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EXISTENCE OF POSITIVE SOLUTIONS FOR SECOND ORDER DIFFERENTIAL EQUATION WITH FOUR POINT BOUNDARY CONDITIONS

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Abstract. We consider positive solutions to second-order four-point boundary value problem

$$\begin{cases} x''(t) + f(t, x, x') = 0, t \in (0, 1) \\ x(0) = \alpha x(\eta), x(1) = \beta x(\xi) \end{cases}$$

By using fixed point theorem, we present sufficient conditions which ensure the existence of three positive solutions to this problem. It's necessary to point out that it's the first time that positive solutions to this problem were established for the general case $\eta, \xi \in (0, 1)$. An examples is given to illustrate the main results.
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1. Introduction

In this paper, we consider the existence of positive solutions of the following four-point boundary value problem

$$x''(t) + f(t, x, x') = 0, t \in (0, 1) \tag{1.1}$$

$$x(0) = \alpha x(\eta), x(1) = \beta x(\xi) \tag{1.2}$$

under following conditions

C_1) $f : [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \to [0, +\infty)$ is continuous.

C_2) $\eta, \xi \in (0, 1)$ with $0 < \alpha < \dfrac{1}{1-\eta}, 0 < \beta < \dfrac{1}{\xi}, (1 - \beta\xi)(1 - \alpha) + \alpha\eta(1 - \beta) > 0.$

Multi-point boundary-value problems for ordinary differential equations arise in different areas of applied mathematics and physics. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have received a great deal of attentions. To identify a few, we refer the reader to $[1 - 12]$ and references along this line.

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As for four point boundary value problem, Liu [8] obtained the existence of one or two positive solutions for the problem:

$$\begin{cases} u''(t) + h(t)f(u) = 0, t \in (0,1) \\ u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \end{cases}$$

under assumptions that

$C_3)$ $f \in C([0,+\infty),[0,+\infty))$;

$C_4)$ $h \in C([0,1],[0,+\infty))$ and there exists $t_0 \in [0,1]$ such that $h(t_0) > 0$;

$C_5)$ $\eta \le \xi, 0 < \alpha < \dfrac{1}{1-\eta}, 0 < \beta < \dfrac{1}{\xi}, \Lambda := (1-\beta\xi)(1-\alpha) + \alpha\eta(1-\beta) > 0.$

In [9], the authors obtained at least three positive solutions for the problem:

$$\begin{cases} u''(t) + q(t)f(t,u,u') = 0, t \in (0,1) \\ u(0) = \alpha u(\eta), u(1) = \beta u(\xi) \end{cases}$$

Recently, second order four point boundary value problems still grasp people's attention, see [10, 11, 12]. However all the above works [8-12] are all established under the condition $\eta \le \xi$. We see in section 2 that the case $\eta \ge \xi$ may cause some difficulties in discussing the positive solutions for second order four point boundary value problems (bvps). In this paper,we overcome this difficulties and extend existence results to arbitrary $\eta, \xi \in (0,1)$. In this sense, we established some general results for positive solutions of second order four point bvps and extend the main results of [8-12].

2. Some lemmas

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces. This definitions can be found in recent literature.

Definition 2.1 Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that (i) $au \in P$, for all $u \in P,\ a \ge 0$ and (ii) $u,\ -u \in P$ implies $u = 0$.

Definition 2.2 An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 2.3 The map α is said to be a nonnegative continuous convex functional on cone P of a real Banach space E provided that $\alpha :\ P \to [0,\ +\infty)$ is continuous and $\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y),$ for all $x,\ y \in P$ *and* $t \in [0,\ 1]$.

Definition 2.4 The map β is said to be a nonnegative continuous concave functional on the cone P of a real Banach space E provided that $\beta :\ P \to [0,\ +\infty)$ is continuous and $\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y),$ for all $x,\ y \in P$ and $t \in [0,\ 1]$.

We introduce a fixed-point theorem due to Avery and Peterson [7], which is the main tool we use in this paper.

Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P. Then for positive numbers $a,\ b,\ c$ and d, we define the following convex sets:

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\},$$

$$P(\gamma, \alpha, b, d) = \{x \in P | b \le \alpha(x), \gamma(x) \le d\},$$

$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P | b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\}$$

and a closed set $R(\gamma, \psi, a, d) = \{x \in P | a \leq \psi(x), \gamma(x) \leq d\}$.

Lemma 2.1. Let P be a cone in a real Banach space E. Let γ, θ be nonnegative continuous convex functionals on P, α be a nonnegative continuous concave functional on P and ψ be a nonnegative continuous functional on P satisfying $\psi(\lambda x) \leq \lambda \psi(x),\ for\ 0 \leq \lambda \leq 1$, such that for some positive numbers l and d we have

$$\alpha(x) \leq \psi(x), \|x\| \leq l\gamma(x),\ \text{for all } x \in \overline{P(\gamma, d)}.$$

Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a < b$ such that

(S_1) $\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in P(\gamma, \theta, \alpha, b, c, d)$;

(S_2) $\alpha(Tx) > b$ for $x \in P(\gamma, \alpha, b, d)$ with $\theta(Tx) > c$;

(S_3) $0 \notin R(\gamma, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\gamma, \psi, a, d)$ with $\psi(x) = a$.

Then T has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d, i = 1, 2, 3;\ b < \alpha(x_1);\ a < \psi(x_2),\ \alpha(x_2) < b; \psi(x_3) < a.$$

Lemma 2.2 Suppose $y(t) \geq 0$ for $t \in [0, 1]$. Then problem

$$x''(t) + y(t) = 0, t \in (0, 1) \tag{2.1}$$

$$x(0) = \alpha u(\eta), x(1) = \beta u(\xi) \tag{2.2}$$

has no positive solution under one of following cases:

(1): $\alpha > \dfrac{1}{1-\eta}$, or $\beta > \dfrac{1}{\xi}$;

(2): $\Lambda = \alpha\eta(1-\beta) + (1-\alpha)(1-\beta\xi) < 0$.

Lemma 2.3. If $\Lambda \neq 0$ then, for $y(t) \in C[0, 1]$, the problem (2.1),(2.2) has a unique solution

$$x(t) = \int_0^1 G(t, s) y(s) ds, \tag{2.3}$$

where $G(t, s)$ is the Green's function of problem

$$\begin{cases} -x''(t) = 0,\ t \in (0, 1) \\ x(0) = \alpha x(\eta),\ x(1) = \beta x(\xi) \end{cases}$$

which is given as follows:

For $\eta \leq \xi$

$$G(t, s) = \frac{1}{\Lambda} \begin{cases} (\alpha\eta + t - \alpha t)[(1-\beta\xi) + (\beta-1)s] + \alpha(s-\eta)[(1-\beta\xi) + (\beta-1)t] \\ \qquad 0 \leq s \leq \eta, t \leq s \\ s[(1-\beta\xi) + (\beta-1)t] \\ \qquad 0 \leq s \leq \eta, t \geq s \\ (\alpha\eta + t - \alpha t)[(1-\beta\xi) + (\beta-1)s] \\ \qquad \eta \leq s \leq \xi, t \leq s \\ (\alpha\eta + s - \alpha s)[(1-\beta\xi) + (\beta-1)t] \\ \qquad \eta \leq s \leq \xi, t \geq s \\ (1-s)[\alpha\eta + (1-\alpha)t] \\ \qquad \xi \leq s \leq 1, t \leq s \\ (\alpha\eta + s - \alpha s)[(1-\beta\xi) + (\beta-1)t] + \beta(\xi - s)[\alpha\eta + (1-\alpha)t] \\ \qquad \xi \leq s \leq 1, t \geq s \end{cases}$$

For $\eta \geq \xi$

$$G(t,s)=\frac{1}{\Lambda}\begin{cases}(\alpha\eta+t-\alpha t)[(1-\beta\xi)+(\beta-1)s]+\alpha(s-\eta)[(1-\beta\xi)+(\beta-1)t] \\ \qquad 0\leq s\leq \xi, t\leq s \\ s[(1-\beta\xi)+(\beta-1)t] \\ \qquad 0\leq s\leq \xi, t\geq s \\ (1-s)[\alpha\eta+(1-\alpha)t]+\alpha(s-\eta)[(1-\beta\xi)+(\beta-1)t] \\ \qquad \xi\leq s\leq \eta, t\leq s \\ s[(1-\beta\xi)+(\beta-1)t]+\beta(\xi-s)[\alpha\eta+(1-\alpha)t] \\ \qquad \xi\leq s\leq \eta, t\geq s \\ (1-s)[\alpha\eta+(1-\alpha)t] \\ \qquad \eta\leq s\leq 1, t\leq s \\ (\alpha\eta+s-\alpha s)[(1-\beta\xi)+(\beta-1)t]+\beta(\xi-s)[\alpha\eta+(1-\alpha)t] \\ \qquad \eta\leq s\leq 1, t\geq s\end{cases}$$

Proof. Considering the definition and properties of the Green's function together with boundary condition (2.2), we can get the expression of the Green's function. □

Denote

$$G_1(t,s)=\frac{1}{\Lambda}\begin{cases}(\alpha\eta+t-\alpha t)[(1-\beta\xi)+(\beta-1)s] & t\leq s \\ (\alpha\eta+s-\alpha s)[(1-\beta\xi)+(\beta-1)t] & t\geq s\end{cases}$$

$$P(t)=\frac{\alpha}{\Lambda}\int_0^\eta[(1-\beta\xi)+(\beta-1)t](s-\eta)y(s)ds+\frac{\beta}{\Lambda}\int_\xi^1[\alpha\eta+(1-\alpha)t](\xi-s)y(s)ds. \quad (2.4)$$

Then (2.3) is given by

$$x(t)=\int_0^1 G_1(t,s)y(s)ds+P(t), 0\leq t\leq 1. \quad (2.5)$$

Remark 2.1 Considering the Green's function is not symmetrical, we can give the expressing of the solution by using a symmetrical kernel function and a linear function. We claim that (2.5) is satisfied under arbitrary $\eta,\xi\in(0,1)$.In fact,if $\eta\leq\xi, P(t)$ is given by (2.6). If $\eta>\xi$, by computation we get

$$P(t)=\frac{\alpha}{\Lambda}\int_0^\xi[(1-\beta\xi)+(\beta-1)t](s-\eta)y(s)ds$$
$$+\frac{1}{\Lambda}\int_\xi^\eta[\alpha(1-\beta\xi+\beta t-t)(s-\eta)+\beta(\alpha\eta+t-\alpha t)(\xi-s)]y(s)ds$$
$$+\frac{\beta}{\Lambda}\int_\eta^1[\alpha\eta+(1-\alpha)t](\xi-s)y(s)ds$$
$$=\frac{\alpha}{\Lambda}\int_0^\eta[(1-\beta\xi)+(\beta-1)t](s-\eta)y(s)ds+\frac{\beta}{\Lambda}\int_\xi^1[\alpha\eta+(1-\alpha)t](\xi-s)y(s)ds.$$

Considering condition (C_2) it's easy to see

$$\alpha\eta+(1-\alpha)t\geq 0, (1-\beta\xi)+(\beta-1)t\geq 0, t\in[0,1],$$

then we can easily prove that $G(t,s)\geq 0,\ if\ \eta\leq\xi$. But we also can see $G(t,s)$ can change sign on $[0,1]\times[0,1]$ if $\eta>\xi$ even if (C_2) is satisfied. In this case we prove that the solution is positive mainly by (2.5).

The following lemma is important in the proof of our main results.

Lemma 2.4 If $y(t) \geq 0$, for all $t \in [0,1]$, if there exists $t_0 \in [0,1]$ with $y(t_0) > 0$ and condition (C_2) holds, then:

(1) $x(t) > 0, t \in [0,1]$.

(2) $x(t) \geq \delta \max\limits_{0 \leq t \leq 1} x(t), t \in [0,1]$, where

$$\delta = \min\{\frac{\beta(1-\xi)}{1-\beta\xi}, \frac{\beta\xi}{\beta\xi + 1 - \beta}, \frac{\alpha\eta}{\alpha\eta + 1 - \alpha}, \frac{\alpha(1-\eta)}{1-\alpha\eta}\} \leq 1.$$

(3) $\max\limits_{0 \leq t \leq 1} |x(t)| \leq l \max\limits_{0 \leq t \leq 1} |x'(t)|$, where $l = \dfrac{\alpha\eta}{(1-\alpha)\delta}$.

Proof. (1) For $\eta \leq \xi$, from (2.5) we have

$$\begin{aligned} x(\eta) = & \frac{1}{\Lambda}\int_0^\eta [(1-\beta\xi)+(\beta-1)\eta][\alpha\eta+(1-\alpha)s]y(s)ds \\ & +\frac{\eta}{\Lambda}\int_\eta^1 [(1-\beta\xi)+(\beta-1)s]y(s)ds \\ & +\frac{\alpha}{\Lambda}\int_0^\eta (s-\eta)[(1-\beta\xi)+(\beta-1)\eta]y(s)ds + \frac{\beta\eta}{\Lambda}\int_\xi^1 (\xi-s)y(s)ds \\ = & \frac{1-\beta\xi+(\beta-1)\eta}{\Lambda}\int_0^\eta sy(s)ds + \frac{\eta}{\Lambda}\int_\eta^\xi [(1-\beta\xi)+(\beta-1)s]y(s)ds \\ & +\frac{\eta}{\Lambda}\int_\xi^1 (1-s)y(s)ds > 0. \end{aligned}$$

Similarly for $\eta \geq \xi$ we have

$$\begin{aligned} x(\eta) = & \frac{1-\beta\xi+(\beta-1)\eta}{\Lambda}\int_0^\xi sy(s)ds + \frac{1}{\Lambda}\int_\xi^\eta [s(1-\eta)+\beta\xi(\eta-s)]y(s)ds \\ & +\frac{\eta}{\Lambda}\int_\eta^1 (1-s)y(s)ds > 0. \end{aligned}$$

So $x(0) = \alpha x(\eta) > 0$.

Then we show $x(1) > 0$. Otherwise, suppose $x(1) \leq 0$. Then $x(\xi) \leq 0$. By the concavity of $x(t)$ and by $\dfrac{x(\xi)-x(0)}{\xi - 0} \geq \dfrac{x(1)-x(0)}{1-0}$, we get $x(1) \leq \dfrac{1}{\xi}x(\xi) + \dfrac{\xi-1}{\xi}x(0) < \dfrac{1}{\xi}x(\xi)$, which means $\beta\xi x(\xi) < x(\xi)$, thus $\beta\xi > 1$. A contradiction to $\beta < \dfrac{1}{\xi}$. So $x(1) > 0$. Considering $x(0) > 0,\ x(1) > 0$ together with the concavity of $x(t)$ we get the conclusion of (1).

(2) Let $\max\limits_{0 \leq t \leq 1} x(t) = x(t_1),\ \min\limits_{0 \leq t \leq 1} x(t) = x(t_2)$. Obviously $t_2 = 0, or\ t_2 = 1$. We distinguish two cases:

(i) $t_2 = 1$. Here $0 < \beta \leq 1, x(0) \geq x(1)$. For $t_1 \leq \xi < 1$, we see

$$\frac{x(t_1)-x(1)}{t_1-1} \geq \frac{x(\xi)-x(1)}{\xi-1}, so\ x(1) \geq \frac{\beta(1-\xi)}{1-\beta\xi}x(t_1).$$

For $0 < \xi \le t_1$, similarly with above we can get

$$x(t_1) \le \frac{t_1 x(\xi) - (t_1 - \xi)x(0)}{\xi} \le \frac{t_1 x(\xi) - (t_1 - \xi)x(1)}{\xi} \le \frac{t_1 - \beta(t_1 - \xi)}{\beta\xi} x(1).$$

Then $x(1) \ge \dfrac{\beta\xi}{\beta\xi + 1 - \beta} x(t_1)$

(ii) $t_2 = 0$. Here $0 < \alpha \le 1, x(0) \le x(1)$. For $\eta \in (0, t_1]$ we have

$$\frac{x(t_1) - x(0)}{t_1} \le \frac{x(\eta) - x(0)}{\eta}, then\ x(0) \ge \frac{\alpha\eta}{\alpha\eta + 1 - \alpha} x(t_1),$$

and for $\eta \in [t_1, 1)$ we get

$$x(t_1) \le \frac{(1 - t_1)x(\eta) - (\eta - t_1)x(1)}{1 - \eta} \le \frac{1 - t_1 - \alpha(\eta - t_1)}{\alpha(1 - \eta)} x(0) \le \frac{1 - \alpha\eta}{\alpha(1 - \eta)} x(0).$$

If $\delta := \min\{\dfrac{\beta(1-\xi)}{1-\beta\xi}, \dfrac{\beta\xi}{\beta\xi+1-\beta}, \dfrac{\alpha\eta}{\alpha\eta+1-\alpha}, \dfrac{\alpha(1-\eta)}{1-\alpha\eta}\}$, then $x(t) \ge \delta \max\limits_{0\le t\le 1} x(t)$.

(3) There exists constant τ such that $x(\eta) - x(0) = x'(\tau)\eta$. Considering the boundary condition $x(0) = \alpha x(\eta)$ and $x(0) > \delta \max\limits_{0\le t\le 1} |x(t)|$, we complete the proof. □

3. Existence results

In this section, we impose growth conditions on f which allow us to apply Lemma 2.1 to establish the existence of three positive solutions of our problem.

Let the nonnegative continuous concave functional α, the nonnegative continuous convex functionals γ, θ and the nonnegative continuous functional ψ be defined on the cone by

$$\gamma(x) = \max_{0\le t\le 1} |x'(t)|, \theta(x) = \psi(x) = \max_{0\le t\le 1} |x(t)|, \alpha(x) = \min_{0\le t\le 1} |x(t)|.$$

By Lemma 5,6 the functionals defined above satisfy:

$$\delta\theta(x) \le \alpha(x) \le \theta(x) = \psi(x), \|x\| = \max\{\theta(x), \gamma(x)\} \le l\gamma(x). \tag{3.1}$$

Let $m = \min_{0\le t\le 1} \int_0^1 |G(t,s)|ds, M = \max_{0\le t\le 1} \int_0^1 |\frac{\partial G(t,s)}{\partial t}|ds$

$N = \max_{0\le t\le 1} \int_0^1 G(t,s)ds, \lambda = min\{\dfrac{m}{M}, \delta l\}$.

To present our main results, we assume there exist constants $0 < a,\ b,\ c,\ d$ with $a < b < \lambda d$ such that

$A_1)$ $f(t,u,v) \le d/M, (t,u,v) \in [0,1] \times [0, ld] \times [-d, d]$;
$A_2)$ $f(t,u,v) > b/m, (t,u,v) \in [0,1] \times [b, b/\delta] \times [-d, d]$;
$A_3)$ $f(t,u,v) < a/N, (t,u,v) \in [0,1] \times [0, a] \times [-d, d]$.

Theorem 1. *Under the assumptions $A_1) - A_3)$, the boundary value problem (1.1)-(1.2) has at least three positive solutions $x_1,\ x_2,\ x_3$ satisfying*

$$\max_{0\le t\le 1} |x_i'(t)| \le d, i = 1,2,3; b < \min_{0\le t\le 1} |x_1(t)|;$$

$$a < \max_{0\le t\le 1} |x_2(t)|, \min_{0\le t\le 1} |x_2(t)| < b; \max_{0\le t\le 1} |x_3(t)| \le a. \tag{3.2}$$

Proof. Problem (1.1)-(1.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = \int_0^1 G(t,s)f(s,x,x')ds = (Tx)(t). \tag{3.3}$$

For $x \in \overline{P(\gamma,d)}$, $\gamma(x) = \max_{0\le t\le 1} |x'(t)| \le d$. Using Lemma 5, assumption (A_1) implies that $f(t,x,x') \le d/M$. On the other hand, for $x \in P$,

$$\gamma(Tx) = \max_{0\le t\le 1} |\int_0^1 \frac{\partial G(t,s)}{\partial t} f(s,x,x')ds| \le M \int_0^1 f(s,x,x')ds \le \frac{d}{M}M = d.$$

Hence $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$ and T is a completely continuous operator. To check condition (S_1) of Lemma 1, we choose $x(t) = \frac{b}{\delta} = c$. It's easy to see $x(t) = \frac{b}{\delta} \in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(\frac{b}{\delta}) > b$. So $\{x \in P(\gamma,\theta,\alpha,b,c,d|\alpha(x) > b)\} \ne \emptyset$.
If $x \in P(\gamma,\theta,\alpha,b,c,d)$,we have $b \le x(t) \le \frac{b}{\delta}$, $|x'(t)| < d$ for $0 \le t \le 1$. From (A_2), we have $f(t,x,x') \ge \frac{b}{m}$. By the definition of α and of the cone P we get

$$\alpha(Tx) = \min_{0\le t\le 1} |\int_0^1 G(t,s)f(s,x,x')ds| \ge > \frac{b}{m}\int_0^1 G(t,s)ds > \frac{b}{m}m = b.$$

Thus $\alpha(Tx) > b$, for all $x \in P(\gamma,\theta,\alpha,b,b/\delta,d)$. Next, from w (4.1) and $b < \lambda d$, we have $\alpha(Tx) \ge \delta\theta(Tx) > \delta b/\delta = b$ for all $x \in P(\gamma,\alpha,b,d)$ with $\theta(Tx) > b/\delta$.
Thus, condition (S_2) of Lemma 1 is satisfied. Finally, we show that (S_3) also holds. Clearly, as $\psi(0) = 0 < a$, we see $0 \notin R(\gamma,\psi,a,d)$. Suppose that $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$. Then, by (A_3), we have

$$\psi(Tx) = max_{0\le t\le 1}|(Tx)(t)| = \max_{0\le t\le 1} |\int_0^1 G(t,s)f(s,x,x')ds|$$

$$< \frac{a}{N}\max_{0\le t\le 1}\int_0^1 |G(t,s)|ds = a. \tag{3.4}$$

Since condition (S_3) is satisfied, Lemma 1 implies the boundary value problem (1.5)-(1.6) has at least three positive solutions x_1, x_2, x_3 satisfying (3.2). □

Remark 3.1.To apply Lemma 1, we only need $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, therefore condition C_1) can be substituted with a weaker condition
$H_1) : f \in C([0,1] \times [0,ld] \times [-d,d], [0,+\infty))$.

4. Application

Finally we present an example to check our main results. Consider the boundary value problem

$$u''(t) + f(t,u,u') = 0, 0 < t < 1, \tag{4.1}$$

$$u(0) = \frac{1}{2}u(\frac{3}{4}), \qquad u(1) = u(\frac{1}{2}), \tag{4.2}$$

where

$$f(t,u,v)=\begin{cases}\frac{1}{6}e^t+\frac{1}{2}u^3+\frac{1}{6}(\frac{v}{5000})^3 & 0\le u\le 10\\ \frac{1}{6}e^t+500+\frac{1}{6}(\frac{v}{5000})^3 & u>10\end{cases}$$

Choose $a=1,b=6,d=4000$, and note that $M=2,m=\frac{1}{18},l=\frac{15}{4},\delta=\frac{1}{5},\lambda=\frac{1}{36},N=\frac{3}{4}$. We can check that conditions $C_2),H_1)$ are satisfied and $f(t,u,v)$ satisfy

$$f(t,u,v)\le 2000,(t,u,v)\in[0,1]\times[0,15000]\times[-4000,4000];$$
$$f(t,u,v)\ge 108,(t,u,v)\in[0,1]\times[6,30]\times[-4000,4000];$$
$$f(t,u,v)\le\frac{4}{3},(t,u,v)\in[0,1]\times[0,1]\times[-4000,4000].$$

Since all the assumptions of Theorem 1 are satisfied, problem $(4.1)-(4.2)$ has at least three positive solutions x_1,x_2,x_3 such that $\max\limits_{0\le t\le 1}|x_i'(t)|\le 4000$, for each $i\in\{1,2,3\}$; $\min_{0\le t\le 1}x_1(t)>6$; $\max\limits_{0\le t\le 1}x_2(t)>1$, $\min\limits_{0\le t\le 1}x_2(t)<6$; $\max\limits_{0\le t\le 1}x_3(t)<1$.

Remark. In (4.1)-(4.2), we have $\eta>\xi$ and, thus, the Green's function can change sign. So, the results in [8-12], concerning the positive solutions, are not applicable.

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