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ITERATIVE METHODS FOR VARIATIONAL INEQUALITIES, EQUILIBRIUM PROBLEMS, AND ASYMPTOTICALLY STRICT PSEUDOCONTRACTIVE MAPPINGS

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Abstract. In this paper, we introduce iterative algorithms for finding a common element of the set of fixed points for an asymptotically strict pseudocontractive mappings in the intermediate sense, the set of solutions of the variational inequalities for a family of α-inverse-strongly monotone mappings and the set of solutions of a system of equilibrium problems in a Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our proposed algorithms. The strong convergence theorems are obtained via the hybrid method.
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1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H. We recall some definitions.
(i) A mapping T of C into H is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

(ii) T is *strictly pseudocontractive* if there exists κ with $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive.
(iii) A mapping $T : C \to C$ is called *asymptotically nonexpansive* (cf. [11]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|T^n x - T^n y\| \leq k_n\|x - y\|, \quad \forall x, y \in C, \ n \geq 1.$$

(iv) $T : C \to C$ is *asymptotically nonexpansive in the intermediate sense* [4] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

(v) A mapping $T : C \to C$ is said to be *asymptotically κ-strict pseudocontractive mapping* with sequence $\{\gamma_n\}$ [13] if there exists a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|T^n x - T^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2$$

for all $x, y \in C$ and $n \geq 1$.
(vi) $T : C \to C$ is *asymptotically κ-strict pseudocontractive mapping in the intermediate sense* with sequence $\{\gamma_n\}$ [21] if there exists a constant $\kappa \in [0,1)$ and a sequence $\{\gamma_n\}$ in $[0,\infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2) \leq 0.$$

Throughout this paper we assume that

$$c_n = \sup_{x,y\in C} \{\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - \kappa\|x - T^n x - (y - T^n y)\|^2\}.$$

Then $c_n \geq 0$ for all $n \geq 1$, $c_n \to 0$ as $n \to \infty$ and the above reduces to the relation

$$\|T^n x - T^n y\|^2 \leq (1+\gamma_n)\|x-y\|^2 + \kappa\|x - T^n x - (y - T^n y)\|^2 + c_n$$

for all $x, y \in C$ and $n \geq 1$.

There are some iterative methods for approximation of fixed points of the mappings defined above; see, for instance, [14, 17, 20, 21, 22, 24, 27].

Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) := \{x \in C : F(x,y) \geq 0 \ \forall y \in C\}.$$

Let $\mathcal{G} = \{F_i\}_{i\in I}$ be a family of bifunctions from $C \times C$ to $\mathbb{R}$. The system of equilibrium problems for $\mathcal{G} = \{F_i\}_{i\in I}$ is to determine common equilibrium points for $\mathcal{G} = \{F_i\}_{i\in I}$, i.e. the set

$$EP(\mathcal{G}) := \{x \in C : F_i(x,y) \geq 0 \ \forall y \in C \ \forall i \in I\}. \quad (1.1)$$

Many problems in applied sciences, such as monotone inclusion problems, saddle point problems, variational inequality problems, minimization problems, Nash equilibria in noncooperative games, vector equilibrium problems, as well as certain fixed point problems reduce into finding some element of $EP(F)$, see [2, 9, 10, 18]. The formulation (1.1), extends this formalism to systems of such problems, covering in particular various forms of feasibility problems [1, 8].

Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone [3], if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \ \ \forall x, y \in C.$$

It is easy to see that if $A : C \to H$ is α-inverse-strongly monotone, then it is a $\frac{1}{\alpha}$-Lipschitzian mapping.

Let $A : C \to H$ be a mapping. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \ \ \forall v \in C. \quad (1.2)$$

The set of solutions of variational inequality (1.2) is denoted by $VI(C,A)$. Put $A = I - T$, where $T : C \to H$ is a strictly pseudocontractive mapping with κ. It is known that A is $\frac{1-\kappa}{2}$-inverse-strongly monotone and $A^{-1}(0) = Fix(T) = \{x \in C : Tx = x\}$.

Recently, weak and strong convergence theorems for finding a common element of $EP(F)$, $VI(C,A)$ and $Fix(T)$, have been studied by many authors (see e.g., [5, 6, 18, 19, 23, 25, 26] and references therein). But, in the case that T is an asymptotically κ-strict pseudocontractive mapping, there were not any strong convergence result for finding an element of $EP(F) \cap VI(C,A) \cap Fix(T)$ (or even $EP(F) \cap Fix(T)$ and $VI(C,A) \cap Fix(T)$).

In this paper, motivated by [18, 19, 21, 23, 25, 26], we introduce iterative algorithms for finding a common element of the set of fixed point for an asymptotically κ-strict pseudocontractive mapping in the intermediate sense, the set of solutions of a system of equilibrium problems $EP(\mathcal{G})$ for a family $\mathcal{G} = \{F_i : i = 1, \dots, M\}$ of bifunctions and the set of solutions of variational inequalities $VI(C, A_j)$ for a family $\{A_j : j = 1 \dots N\}$ of α-inverse-strongly monotone mappings from C into H in a Hilbert space H. We establish some weak and strong convergence theorems of the sequences generated by our proposed algorithms. We obtain our strong convergence results via the hybrid method. Our results are new even for asymptotically κ-strict pseudocontractive mappings.

2. Preliminaries

Let C be a nonempty closed and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) := \{x \in C : F(x,y) \geq 0 \; \forall y \in C\}.$$

Given any $r > 0$. The operator $J_r^F : H \to C$ defined by

$$J_r^F(x) := \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \; \forall y \in C\}$$

is called the resolvent of F.

Lemma 2.1 ([9]) *Let C be a nonempty closed convex subset of H and $F : C \times C \to \mathbb{R}$ satisfy*
(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e. $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$.
(A3) for all $x, y, z \in C$,

$$\liminf_{t \to 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for all $x \in C$, $y \longmapsto F(x,y)$ is convex and lower semicontinuous.
Then:
(1) J_r^F is single-valued;
(2) J_r^F is firmly nonexpansive, i.e.

$$\|J_r^F x - J_r^F y\|^2 \leq \langle J_r^F x - J_r^F y, x - y\rangle, \; \textit{for all } x, y \in H;$$

(3) $Fix(J_r^F) = EP(F)$;
(4) $EP(F)$ is closed and convex.

Recall the metric (nearest point) projection P_C from a Hilbert space H to a closed convex subset C of H is defined as follows: given $x \in H$, $P_C x$ is the only point in C with the property

$$\|x - P_C x\| = \inf\{\|x - y\| : y \in C\}.$$

It is known that P_C is a nonexpansive mapping and satisfies:

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H. \quad (2.1)$$

P_C is characterized as follows.

$$y = P_C x \iff \langle x - y, y - z\rangle \geq 0, \quad \forall z \in C. \quad (2.2)$$

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au), \quad \forall \lambda > 0. \quad (2.3)$$

A set-valued mapping $T : H \to 2^H$ is said to be monotone, if for all $x, y \in H$, $f \in Tx$, and $g \in Ty$ imply that $\langle f - g, x - y\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is said to be maximal, if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping is maximal, if and only if for $(x, f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$, $\forall (y, g) \in G(T)$ imply that $f \in Tx$. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\},$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C; \\ \varnothing, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [12, 16]). It is easy to show that for given $\lambda \in [0, 2\alpha]$, the mapping $(I - \lambda A) : C \to H$ is nonexpansive.

Lemma 2.2 ([15]) *Let $\{\delta_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three sequences of nonnegative numbers satisfying the recursive inequality:*

$$\delta_{n+1} \leq \beta_n \delta_n + \gamma_n \text{ for all } n \in \mathbb{N}.$$

If $\beta_n \geq 1$, $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Lemma 2.3 ([21]) *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense. Then*
(a) if T is uniformly continuous and $\{x_n\}$ is a sequence in C such that $\|x_{n+1} - x_n\| \to 0$ and $\|x_n - T^n x_n\| \to 0$, as $n \to \infty$, then $\|x_n - Tx_n\| \to 0$, as $n \to \infty$;
(b) $I - T$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty} \limsup_{n\to\infty} \|x_n - T^m x_n\| = 0$, then $(I - T)x = 0$
(c) $F(T)$ is closed and convex.

3. Strong convergence

The following is our main strong convergence result, which is a generalization of [21, Theorem 4.1].

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$, $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), $\{A_k : k = 1 \ldots N\}$ a finite family of α-inverse-strongly monotone mappings from C into H, and $\mathcal{F} := \cap_{k=1}^{N} VI(C, A_k) \cap Fix(T) \cap EP(\mathcal{G})$ nonempty and bounded.*

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa$ for all $n \in \mathbb{N}$, $\{\lambda_{k,n}\}_{k=1}^{N}$ sequences in $[c, d] \subset (0, 2\alpha)$ such that $\lim_n |\lambda_{k,n} - \lambda_{k,n+1}| = 0$ for every $k \in \{1, \ldots, N\}$ and $\{r_{j,n}\}_{j=1}^{M}$ sequences in $(0, \infty)$ such that $\liminf_n r_{j,n} > 0$ and $\lim_n r_{j,n}/r_{j,n+1} = 1$ for every $j \in \{1, \ldots, M\}$.

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \geq 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = P_C(I - \lambda_{N,n}A_N) \ldots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1)u_n, \\ y_n = (1 - \alpha_n)v_n + \alpha_n T^n v_n, \\ C_n = \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \end{cases}$$

where $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - p\|^2 : p \in \mathcal{F}\} < \infty$, then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.

Proof. Take

$$\mathcal{J}^k_n := J^{F_k}_{r_{k,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}}, \quad (k = 1, \ldots, M),$$

$$\mathcal{P}^k_n := P_C(I - \lambda_{k,n}A_k) \ldots P_C(I - \lambda_{2,n}A_2)P_C(I - \lambda_{1,n}A_1), \quad (k = 1, \ldots, N),$$

and let $\mathcal{J}^0_n := I$ and $\mathcal{P}^0_n := I$. So, we can write

$$y_n = (1 - \alpha_n)\mathcal{P}^N_n \mathcal{J}^M_n x_n + \alpha_n T^n \mathcal{P}^N_n \mathcal{J}^M_n x_n.$$

We shall divide the proof into several steps.
Step 1. The sequence $\{x_n\}$ is well defined.
Proof of Step 1. The sets C_n and Q_n are closed and convex subsets of H for every $n \in \mathbb{N}$; see [21]. So, $C_n \cap Q_n$ is a closed convex subset of H for any $n \in \mathbb{N}$. Let $p \in \mathcal{F}$. Since, for each $k \in \{1, \ldots, M\}$, $J^{F_k}_{r_{k,n}}$ is nonexpansive, it follows, by Lemma 2.1,

$$\|u_n - p\| = \|\mathcal{J}^M_n x_n - p\| = \|\mathcal{J}^M_n x_n - \mathcal{J}^M_n p\| \leq \|x_n - p\|. \quad (3.1)$$

On the other hand, because $A_k : C \to H$ is α-inverse-strongly monotone and $\lambda_{n,k} \in [c, d] \subset [0, 2\alpha]$, $P_C(I - \lambda_{n,k}A_k)$ is nonexpansive. Thus, $\mathcal{P}^N_n$ is nonexpansive. Also, by (2.3), we have $\mathcal{P}^N_n p = p$. Thus,

$$\|v_n - p\| = \|\mathcal{P}^N_n u_n - \mathcal{P}^N_n p\| \leq \|u_n - p\| \leq \|x_n - p\|. \quad (3.2)$$

So, because $\alpha_n \leq 1 - \kappa$, we get

$$\|y_n - p\|^2 = \|(1 - \alpha_n)(v_n - p) + \alpha_n(T^n v_n - p)\|^2$$

$$
\begin{aligned}
&= (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|T^n v_n - p\|^2 - \alpha_n(1-\alpha_n)\|v_n - T^n v_n\|^2 \\
&\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n((1+\gamma_n)\|v_n - p\|^2 + \kappa\|v_n - T^n v_n\|^2 + c_n) \\
&\quad -\alpha_n(1-\alpha_n)\|v_n - T^n v_n\|^2 \\
&\le \|x_n - p\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|v_n - T^n v_n\|^2 + c_n + \gamma_n\Delta_n \\
&\le \|x_n - p\|^2 + \theta_n. \quad (3.3)
\end{aligned}
$$

So, we have $p \in C_n$; thus, $\mathcal{F} \subset C_n$, for every $n \in \mathbb{N}$. Next, we show by induction that

$$\mathcal{F} \subset C_n \cap Q_n,$$

for each $n \in \mathbb{N}$. Since $\mathcal{F} \subset C_1$ and $Q_1 = H$, we get $\mathcal{F} \subset C_1 \cap Q_1$. Suppose that $\mathcal{F} \subset C_k \cap Q_k$ for $k \in \mathbb{N}$. Then, there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = P_{C_k \cap Q_k}(x)$. Therefore, for each $z \in C_k \cap Q_k$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0.$$

So, we get

$$\mathcal{F} \subset C_k \cap Q_k \subset Q_{k+1}.$$

From this and $\mathcal{F} \subset C_n$ $(\forall n)$, we have

$$\mathcal{F} \subset C_{k+1} \cap Q_{k+1}.$$

This means that the sequence $\{x_n\}$ is well defined.

Step 2. The sequences $\{x_n\}$, $\{y_n\}$, $\{\mathcal{J}_n^k x_n\}_{k=1}^M$ and $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ are bounded and

$$\lim_{n\to\infty} \|x_n - x\| = c, \text{ for some } c \in \mathbb{R}. \ (3.4)$$

Proof of Step 2. From $x_{n+1} = P_{C_n \cap Q_n}(x)$, we have

$$\|x_{n+1} - x\| \le \|z - x\|, \ \forall z \in C_n \cap Q_n.$$

Since $P_{\mathcal{F}}(x) \in \mathcal{F} \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \le \|P_{\mathcal{F}}(x) - x\|, \ (3.5)$$

for every $n \in \mathbb{N}$. Therefore $\{x_n\}$ is bounded. From this, the sequence $\{\Delta_n\}$ is bounded and consequently $\theta_n \to 0$ as $n \to \infty$. So, from (3.1), (3.2) and (3.3), the sequences $\{\mathcal{J}_n^k x_n\}_{k=1}^M$, $\{\mathcal{P}_n^k u_n\}_{k=1}^N$ and $\{y_n\}$ are also bounded.

It is easy to show that $x_n = P_{Q_n}(x)$. From this and $x_{n+1} \in Q_n$, we have

$$\|x - x_n\| \le \|x - x_{n+1}\|,$$

for every $n \in \mathbb{N}$. Since $\{x_n\}$ is bounded, there exists $c \in \mathbb{R}$ such that (3.4) holds.

Step 3. $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Proof of Step 3. Since $x_n = P_{Q_n}(x)$, $x_{n+1} \in Q_n$ and $(x_n + x_{n+1})/2 \in Q_n$, we have

$$
\begin{aligned}
\|x - x_n\|^2 &\le \|x - \frac{x_n + x_{n+1}}{2}\|^2 \\
&= \|\frac{1}{2}(x - x_n) + \frac{1}{2}(x - x_{n+1}\|^2 \\
&= \frac{1}{2}\|x - x_n\|^2 + \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{4}\|x_n - x_{n+1}\|^2.
\end{aligned}
$$

So, we get

$$\frac{1}{4}\|x_n - x_{n+1}\|^2 \le \frac{1}{2}\|x - x_{n+1}\|^2 - \frac{1}{2}\|x - x_n\|^2.$$

From (3.4), we obtain $\lim_{n\to\infty} \|x_n - x_{n+1}\|^2 = 0$.
Step 4. Let $\{\omega_n\}$ be a bounded sequence in H. Then

$$\lim_{n\to\infty} \|\mathcal{J}^k_{n+1} w_n - \mathcal{J}^k_n \omega_n\| = 0 \quad (3.6)$$

for every $k \in \{1, \dots, M\}$.
Proof of Step 4. From [7], we have that

$$\lim_n \|J^{F_k}_{r_{k,n+1}} \omega_n - J^{F_k}_{r_{k,n}} \omega_n\| = 0 \quad (3.7)$$

for for every $k \in \{1, \dots, M\}$. Note that for every $k \in \{1, \dots, M\}$, we have

$$\mathcal{J}^k_n = J^{F_k}_{r_{k,n}} \mathcal{J}^{k-1}_n.$$

So,

$$\begin{aligned}
\|\mathcal{J}^k_{n+1} w_n - \mathcal{J}^k_n \omega_n\|\| &\le \|J^{F_k}_{r_{k,n+1}} \mathcal{J}^{k-1}_{n+1} w_n - J^{F_k}_{r_{k,n}} \mathcal{J}^{k-1}_{n+1} w_n\| \\
&+ \|J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n+1}} \mathcal{J}^{k-2}_{n+1} w_n - J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \mathcal{J}^{k-2}_{n+1} w_n\| + \dots \\
&+ \|J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n+1}} J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n+1}} \omega_n\| \\
&+ \|J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_k}_{r_{k,n}} J^{F_{k-1}}_{r_{k-1,n}} \dots J^{F_3}_{r_{3,n}} J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} \omega_n\| \\
&\le \|J^{F_k}_{r_{k,n+1}} \mathcal{J}^{k-1}_{n+1} w_n - J^{F_k}_{r_{k,n}} \mathcal{J}^{k-1}_{n+1} w_n\| + \|J^{F_{k-1}}_{r_{k-1,n+1}} \mathcal{J}^{k-2}_{n+1} w_n - J^{F_{k-1}}_{r_{k-1,n}} \mathcal{J}^{k-2}_{n+1} w_n\| \\
&+ \dots + \|J^{F_2}_{r_{2,n+1}} J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n+1}} \omega_n\| + \|J^{F_1}_{r_{1,n+1}} \omega_n - J^{F_1}_{r_{1,n}} \omega_n\| \\
&= \sum_{j=1}^k \|J^{F_j}_{r_{j,n+1}} (\mathcal{J}^{j-1}_{n+1} w_n) - J^{F_j}_{r_{j,n}} (\mathcal{J}^{j-1}_{n+1} w_n)\|.
\end{aligned}$$

From this and (3.7), it is easy to conclude (3.6).
Step 5. Let $\{\omega_n\}$ be a bounded sequence in C. Then

$$\lim_{n\to\infty} \|P_C(I - \lambda_{k,n+1} A_k) w_n - P_C(I - \lambda_{k,n} A_k) \omega_n\| = 0,$$

and

$$\lim_{n\to\infty} \|\mathcal{P}^k_{n+1} w_n - \mathcal{P}^k_n \omega_n\| = 0$$

for every $k \in \{1, \dots, N\}$.
Proof of Step 5. Since $\{\omega_n\}$ is bounded and A_k for $k \in \{1, \dots, N\}$ a Lipschitzian mapping, we know that

$$L := \sup_n \{\|A_k \omega_n\|\} < \infty.$$

Now,

$$\begin{aligned}
&\|P_C(I - \lambda_{k,n+1} A_k) w_n - P_C(I - \lambda_{k,n} A_k) \omega_n\| \\
&\le \|(I - \lambda_{k,n+1} A_k) w_n - (I - \lambda_{k,n} A_k) \omega_n\| \\
&= |\lambda_{k,n+1} - \lambda_{k,n}| \|A_k \omega_n\| \le |\lambda_{k,n+1} - \lambda_{k,n}| L \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Now, applying a technique similar to that used in proof of Step 4, it is easy to prove the second assertion.
Step 6. $\lim_{n\to\infty} \|x_n - y_n\| = 0$.
Proof of Step 6. From the convexity of $\|.\|^2$ and $x_{n+1} \in C_n$, we have

$$\|\frac{x_n - y_n}{2}\|^2 \le \frac{1}{2}\|x_n - x_{n+1}\|^2 + \frac{1}{2}\|x_{n+1} - y_n\|^2 \le \|x_n - x_{n+1}\|^2 + \frac{1}{2}\theta_n.$$

Since $\theta_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$, the desired result follows.

Step 7. $\lim_{n\to\infty} \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\| = 0$, $\forall k \in \{0, 1, \dots, M-1\}$.

Proof of Step 7. Let $p \in \mathcal{F}$ and $k \in \{0, 1, \dots, M-1\}$. Since $J_{r_{k+1,n}}^{F_{k+1}}$ is firmly nonexpansive, we obtain

$$\begin{aligned}\|p - \mathcal{J}_n^{k+1} x_n\|^2 &= \|J_{r_{k+1,n}}^{F_{k+1}} p - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2 \\ &\leq \langle J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - p, \mathcal{J}_n^k x_n - p\rangle \\ &= \frac{1}{2}(\|J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n - p\|^2 + \|\mathcal{J}_n^k x_n - p\|^2 - \|\mathcal{J}_n^k x_n - J_{r_{k+1,n}}^{F_{k+1}} \mathcal{J}_n^k x_n\|^2).\end{aligned}$$

It follows that

$$\|\mathcal{J}_n^{k+1} x_n - p\|^2 \leq \|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2.$$

Therefore, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}\|y_n - p\|^2 &= (1-\alpha_n)\|v_n - p\|^2 + \alpha_n\|T^n v_n - p\|^2 - \alpha_n(1-\alpha_n)\|v_n - T^n v_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n((1+\gamma_n)\|v_n - p\|^2 + \kappa\|v_n - T^n v_n\|^2 + c_n) \\ &\quad -\alpha_n(1-\alpha_n)\|v_n - T^n v_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|v_n - p\|^2 + \alpha_n(\kappa - (1-\alpha_n))\|v_n - T^n v_n\|^2 + c_n + \gamma_n\Delta_n \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|v_n - p\|^2 + \theta_n \qquad (3.8) \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{J}_n^{k+1} x_n - p\|^2 + \theta_n \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(\|x_n - p\|^2 - \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2) + \theta_n \\ &= \|x_n - p\|^2 - \alpha_n\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 + \theta_n.\end{aligned}$$

Since $\{\alpha_n\} \subset [\delta, 1]$, we get

$$\begin{aligned}\delta\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 &\leq \alpha_n\|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|^2 \\ &\leq \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n \leq \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.\end{aligned}$$

From this and Step 6, we get the desired result.

Step 8. $\lim_{n\to\infty} \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\| = 0$, $\forall k \in \{0, 1, \dots, N-1\}$.

Proof of Step 8. Since $\{A_k : k = 1 \dots N\}$ are α-inverse-strongly monotone, by the assumptions imposed on $\{\lambda_{k,n}\}$ for given $p \in \mathcal{F}$ and $k \in \{0, 1, \dots, N-1\}$ we have

$$\begin{aligned}&\|\mathcal{P}_n^{k+1} u_n - p\|^2 \\ &= \|P_C(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n} A_{k+1})p\|^2 \\ &\leq \|(I - \lambda_{k+1,n} A_{k+1})\mathcal{P}_n^k u_n - (I - \lambda_{k+1,n} A_{k+1})p\|^2 \\ &\leq \|\mathcal{P}_n^k u_n - p\|^2 + \lambda_{k+1,n}(\lambda_{k+1,n} - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 \\ &\leq \|x_n - p\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2.\end{aligned}$$

From this and (3.8), we have

$$\begin{aligned}\|y_n - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|v_n - p\|^2 + \theta_n \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{P}_n^{k+1} u_n - p\|^2 + \theta_n \qquad (3.9) \\ &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n(\|x_n - p\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2) + \theta_n \\ &= \|x_n - p\|^2 + c(d - 2\alpha)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 + \theta_n.\end{aligned}$$

So,

$$c(2\alpha - d)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2 \le \|x_n - p\|^2 - \|y_n - p\|^2 + \theta_n$$
$$\le \|x_n - y_n\|(\|x_n - p\| + \|y_n - p\|) + \theta_n.$$

Since $\alpha_n \subset [\delta, 1]$, $\theta_n \to 0$ and Step 6, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\| \to 0 \ \ (n \to \infty). \quad (3.10)$$

From (2.1) and the fact that $I - \lambda_{k+1,n}A_{k+1}$ is nonexpansive, we have

$$\|\mathcal{P}_n^{k+1}u_n - p\|^2 = \|P_C(I - \lambda_{k+1,n}A_{k+1})\mathcal{P}_n^k u_n - P_C(I - \lambda_{k+1,n}A_{k+1})p\|^2$$
$$\le \langle(\mathcal{P}_n^k u_n - \lambda_{k+1,n}A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n}A_{k+1}p), \mathcal{P}_n^{k+1}u_n - p\rangle$$
$$= \frac{1}{2}\{\|(\mathcal{P}_n^k u_n - \lambda_{k+1,n}A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n}A_{k+1}p)\|^2 + \|\mathcal{P}_n^{k+1}u_n - p\|^2$$
$$-\|(\mathcal{P}_n^k u_n - \lambda_{k+1,n}A_{k+1}\mathcal{P}_n^k u_n) - (p - \lambda_{k+1,n}A_{k+1}p) - (\mathcal{P}_n^{k+1}u_n - p)\|^2\}$$
$$\le \frac{1}{2}\{\|\mathcal{P}_n^k u_n - p\|^2 + \|\mathcal{P}_n^{k+1}u_n - p\|^2$$
$$-\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n - \lambda_{k+1,n}(A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p)\|^2\}$$
$$= \frac{1}{2}\{\|\mathcal{P}_n^k u_n - p\|^2 + \|\mathcal{P}_n^{k+1}u_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle$$
$$-\lambda_{k+1,n}^2\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2\}.$$

This implies that

$$\|\mathcal{P}_n^{k+1}u_n - p\|^2 \le \|\mathcal{P}_n^k u_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle$$
$$-\lambda_{k+1,n}^2\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\|^2$$
$$\le \|x_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle.$$

Then, from this and (3.9), we have

$$\|y_n - p\|^2 \le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|\mathcal{P}_n^{k+1}u_n - p\|^2 + \theta_n$$
$$\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\{\|x_n - p\|^2 - \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\rangle\} + \theta_n$$
$$\le \|x_n - p\|^2 - \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\| + \theta_n,$$

which implies that

$$\delta\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \le \alpha_n\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|^2 \le \|x_n - p\|^2 - \|y_n - p\|^2$$
$$+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n - A_{k+1}p\| + \theta_n.$$

Hence it follows from $\theta_n \to 0$, Step 6 and (3.10) that $\|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1}u_n\| \to 0$.

Step 9. $\lim_{n\to\infty}\|v_n - Tv_n\| = 0$.

Proof of Step 9. Observe that

$$\|u_n - v_n\| = \|\mathcal{P}_n^0 u_n - \mathcal{P}_n^N u_n\| \le \sum_{k=0}^{N-1} \|\mathcal{P}_n^k u_n - \mathcal{P}_n^{k+1} u_n\|,$$

and

$$\|x_n - u_n\| = \|\mathcal{J}_n^0 x_n - \mathcal{J}_n^M x_n\| \le \sum_{k=0}^{M-1} \|\mathcal{J}_n^k x_n - \mathcal{J}_n^{k+1} x_n\|.$$

So, by Steps 7 and 8, we have $\|u_n - v_n\| \to 0$ and $\|x_n - u_n\| \to 0$, as $n \to \infty$. On the other hand,

$$\alpha_n T^n v_n = y_n - (1-\alpha_n)v_n.$$

So, we have

$$\begin{aligned}\delta\|v_n - T^n v_n\| &\le \alpha_n \|v_n - T^n v_n\| \\ &= \|y_n - (1-\alpha_n)v_n - \alpha_n v_n\| = \|y_n - v_n\| \\ &\le \|y_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\| \to 0, \text{ as } n \to \infty.\end{aligned}$$

From these and Step 6, we obtain

$$\lim_{n\to\infty} \|v_n - T^n v_n\| = 0. \quad (3.11)$$

In this stage, note that

$$\begin{aligned}\|v_n - v_{n+1}\| &= \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_{n+1}\| \\ &\le \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_n\| + \|\mathcal{P}_{n+1}^N u_n - \mathcal{P}_{n+1}^N u_{n+1}\| \\ &\le \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_n\| + \|u_n - u_{n+1}\| \\ &= \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_n\| + \|\mathcal{J}_n^M x_n - \mathcal{J}_{n+1}^M x_{n+1}\| \\ &\le \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_n\| + \|\mathcal{J}_n^M x_n - \mathcal{J}_{n+1}^M x_n\| + \|\mathcal{J}_{n+1}^M x_n - \mathcal{J}_{n+1}^M x_{n+1}\| \\ &\le \|\mathcal{P}_n^N u_n - \mathcal{P}_{n+1}^N u_n\| + \|\mathcal{J}_n^M x_n - \mathcal{J}_{n+1}^M x_n\| + \|x_n - x_{n+1}\|.\end{aligned}$$

It follows from this and Steps 3, 4 and 5 that $\|v_n - v_{n+1}\| \to 0$, as $n \to \infty$. By (3.11), we obtain from Lemma 2.3 that $\lim_{n\to\infty} \|v_n - Tv_n\| = 0$.

Step 10. The weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Proof of Step 10. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From Steps 7 and 8, we obtain also that

$$\mathcal{J}_{n_m}^k x_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \dots, M\}$, and

$$\mathcal{P}_{n_m}^k u_{n_m} \rightharpoonup z_0,$$

for all $k \in \{1, \dots, N\}$. In particular, $u_{n_m} \rightharpoonup z_0$ and $v_{n_m} \rightharpoonup z_0$. We need to show that $z_0 \in \mathcal{F}$. First, we prove $z_0 \in Fix(T)$. Since $\lim_{n\to\infty} \|v_n - Tv_n\| = 0$ by Step 9, it follows from uniform continuity of T that $\lim_{n\to\infty} \|v_n - T^m v_n\| = 0$ for all $m \in \mathbb{N}$. So, from $v_{n_m} \rightharpoonup z_0$ and Lemma 2.3, we get

$$z_0 \in Fix(T).$$

Now, we prove $z_0 \in \cap_{i=1}^N VI(C, A_i)$. For this purpose, let $k \in \{1, \ldots, N\}$ and T_k be the maximal monotone mapping defined by

$$T_k x = \begin{cases} A_k z + N_C z, & z \in C; \\ \varnothing, & z \notin C. \end{cases}$$

For any given $(z, u) \in G(T_k)$, hence $u - A_k z \in N_C z$. Since $\mathcal{P}_n^k u_n \in C$, by the definition of N_C, we have

$$\langle z - \mathcal{P}_n^k u_n, u - A_k z\rangle \geq 0. \qquad (3.12)$$

On the other hand, since $\mathcal{P}_n^k u_n = P_C(\mathcal{P}_n^{k-1} u_n - \lambda_{k,n} A_k \mathcal{P}_n^{k-1} u_n)$, we have

$$\langle z - \mathcal{P}_n^k u_n, \mathcal{P}_n^k u_n - (\mathcal{P}_n^{k-1} u_n - \lambda_{k,n} A_k \mathcal{P}_n^{k-1} u_n)\rangle \geq 0.$$

So

$$\langle z - \mathcal{P}_n^k u_n, \frac{\mathcal{P}_n^k u_n - \mathcal{P}_n^{k-1} u_n}{\lambda_{k,n}} + A_k \mathcal{P}_n^{k-1} u_n\rangle \geq 0.$$

By (3.12) and the α-inverse monotonicity, we have

$$\begin{aligned} \langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle &\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\ &\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z\rangle \\ &\quad -\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}} + A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\ &= \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k z - A_k \mathcal{P}_{n_m}^k u_{n_m}\rangle \\ &\quad +\langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\ &\quad -\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle \\ &\geq \langle z - \mathcal{P}_{n_m}^k u_{n_m}, A_k \mathcal{P}_{n_m}^k u_{n_m} - A_k \mathcal{P}_{n_m}^{k-1} u_{n_m}\rangle \\ &\quad -\langle z - \mathcal{P}_{n_m}^k u_{n_m}, \frac{\mathcal{P}_{n_m}^k u_{n_m} - \mathcal{P}_{n_m}^{k-1} u_{n_m}}{\lambda_{k,n_m}}\rangle. \end{aligned}$$

Since $\|\mathcal{P}_n^k \mathcal{J}_n^M x_n - \mathcal{P}_n^{k-1} \mathcal{J}_n^M x_n\| \to 0$, $\mathcal{P}_{n_m}^k u_{n_m} \rightharpoonup z_0$ and $\{A_k : k = 1, \ldots, N\}$ are Lipschitz continuous, we have

$$\lim_{m\to\infty} \langle z - \mathcal{P}_{n_m}^k u_{n_m}, u\rangle = \langle z - z_0, u\rangle \geq 0.$$

Again since T_k is maximal monotone, hence $0 \in T_k z_0$. This shows that $z_0 \in VI(C, A_k)$. From this, it follows that

$$z_0 \in \cap_{i=1}^N VI(C, A_i).$$

Note that by (A2) for given $y \in C$ and $k \in \{0, 1, \ldots, M-1\}$, we have

$$\frac{1}{r_{k+1,n}} \langle y - \mathcal{J}_n^{k+1} x_n, \mathcal{J}_n^{k+1} x_n - \mathcal{J}_n^k x_n\rangle \geq F_{k+1}(y, \mathcal{J}_n^{k+1} x_n).$$

Thus

$$\langle y - \mathcal{J}_{n_m}^{k+1} x_{n_m}, \frac{\mathcal{J}_{n_m}^{k+1} x_{n_m} - \mathcal{J}_{n_m}^k x_{n_m}}{r_{k+1,n_m}}\rangle \geq F_{k+1}(y, \mathcal{J}_{n_m}^{k+1} x_{n_m}). \quad (3.13)$$

By condition (A4), $F_i(y,.)$, $\forall i$, is lower semicontinuous and convex, and thus weakly semicontinuous. Step 7 and condition $\liminf_n r_{j,n} > 0$ imply that

$$\frac{\mathcal{J}^{k+1}_{n_m} x_{n_m} - \mathcal{J}^{k}_{n_m} x_{n_m}}{r_{k+1,n_m}} \to 0,$$

in norm. Therefore, letting $m \to \infty$ in (3.13) yields

$$F_{k+1}(y, z_0) \le \lim_m F_{k+1}(y, \mathcal{J}^{k+1}_{n_m} x_{n_m}) \le 0,$$

for all $y \in C$ and $k \in \{0, 1, \dots, M-1\}$. Replacing y with $y_t := ty + (1-t)z_0$ with $t \in (0,1)$ and using (A1) and (A4), we obtain

$$0 = F_{k+1}(y_t, y_t) \le tF_{k+1}(y_t, y) + (1-t)F_{k+1}(y_t, z_0) \le tF_{k+1}(y_t, y).$$

Hence $F_{k+1}(ty + (1-t)z_0, y) \ge 0$, for all $t \in (0,1)$ and $y \in C$. Letting $t \to 0^+$ and using (A3), we conclude $F_{k+1}(z_0, y) \ge 0$, for all $y \in C$ and $k \in \{0, \dots, M-1\}$. Therefore

$$z_0 \in \bigcap_{k=1}^{M} EP(F_k) = EP(\mathcal{G}).$$

Step 11. The sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $P_{\mathcal{F}}(x)$.
Proof of Step 11. Let $z_0 \in \omega_w(x_n)$ and let $\{x_{n_m}\}$ be a subsequence of $\{x_n\}$ weakly converging to z_0. From (3.5) and Step 10, we have

$$\|x - P_{\mathcal{F}}(x)\| \le \|x - z_0\| \le \liminf_{m\to\infty} \|x - x_{n_m}\|$$

$$\le \limsup_{m\to\infty} \|x - x_{n_m}\| \le \|x - P_{\mathcal{F}}(x)\|.$$

Hence

$$\lim_{m\to\infty} \|x - x_{n_m}\| = \|x - z_0\| = \|x - P_{\mathcal{F}}(x)\|.$$

Since $z_0 \in \mathcal{F}$ and H is a Hilbert space, we obtain

$$x_{n_m} \longrightarrow z_0 = P_{\mathcal{F}}(x).$$

Since $z_0 \in \omega_w(x_n)$ was arbitrary, we get $x_n \longrightarrow P_{\mathcal{F}}(x)$. $\square$

Corollary 3.2 *Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$, $\psi = \{T_j : j = 1 \dots N\}$ a finite family of strictly pseudocontractive mappings with $0 \le \kappa < 1$ from C into C, $\mathcal{G} = \{F_j : j = 1, \dots, M\}$ a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), and $\mathcal{F} := Fix(T) \cap Fix(\psi) \cap EP(\mathcal{G})$ nonempty and bounded.*

Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \delta \le \alpha_n \le 1-\kappa$ for all $n \in \mathbb{N}$, $\{\lambda_{k,n}\}_{k=1}^N$ sequences in $[c,d] \subset (0, 1-\kappa)$ such that $\lim_n |\lambda_{k,n} - \lambda_{k,n+1}| = 0$ for every $k \in \{1, \dots, N\}$ and $\{r_{j,n}\}_{j=1}^M$ sequences in $(0,\infty)$ such that $\liminf_n r_{j,n} > 0$ and $\lim_n r_{j,n}/r_{j,n+1} = 1$ for every $j \in \{1, \dots, M\}$.

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \ge 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \dots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = ((1-\lambda_{N,n})I + \lambda_{N,n}T_N) \dots ((1-\lambda_{1,n})I + \lambda_{1,n}T_1)u_n, \\ y_n = (1-\alpha_n)v_n + \alpha_n T^n v_n, \\ C_n = \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x), \end{cases}$$

where $\theta_n = c_n + \gamma_n \Delta_n$ *and* $\Delta_n = \sup\{\|x_n - p\|^2 : p \in \mathcal{F}\} < \infty$*, then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *converge strongly to* $P_{\mathcal{F}}(x)$*.*

Proof. Put $A_j = I - T_j$ for every $j \in \{1, \dots, N\}$. Then A_j is $\frac{1-k}{2}$-inverse-strongly monotone. We have that $Fix(T_j)$ is the solution set of $VI(C, A_j)$; i.e., $Fix(T_j) = VI(C, A_j)$. Therefore, $Fix(\psi) = \cap_{k=1}^N VI(C, A_k)$ and it suffices to apply Theorem 3.1. □

At this stage, considering Theorem 3.1, we present a numerical example in Hilbert space l^2:

Example 3.3 Let $H = l^2$ and $C = l^2 \cap \prod_{i=1}^{\infty}[0,1]$. For each $(w_1, w_2, \dots) \in C$, we define $T(w_1, w_2, \dots) = (T_1(w_1), T_2(w_2), \dots)$, where for any natural number k,

$$T_k(t) = \begin{cases} \frac{1}{2^k} t & \text{if } t \in [0, \frac{1}{2^k}], \\ \\ 0 & \text{if } t \in (\frac{1}{2^k}, 1]. \end{cases}$$

It is easy to show that

$$|T_k^n t - T_k^n s|^2 \le (\frac{1}{2^{kn}}|t-s| + \frac{1}{2^{kn}})^2 \le |t-s|^2 + \frac{3}{4^{kn}},$$

for all $t, s \in [0,1]$ and $k, n \in \mathbb{N}$. (See also [21, Example 1.6]). Hence, for (w_k), $(z_k) \in C$ and $n \in \mathbb{N}$, we have

$$\|T^n(w_k) - T^n(z_k)\|^2 \le \|(T_k^n w_k) - (T_k^n z_k)\|^2$$
$$= \sum_{k=1}^{\infty} |T_k^n w_k - T_k^n z_k|^2 \le \sum_{k=1}^{\infty}\{|w_k - z_k|^2 + \frac{3}{4^{kn}}\}$$
$$= \|(w_k) - (z_k)\|^2 + \frac{3}{4^n - 1}.$$

Therefore, $T : C \to C$ is an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with $\kappa = 0$, $\gamma_n = 0$ and $c_n = \frac{3}{4^n-1}$, for all n. It is easy to see that T is discontinuous at $(\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^k}, \dots)$. Now, taking $\alpha_n = 1$, for all n, $A_k \equiv 0$, for $k = 1, ..., N$, $F_j \equiv 0$, for $j = 1, ..., M$ and $x_1 = x = (\frac{1}{2}, 0, 0, \dots)$, in Theorem 3.1, we have $\theta_n = \frac{3}{4^n-1}$, $y_n = T^n v_n$, $u_n = P_C x_n$ and $v_n = u_n$, for all n. In particular, we may compute $\{x_n\}$ as follows:

$$x \in C_1,\ D_1 = l^2 \Longrightarrow x_2 = P_{C_1 \cap Q_1}(x) = x;$$
$$x \in C_2,\ D_2 = l^2 \Longrightarrow x_3 = P_{C_2 \cap Q_2}(x) = x;$$
$$y_3 = \frac{1}{2^3}(\frac{1}{2}, 0, 0, \dots),$$

$$C_3 = \{z \in l^2 : \|y_3 - z\|^2 \le \|x_3 - z\|^2 + \frac{3}{4^3-1}\}$$
$$= \{(z_i) \in l^2 : |\frac{1}{2^4} - z_1|^2 \le \|\frac{1}{2} - z_1\|^2 + \frac{3}{4^3-1}\}$$
$$= \{(z_i) \in l^2 : z_1 \le 0/3356717\},\ D_3 = l^2$$
$$\Longrightarrow x_4 = P_{C_3 \cap Q_3}(x) = (0/3356717, 0, 0, \dots);$$
$$y_4 = \frac{1}{2^4}(0/3356717, 0, 0, \dots) = (0/0209795, 0, 0, \dots),$$
$$C_4 = \{(z_i) \in l^2 : z_1 \le 0/197018\},$$
$$D_4 = \{(z_i) \in l^2 : z_1 \le 0/3356717\}$$
$$\Longrightarrow x_5 = P_{C_4 \cap Q_4}(x) = (0/197018, 0, 0, \dots);$$
$$y_5 = \frac{1}{2^5}(0/197018, 0, 0, \dots) = (0/0061568, 0, 0, \dots),$$
$$C_5 = \{(z_i) \in l^2 : z_1 \le 0/1092623\},$$
$$D_5 = \{(z_i) \in l^2 : z_1 \le 0/197018\}$$
$$\Longrightarrow x_6 = P_{C_5 \cap Q_5}(x) = (0/1092623, 0, 0, \dots);$$
$$y_6 = \frac{1}{2^6}(0/1092623, 0, 0, \dots) = (0/0017072, 0, 0, \dots),$$
$$C_6 = \{(z_i) \in l^2 : z_1 \le 0/0588905\},$$
$$D_6 = \{(z_i) \in l^2 : z_1 \le 0/1092623\}$$
$$\Longrightarrow x_7 = P_{C_6 \cap Q_6}(x) = (0/0588905, 0, 0, \dots);$$
$$\vdots$$

Note that the argument used for computing C_4 have been used for computing C_5 and C_6. As we expected, it is easy to see that $\{x_n\}$ tends to 0.

4. Weak convergence

The following is a weak convergence theorem which extends [21, Theorem 3.4].

Theorem 4.1 *Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\sum_{n=1}^{\infty}(\gamma_n + c_n) < \infty$, $\mathcal{G} = \{F_j : j = 1, \dots, M\}$ a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), $\{A_k : k = 1 \dots N\}$ a finite family of α-inverse-strongly monotone mappings from C into H, and $\mathcal{F} := \cap_{k=1}^{N} VI(C, A_k) \cap Fix(T) \cap EP(\mathcal{G}) \neq \infty$.*

Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $0 < \delta \le \alpha_n \le 1 - \kappa - \delta$, $\{\lambda_{k,n}\}_{k=1}^{N}$ sequences in $[c,d] \subset (0, 2\alpha)$ such that $\lim_n |\lambda_{k,n} - \lambda_{k,n+1}| = 0$ for every $k \in \{1, \dots, N\}$ and $\{r_{j,n}\}_{j=1}^{M}$ sequences in $(0, \infty)$ such that $\liminf_n r_{j,n} > 0$ and $\lim_n r_{j,n}/r_{j,n+1} = 1$ for every $j \in \{1, \dots, M\}$.

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \ge 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \dots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = P_C(I - \lambda_{N,n} A_N) \dots P_C(I - \lambda_{2,n} A_2) P_C(I - \lambda_{1,n} A_1) u_n, \\ x_{n+1} = (1 - \alpha_n) v_n + \alpha_n T^n v_n, \end{cases}$$

then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Proof. We will apply the notations used in proof of Theorem 3.1.

Step 1. $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.

Proof of Step 1. Let $p \in \mathcal{F}$. Then

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n)(v_n - p) + \alpha_n(T^n v_n - p)\|^2 \\ &= (1 - \alpha_n)\|v_n - p\|^2 + \alpha_n\|T^n v_n - p\|^2 - \alpha_n(1 - \alpha_n)\|v_n - T^n v_n\|^2 \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n((1 + \gamma_n)\|v_n - p\|^2 + \kappa\|v_n - T^n v_n\|^2 + c_n) \\ &\quad -\alpha_n(1 - \alpha_n)\|v_n - T^n v_n\|^2 \\ &= (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\|v_n - p\|^2 - \alpha_n(1 - \alpha_n - \kappa)\|v_n - T^n v_n\|^2 + \alpha_n c_n \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\|v_n - p\|^2 - \delta^2\|v_n - T^n v_n\|^2 + c_n \quad (4.1) \\ &\le (1 + \gamma_n)\|x_n - p\|^2 - \alpha_n(1 - \alpha_n - \kappa)\|v_n - T^n v_n\|^2 + c_n \\ &\le (1 + \gamma_n)\|x_n - p\|^2 - \delta^2\|v_n - T^n v_n\|^2 + c_n \quad (4.2) \\ &\le (1 + \gamma_n)\|x_n - p\|^2 + c_n \quad (4.3) \end{aligned}$$

By Lemma 2.2, (4.3) and the assumption $\sum_{n=1}^{\infty}(\gamma_n + c_n) < \infty$, we obtain that

$$\lim_{n\to\infty} \|x_n - p\| \text{ exists.} \quad (4.4)$$

Hence, $\{x_n\}$ is bounded.

Step 2. $\lim_{n\to\infty} \|\mathcal{J}^k_n x_n - \mathcal{J}^{k+1}_n x_n\| = 0$, $\forall k \in \{0, 1, \dots, M-1\}$.

Proof of Step 2. For $p \in \mathcal{F}$, as in Step 7 of Theorem 3.1, we get

$$\|\mathcal{J}^{k+1}_n x_n - p\|^2 \le \|x_n - p\|^2 - \|\mathcal{J}^k_n x_n - \mathcal{J}^{k+1}_n x_n\|^2,$$

for all $k \in \{0, 1, \dots, M-1\}$. Therefore, by (4.1), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\|v_n - p\|^2 + c_n \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\|\mathcal{J}^{k+1}_n x_n - p\|^2 + c_n \\ &\le (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n(1 + \gamma_n)\{\|x_n - p\|^2 - \|\mathcal{J}^k_n x_n - \mathcal{J}^{k+1}_n x_n\|^2\} + c_n \\ &\le \|x_n - p\|^2 - \delta\|\mathcal{J}^k_n x_n - \mathcal{J}^{k+1}_n x_n\|^2 + \gamma_n\|x_n - p\|^2 + c_n. \end{aligned}$$

Since $\gamma_n \to 0$ and $c_n \to 0$, applying (4.4), we have

$$\begin{aligned} &\delta\|\mathcal{J}^k_n x_n - \mathcal{J}^{k+1}_n x_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n\|x_n - p\|^2 + c_n \to 0. \end{aligned}$$

So, we get the desired result.

Step 3. $\lim_{n\to\infty} \|\mathcal{P}^k_n u_n - \mathcal{P}^{k+1}_n u_n\| = 0$, $\forall k \in \{0, 1, \dots, N-1\}$.

Proof of Step 3. For $p \in \mathcal{F}$ and $k \in \{0, 1, \dots, N-1\}$, like that in Step 8 of Theorem 3.1, we get

$$\|\mathcal{P}^{k+1}_n u_n - p\|^2 \le \|x_n - p\|^2 + c(d - 2\alpha)\|A_{k+1}\mathcal{P}^k_n u_n - A_{k+1}p\|^2.$$

From this and (4.1), we have

$$\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\|v_n-p\|^2+c_n$$
$$\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\|\mathcal{P}_n^{k+1}u_n-p\|^2+c_n \quad (4.5)$$
$$\le (1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\{\|x_n-p\|^2+c(d-2\alpha)\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2\}+c_n$$
$$=\|x_n-p\|^2+c(d-2\alpha)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2+\gamma_n\|x_n-p\|^2+c_n.$$

So,

$$c(2\alpha-d)\alpha_n\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|^2$$
$$\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\gamma_n\|x_n-p\|^2+c_n\to 0.$$

Since $0<\delta\le\alpha_n\le 1-\kappa-\delta$, we obtain

$$\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|\to 0 \ \ (n\to\infty). \quad (4.6)$$

Again, like that in Step 8 of Theorem 3.1, we have

$$\|\mathcal{P}_n^{k+1}u_n-p\|^2\le\|x_n-p\|^2-\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\rangle.$$

Then, from this and (4.5), we have

$$\|x_{n+1}-p\|^2\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\|\mathcal{P}_n^{k+1}u_n-p\|^2+c_n$$
$$\le(1-\alpha_n)\|x_n-p\|^2+\alpha_n(1+\gamma_n)\{\|x_n-p\|^2-\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2$$
$$+2\lambda_{k+1,n}\langle\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n, A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\rangle\}+c_n$$
$$\le\|x_n-p\|^2-\delta\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2+\gamma_n\|x_n-p\|^2$$
$$+2\lambda_{k+1,n}(1-\gamma_n)\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|+c_n,$$

which implies that

$$\delta\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|^2\le\|x_n-p\|^2-\|x_{n+1}-p\|^2$$
$$+\gamma_n\|x_n-p\|^2+2\lambda_{k+1,n}\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\|A_{k+1}\mathcal{P}_n^k u_n-A_{k+1}p\|+c_n.$$

Hence it follows from $c_n\to 0$, $\gamma_n\to 0$, Step 1 and (4.6) that $\|\mathcal{P}_n^k u_n-\mathcal{P}_n^{k+1}u_n\|\to 0$.
Step 4. $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$.
Proof of Step 4. It is easy to see from (4.2) that

$$\delta^2\|v_n-T^n v_n\|^2\le(1+\gamma_n)\|x_n-p\|^2-\|x_{n+1}-p\|^2+c_n,$$

which implies that

$$\lim_{n\to\infty}\|v_n-T^n v_n\|=0. \ \ (4.7)$$

Therefore,

$$\|x_{n+1}-v_n\|=\alpha_n\|v_n-T^n v_n\|\to 0, \text{ as } n\to\infty. \ \ (4.8)$$

From Steps 2 and 3 we see that $\|x_n-u_n\|\to 0$ and $\|u_n-v_n\|\to 0$, as $n\to\infty$. So, from this and (4.8), we obtain

$$\|x_{n+1}-x_n\|\le\|x_{n+1}-v_n\|+\|v_n-u_n\|+\|u_n-x_n\|\to 0,$$

as $n\to\infty$.
Step 5. $\lim_{n\to\infty}\|v_n-Tv_n\|=0$.
Proof of Step 5. It is easy to see that the assertions of Steps 4 and 5 of Theorem 3.1 hold in our assumptions. From this fact and $\|x_n-x_{n+1}\|\to 0$, by a proof like that in

Step 9 of Theorem 3.1, we can get $\|v_{n+1} - v_n\| \to 0$, as $n \to \infty$. By (4.7), we obtain from Lemma 2.3 that $\lim_{n\to\infty} \|v_n - Tv_n\| = 0$.

Step 6. $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Proof of Step 6. Applying Steps 2, 3 and 5, by a proof similar to Step 10 of Theorem 3.1, we can show that the weak ω-limit set of $\{x_n\}$, $\omega_w(x_n)$, is a subset of $\mathcal{F}$.

Now, (4.4) and the Opial's property of Hilbert space imply that $\omega_w(x_n)$ is singleton. Therefore, $x_n \rightharpoonup z_0$ for some $z_0 \in \mathcal{F}$. □

Corollary 4.2 *Let C be a nonempty closed convex subset of a Hilbert space H, $T : C \to C$ a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\sum_{n=1}^{\infty}(\gamma_n + c_n) < \infty$, $\psi = \{T_j : j = 1 \ldots N\}$ a finite family of strictly pseudocontractive mappings with $0 \le \kappa < 1$ from C into C, $\mathcal{G} = \{F_j : j = 1, \ldots, M\}$ a finite family of bifunctions from $C \times C$ into $\mathbb{R}$ which satisfy (A1)-(A4), and $\mathcal{F} := Fix(T) \cap Fix(\psi) \cap EP(\mathcal{G}) \neq \infty$.*

Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \le \alpha_n \le 1 - \kappa - \delta$, $\{\lambda_{k,n}\}_{k=1}^N$ sequences in $[c, d] \subset (0, 1 - \kappa)$ such that $\lim_n |\lambda_{k,n} - \lambda_{k,n+1}| = 0$ $(1 \le k \le N)$ and $\{r_{j,n}\}_{j=1}^M$ sequences in $(0, \infty)$ such that $\liminf_n r_{j,n} > 0$ and $\lim_n \frac{r_{j,n}}{r_{j,n+1}} = 1$ for every $j \in \{1, \ldots, M\}$.

If $\{x_n\}$ is the sequence generated by $x_1 = x \in H$ and $\forall n \ge 1$,

$$\begin{cases} u_n = J^{F_M}_{r_{M,n}} \ldots J^{F_2}_{r_{2,n}} J^{F_1}_{r_{1,n}} x_n, \\ v_n = ((1 - \lambda_{N,n})I + \lambda_{N,n}T_N) \ldots ((1 - \lambda_{1,n})I + \lambda_{1,n}T_1)u_n, \\ x_{n+1} = (1 - \alpha_n)v_n + \alpha_n T^n v_n, \end{cases}$$

then the sequence $\{x_n\}$ converges weakly to an element of $\mathcal{F}$.

Remark 4.3 *We may put*

$$v_n = P_C(I - \lambda_{N,n}(I - T_N)) \ldots P_C(I - \lambda_{2,n}(I - T_2))P_C(I - \lambda_{1,n}(I - T_1))u_n,$$

in the schemes of Corollaries 3.2 and 4.2, and obtain schemes for families of non-self strictly pseudocontractive mappings.

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