \in C$, we have

$$\begin{aligned}
\|(I-\lambda A)x - (I-\lambda A)y\|^2 &= \|(x-y) - \lambda(Ax - Ay)\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\langle Ax - Ay, J(x-y)\rangle + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + 2K^2\lambda^2\|Ax - Ay\|^2 \\
&= \|x-y\|^2 + 2\lambda(\lambda K^2 - \alpha)\|Ax - Ay\|^2 \leq \|x-y\|^2.
\end{aligned}$$

This shows that $I - \lambda A$ is a nonexpansive mapping. Letting $x^* \in BVI(C, A) \cap F(S)$, we have $x^* = Sx^* = Q_C(x^* - \lambda A x^*)$. It follows that

$$\begin{aligned}
\|y_n - x^*\| &= \|\delta_n x_n + (1-\delta_n)Q_C(x_n - \lambda A x_n) - x^*\| \\
&\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|Q_C(x_n - \lambda A x_n) - Q_C(x^* - \lambda A x^*)\| \\
&\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|x_n - x^*\| = \|x_n - x^*\|.
\end{aligned} \tag{2.2}$$

Putting $t_n = \mu_n S x_n + (1-\mu_n)y_n$, we see that

$$\begin{aligned}
\|t_n - x^*\| &= \|\mu_n S x_n + (1-\mu_n)y_n - [\mu_n S x^* + (1-\mu_n)x^*]\| \\
&\leq \mu_n\|x_n - x^*\| + (1-\mu_n)\|y_n - x^*\| \\
&= \mu_n\|x_n - x^*\| + (1-\mu_n)\|\delta_n x_n + (1-\delta_n)Q_C(x_n - \lambda A x_n) - x^*\| \\
&\leq \mu_n\|x_n - x^*\| + (1-\mu_n)\delta_n\|x_n - x^*\| \\
&\quad + (1-\mu_n)(1-\delta_n)\|Q_C(x_n - \lambda A x_n) - Q_C(x^* - \lambda A x^*)\| \\
&\leq \|x_n - x^*\|,
\end{aligned} \tag{2.3}$$

from which it follows that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|t_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + \beta_n\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\
&\leq \alpha_n\|u - x^*\| + (1-\alpha_n)\|x_n - x^*\| \\
&\leq \max\{\|u - x^*\|, \|x_1 - x^*\|\} \\
&= \|u - x^*\|.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{t_n\}$. Notice that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})Q_C(I-\lambda A)x_{n+1} \\
&\quad - [\delta_n x_n + (1-\delta_n)Q_C(I-\lambda A)x_n]\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + \|x_n - Q_C(I-\lambda A)x_n\||\delta_{n+1} - \delta_n| \\
&\quad + (1-\delta_{n+1})\|Q_C(I-\lambda A)x_{n+1} - Q_C(I-\lambda A)x_n\| \\
&\leq \|x_{n+1} - x_n\| + R_1|\delta_{n+1} - \delta_n|,
\end{aligned} \tag{2.4}$$

where R_1 is an appropriate constant such that $R_1 \geq \sup_{n\geq 1}\{\|x_n - Q_C(I-\lambda A)x_n\|\}$. On the other hand, we have

$$\begin{aligned} \|t_{n+1}-t_n\| &= \|\mu_{n+1}Sx_{n+1} + (1-\mu_{n+1})y_{n+1} - [\mu_n Sx_n + (1-\mu_n)y_n]\| \\ &\leq \mu_{n+1}\|Sx_{n+1}-Sx_n\| + \|Sx_n - y_n\||\mu_{n+1}-\mu_n| \\ &\quad + (1-\mu_{n+1})\|y_{n+1}-y_n\| \\ &\leq \mu_{n+1}\|x_{n+1}-x_n\| + \|Sx_n - y_n\||\mu_{n+1}-\mu_n| \\ &\quad + (1-\mu_{n+1})\|y_{n+1}-y_n\|. \end{aligned} \tag{2.5}$$

Substituting (2.4) into (2.5), we see that

$$\begin{aligned} \|t_{n+1}-t_n\| &\leq \mu_{n+1}\|x_{n+1}-x_n\| + \|Sx_n - y_n\||\mu_{n+1}-\mu_n| \\ &\quad + (1-\mu_{n+1})(\|x_{n+1}-x_n\| + R_1|\delta_{n+1}-\delta_n|) \\ &\leq \|x_{n+1}-x_n\| + R_2(|\mu_{n+1}-\mu_n| + |\delta_{n+1}-\delta_n|), \end{aligned} \tag{2.6}$$

where R_2 is an appropriate constant such that $R_2 \geq \max\{\sup_{n\geq 1}\{\|Sx_n - y_n\|\}, R_1\}$.

Setting $l_n = \frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$, $\forall n \geq 1$, we have

$$x_{n+1} = (1-\beta_n)l_n + \beta_n x_n, \quad \forall n \geq 1. \tag{2.7}$$

Next, we estimate $\|l_{n+1}-l_n\|$. In view of

$$\begin{aligned} l_{n+1}-l_n &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}u + \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}}t_{n+1} - \frac{\alpha_n}{1-\beta_n}u - \frac{1-\beta_n-\alpha_n}{1-\beta_n}t_n \\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(u-t_{n+1}) + \frac{\alpha_n}{1-\beta_n}(t_n-u) + t_{n+1}-t_n, \end{aligned}$$

we obtain that

$$\|l_{n+1}-l_n\| \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|t_n-u\| + \|t_{n+1}-t_n\|. \tag{2.8}$$

Substituting (2.6) into (2.8), we arrive at

$$\begin{aligned} &\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\| \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|u-t_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|t_n-u\| + R_2(|\delta_{n+1}-\delta_n| + |\mu_n-\mu_{n+1}|). \end{aligned}$$

It follows from the conditions (b)-(d) that

$$\limsup_{n\to\infty}\left(\|l_{n+1}-l_n\| - \|x_{n+1}-x_{n+1}\|\right) < 0.$$

From Lemma 1.2, we obtain that

$$\lim_{n\to\infty}\|l_n - x_n\| = 0. \tag{2.9}$$

Thanks to (2.7), we see that $x_{n+1}-x_n = (1-\beta_n)(l_n-x_n)$. Combining the condition (c) and (2.9), we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{2.10}$$

On other hand, we have $x_{n+1} - x_n = \alpha_n(u - x_n) + \gamma_n(t_n - x_n)$, which together with (2.10) and the conditions (b), (c) implies that

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{2.11}$$

Next, we show that

$$\limsup_{n\to\infty} \langle u - q, J(x_n - q)\rangle \le 0, \tag{2.12}$$

where $q = Q_{\mathcal{F}}u$, $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto F. Define a mapping $M : C \to C$ by

$$Mx = \mu Sx + (1-\mu)[\delta I + (1-\delta)Q_C(I - \lambda A)]x, \quad \forall x \in C.$$

From Lemma 1.6, we see that M is a nonexpansive mapping with

$$\begin{aligned} F(M) &= F(S) \cap F(\delta I + (1-\delta)Q_C(I - \lambda A)) \\ &= F(S) \cap F(Q_C(I - \lambda A)) \\ &= F(S) \cap BVI(C, A) \\ &= \mathcal{F}. \end{aligned}$$

Note that $\|y_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n]\| \le R_1|\delta_n - \delta|$. It follows that

$$\begin{aligned} &\|x_n - Mx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Mx_n\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - Mx_n\| + \beta_n\|x_n - Mx_n\| + \gamma_n\|t_n - Mx_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n\|u - Mx_n\| + \beta_n\|x_n - Mx_n\| \\ &\quad + \gamma_n\|(\mu_n - \mu)(Sx_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n]) \\ &\quad + (1-\mu_n)(y_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n])\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - Mx_n\| + \beta_n\|x_n - Mx_n\| \\ &\quad + \gamma_n|\mu_n - \mu|\|Sx_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n]\| \\ &\quad + \gamma_n(1-\mu_n)\|y_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n]\| \\ &\le \|x_n - x_{n+1}\| + \alpha_n\|u - Mx_n\| + \beta_n\|x_n - Mx_n\| \\ &\quad + R_3(|\mu_n - \mu| + |\delta_n - \delta|). \end{aligned}$$

where R_3 is an appropriate constant such that

$$R_3 = \max\{\sup_{n\ge 1}\{\|Sx_n - [\delta x_n + (1-\delta)Q_C(I - \lambda A)x_n]\|\}, R_1\}.$$

This implies that

$$(1-\beta_n)\|x_n - Mx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|u - Mx_n\| \quad + R_3(|\mu_n - \mu| + |\delta_n - \delta|).$$

It follows, from the conditions (b)-(d) and (2.10), that

$$\lim_{n\to\infty} \|x_n - Mx_n\| = 0. \tag{2.13}$$

Let z_t be the fixed point of the contraction $z \mapsto tu + (1-t)Mz$, where $t \in (0, 1)$. That is, $z_t = tu + (1-t)Mz_t$. It follows that $\|z_t - x_n\| = \|(1-t)(Mz_t - x_n) + t(u - x_n)\|$.

On the other hand, for any $t \in (0,1)$, we see that

$$\begin{aligned}\|z_t - x_n\|^2 &= (1-t)\langle Mz_t - x_n, J(z_t - x_n)\rangle + t\langle u - x_n, J(z_t - x_n)\rangle \\ &= (1-t)\big(\langle Mz_t - Mx_n, J(z_t - x_n)\rangle + \langle Mx_n - x_n, J(z_t - x_n)\rangle\big) \\ &\quad + t\langle u - z_t, J(z_t - x_n)\rangle + t\langle z_t - x_n, J(z_t - x_n)\rangle \\ &\le (1-t)\big(\|z_t - x_n\|^2 + \|Mx_n - x_n\|\|z_t - x_n\|\big) \\ &\quad + t\langle u - z_t, J(z_t - x_n)\rangle + t\|z_t - x_n\|^2 \\ &\le \|z_t - x_n\|^2 + \|Mx_n - x_n\|\|z_t - x_n\| + t\langle u - z_t, J(z_t - x_n)\rangle.\end{aligned}$$

It follows that $\langle z_t - u, J(z_t - x_n)\rangle \le \frac{1}{t}\|Mx_n - x_n\|\|z_t - x_n\| \quad \forall t \in (0,1)$. In view of (2.13), we see that

$$\limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle \le 0. \tag{2.14}$$

On the other hand, we see that $Q_{F(M)}u = \lim_{t\to 0} z_t$ and $F(M) = \mathcal{F}$. It follows that $z_t \to q = Q_{\mathcal{F}}u$ as $t \to 0$. Since the fact that J is strong to weak* uniformly continuous on bounded subsets of E, we see that

$$\begin{aligned}&|\langle u - q, J(x_n - q)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\le |\langle u - q, J(x_n - q)\rangle - \langle u - q, J(x_n - z_t)\rangle| \\ &\quad + |\langle u - q, J(x_n - z_t)\rangle - \langle z_t - u, J(z_t - x_n)\rangle| \\ &\le |\langle u - q, J(x_n - q) - J(x_n - z_t)\rangle| + |\langle z_t - q, J(x_n - z_t)\rangle| \\ &\le \|u - q\|\|J(x_n - q) - J(x_n - z_t)\| + \|z_t - q\|\|x_n - z_t\| \to 0 \quad \text{as } t \to 0.\end{aligned}$$

Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $t \in (0,\delta)$ the following inequality holds $\langle u - q, J(x_n - q)\rangle \le \langle z_t - u, J(z_t - x_n)\rangle + \epsilon$. This implies that $\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le \limsup_{n\to\infty}\langle z_t - u, J(z_t - x_n)\rangle + \epsilon$. Since ϵ is arbitrary and using (2.14), we see that $\limsup_{n\to\infty}\langle u - q, J(x_n - q)\rangle \le 0$. That is,

$$\limsup_{n\to\infty}\langle u - q, J(x_{n+1} - q)\rangle \le 0. \tag{2.15}$$

Finally, we show that $x_n \to q$ as $n \to \infty$. Observe that

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \alpha_n\langle u - q, J(x_{n+1} - q)\rangle + \beta_n\langle x_n - q, J(x_{n+1} - q)\rangle \\ &\quad + \gamma_n\langle t_n - q, J(x_{n+1} - q)\rangle \\ &\le \alpha_n\langle u - q, J(x_{n+1} - q)\rangle + \beta_n\|x_n - q\|\|x_{n+1} - q\| \\ &\quad + \gamma_n\|t_n - q\|\|x_{n+1} - q\| \\ &\le \alpha_n\langle u - q, J(x_{n+1} - q)\rangle + (1-\alpha_n)\|x_n - q\|\|x_{n+1} - q\| \\ &\le \alpha_n\langle u - q, J(x_{n+1} - q)\rangle + \frac{1-\alpha_n}{2}(\|x_n - q\|^2 + \|x_{n+1} - q\|^2),\end{aligned}$$

which implies that

$$\|x_{n+1} - q\|^2 \le (1-\alpha_n)\|x_n - q\|^2 + 2\alpha_n\langle u - q, J(x_{n+1} - q)\rangle. \tag{2.16}$$

From the condition (b), (2.15) and applying Lemma 1.3 to (2.16), we obtain that $\lim_{n\to\infty}\|x_n - q\| = 0$. This completes the proof. □

If $S = I$, the identity mapping, then Theorem 2.1 is reduced to the following result.

Corollary 2.2. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse-strongly accretive mapping with $BVI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = \delta_n x_n + (1 - \delta_n) Q_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + (\beta_n + \gamma_n \mu_n) x_n + \gamma_n (1 - \mu_n) y_n, \quad n \geq 1, \end{cases} \tag{2.17}$$

where u is a fixed element in C, $\lambda \in (0, \alpha/K^2]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\mu_n\}$ are sequences in $(0, 1)$. Assume that the above control sequences satisfied the conditions (a)-(d). Then the sequence $\{x_n\}$ defined by (2.17) converges strongly to $q = Q_{BVI(C,A)} u$, where $Q_{BVI(C,A)}$ is a sunny nonexpansive retraction of C onto $BVI(C, A)$.

Further, if the sequence $\{\delta_n\} \equiv 0$, then Corollary 2.2 is reduced to the following which is an analogue of Theorem 3.1 of Cho et al. [6].

Corollary 2.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $A : C \to E$ be an α-inverse-strongly accretive mapping with $BVI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated, for each $n \geq 1$, by*

$$x_1 = u \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n [\mu_n x_n + (1 - \mu_n) Q_C(x_n - \lambda A x_n)], \tag{2.18}$$

where u is a fixed element in C, $\lambda \in (0, \alpha/K^2]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\mu_n\}$ are sequences in $(0, 1)$. Assume that the above control sequences satisfied the conditions (a)-(d). Then the sequence $\{x_n\}$ defined by (2.18) converges strongly to $q = Q_{BVI(C,A)} u$, where $Q_{BVI(C,A)}$ is a sunny nonexpansive retraction of C onto $BVI(C, A)$.

3. Applications

Let E be a smooth Banach space and C a nonempty closed convex subset of E. Recall that an operator B with domain $D(B)$ and range $R(B)$ in E is accretive, if for each $x_i \in D(B)$ and $y_i \in B x_i (i = 1, 2)$,

$$\langle y_2 - y_1, J(x_2 - x_1) \rangle \geq 0,$$

An accretive operator B is m-accretive if $R(I + rB) = E$ for each $r > 0$. Next, we assume that B is m-accretive and has a zero (i.e., the inclusion $0 \in B(z)$ is solvable). The set of zeros of B is denoted by Ω. Hence, $\Omega = \{z \in D(B) : 0 \in B(z)\} = B^{-1}(0)$.

For each $r > 0$, we denote by J_r^B the resolvent of B, i.e., $J_r^B = (I + rB)^{-1}$. Note that if B is m-accretive, then $J_r^B : E \to E$ is nonexpansive and $F(J_r^B) = \Gamma$ for all $r > 0$.

For the variational inequality (1.2), In the case when $C = E$, we see that $BVI(E, A) = A^{-1}(0)$ holds, where $A^{-1}(0) = \{u \in E : Au = 0\}$.

From the above, we have the following.

Theorem 3.1. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K. Let A be an α-inverse-strongly accretive mapping and B an m-accretive mapping. Assume that $\mathcal{F} = A^{-1}(0) \cap B^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = u \in E, \\ y_n = x_n - (1-\delta_n)\lambda A x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\mu_n J_r^B x_n + (1-\mu_n) y_n], \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\lambda \in (0, \alpha/K^2]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\mu_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfied the conditions (a)-(d). Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $q = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto $\mathcal{F}$.

Next, we consider another class of mappings: strict pseudo-contractions.

Recall that $T : C \to C$ is said to be a λ-strict pseudo-contraction [5] if there exists a constant $\lambda \in (0,1)$ such that

$$\langle Tx - Ty, J(x-y)\rangle \leq \|x-y\|^2 - \lambda\|(I-T)x - (I-T)y\|^2 \tag{3.3}$$

for every $x, y \in C$. From (3.3), we see that

$$\langle (I-T)x - (I-T)y, J(x-y)\rangle = \|x-y\|^2 - \langle Tx - Ty, J(x-y)\rangle$$
$$\geq \|x-y\|^2 - (\|x-y\|^2 - \lambda\|(I-T)x-(I-T)y\|^2) = \lambda\|(I-T)x-(I-T)y\|^2.$$

This implies that $(I-T)$ is λ-inverse-strongly accretive. Thus we obtain the following.

Theorem 3.3. *Let E be a uniformly convex and 2-uniformly smooth Banach space with the best smooth constant K, C a nonempty closed convex subset of E and Q_C a sunny nonexpansive retraction from E onto C. Let $T : C \to C$ be an α-strict pseudo-contraction and $S : C \to C$ a nonexpansive mapping with a fixed point. Assume that $\mathcal{F} = F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = u \in C, \\ y_n = (1-\lambda)x_n + \lambda T x_n, \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n[\mu_n S x_n + (1-\mu_n) y_n], \quad n \geq 1, \end{cases} \tag{3.4}$$

where $\lambda \in (0, \alpha/K^2]$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{\mu_n\}$ are sequences in $(0,1)$. Assume that the above control sequences satisfied the conditions (a)-(d). Then the sequence $\{x_n\}$ defined by (3.4) converges strongly to $q = Q_{\mathcal{F}}u$, where $Q_{\mathcal{F}}$ is a sunny nonexpansive retraction of C onto $\mathcal{F}$.

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