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TWO ALGORITHMS FOR NONEXPANSIVE MAPPINGS

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Abstract. We study two algorithms for approximating fixed points of nonexpansive mappings in Banach spaces. One of them is implicit and the other is explicit. We prove strong convergence theorems for both of them.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Banach space $(E, \|\cdot\|)$. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Nonlinear fixed point theory continues to be an important and active research area. In particular, iterative methods for finding fixed points of nonexpansive mappings have been investigated intensively. In this paper we study two algorithms for approximating fixed points of nonexpansive mappings in Banach spaces. The first one is implicit and the second is explicit. In Hilbert space these algorithms have recently been studied in [11]. Similar algorithms in Banach spaces have been studied in [8]. We establish strong convergence theorems for both algorithms under weaker assumptions. In the next section we recall a few preliminary results. Our main results, Theorems 3.1, 3.3 and 3.4 below, are stated and proved in Section 3.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a Banach space E. We assume that C is a nonexpansive retract of E. That is, we assume that there exists a retraction R_C of E onto C which is a nonexpansive mapping. See [2] and [3] for information regarding nonexpanisve retracts in Banach spaces.
Let E^* be the dual space of E. The duality mapping J from E into the family of nonempty, weak*-compact and convex subsets of E^* is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 \text{ and } \|x^*\| = \|x\|\} \text{ for each } x \in E.$$

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The mapping J is single-valued if and only if E is smooth. If E has a uniformly convex dual (equivalently, if E is uniformly smooth), then we have for all $x, y \in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + \max\{\|x\|, 1\}\|y\|b(\|y\|),$$

where $b : (0, \infty) \to [0, \infty)$ is an increasing and continuous function, defined by

$$b(t) = \sup\{(\|x+ty\|^2 - \|x\|^2)/t - 2\langle y, J(x)\rangle : \|x\| \leq 1, \|y\| \leq 1\},$$

which satisfies $\lim_{t\to 0^+} b(t) = 0$. See [4, page 90] for more details.
If E is smooth, the duality mapping J is said to be weakly sequentially continuous at zero if $x_n \rightharpoonup 0$ in E implies that $\{J(x_n)\}$ converges weak* to 0 in E^*. The duality mapping of each ℓ^p space, $1 < p < \infty$, has this property.
In order to prove our main results, we also need the following lemmas.

Lemma 2.1. [1] *(Demiclosedness Principle) Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E, and let $T : C \to E$ be a nonexpansive mapping. Then $I - T$ is demiclosed at 0, i. e., if $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.2. [9] *Let $\{x_n\}$ and $\{z_n\}$ be two bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n$, for all $n \geq 0$ and that $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

Lemma 2.3. [10] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n$, $n \geq 0$, where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\delta_n \gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Let C be a nonempty, closed and convex subset of a real Banach E. Assume that C is a nonexpansive retract of E and let $R_C : E \to C$ be a nonexpansive retraction of E onto C. Let $T : C \to C$ be a nonexpansive mapping. We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T. For each $t \in [0, 1)$, consider the mapping $T_t : E \to C$, defined by $T_t x := TR_C(tx)$, $x \in E$. It is easy to check that $\|T_t x - T_t y\| \leq t\|x - y\|$, so that T_t is a strict contraction. By the Banach contraction principle, there exists a unique fixed point x_t of T_t in E, that is, a point x_t such that

$$x_t = TR_C(tx_t). \tag{1}$$

Theorem 3.1. *Let E be a real uniformly convex Banach space with a uniformly convex dual and a duality mapping J which is weakly sequentially continuous at zero. Let C be a nonexpansive retract of E and let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let the net $\{x_t : 0 \leq t < 1\}$ be generated by (1). Then, as $t \to 1^-$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Proof. First, we show that $\{x_t\}$ is bounded. Take $u \in \mathrm{Fix}(T)$. From (1), we have $\|x_t - u\| = \|TR_C(tx_t) - TR_C u\| \le \|tx_t - u\| = \|t(x_t - u) + (t-1)u\| \le t\|x_t - u\| + (1-t)\|u\|$, that is, $\|x_t - u\| \le \|u\|$. Hence $\{x_t\}$ is indeed bounded.

Again from (1), we obtain

$$\|x_t - Tx_t\| = \|TR_C(tx_t) - TR_C x_t\| \le (1-t)\|x_t\| \to 0 \quad \text{as} \quad t \to 1^-. \tag{2}$$

Next, let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 1^-$ as $n \to \infty$, and put $x_n := x_{t_n}$. From (1), we have

$$\|x_n - Tx_n\| \to 0. \tag{3}$$

From (2), we get

$$\begin{aligned}
\|x_t - u\|^2 &= \|TR_C(tx_t) - TR_C u\|^2 \le \|tx_t - u\|^2 \\
&= \|x_t - u - (1-t)x_t\|^2 \\
&\le \|x_t - u\|^2 - 2(1-t)\langle x_t, J(x_t - u)\rangle \\
&\quad + \max\{\|x_t - u\|, 1\}(1-t)\|x_t\| b((1-t)\|x_t\|) \\
&\le \|x_t - u\|^2 - 2(1-t)\langle x_t - u, J(x_t - u)\rangle - 2(1-t)\langle u, J(x_t - u)\rangle \\
&\quad + (1-t)b((1-t)M)M,
\end{aligned}$$

where $M := \sup_{t\in(0,1)}\{\max\{\|x_t - u\|, 1\}\|x_t\|\}$.
Hence

$$\|x_t - u\|^2 \le \langle u, J(u - x_t)\rangle + \frac{M}{2} b((1-t)M). \tag{4}$$

In particular,

$$\|x_n - u\|^2 \le \langle u, J(u - x_n)\rangle + \frac{M}{2} b((1-t_n)M), \quad u \in \mathrm{Fix}(T). \tag{5}$$

Since E is reflexive and $\{x_n\}$ is bounded, it has weak subsequential limits. We claim that each such limit is, in fact, a strong one. To see this we may assume, without any loss of generality, that $\{x_n\}$ itself converges weakly to a point $x^* \in C$. In view of (3), we can use Lemma 2.1 to get $x^* \in \mathrm{Fix}(T)$. Therefore we can substitute x^* for u in (5) to get

$$\|x_n - x^*\|^2 \le \langle x^*, J(x^* - x_n)\rangle + \frac{M}{2} b((1-t_n)M).$$

Hence the weak convergence of $\{x_n\}$ to x^* implies that $x_n \to x^*$ strongly, as claimed. To show that the entire net $\{x_t\}$ converges to x^*, assume $x_{s_n} \to \tilde{x} \in \mathrm{Fix}(T)$, where $s_n \to 1^-$. Put $y_n := x_{s_n}$. Substituting $t := s_n$ and $u = x^*$ in (4), we get

$$\|y_n - x^*\|^2 \le \langle x^*, J(x^* - y_n)\rangle + \frac{M}{2} b((1-s_n)M).$$

Therefore

$$\|\tilde{x} - x^*\|^2 \le \langle x^*, J(x^* - \tilde{x})\rangle. \tag{6}$$

Interchanging x^* and $\tilde{x}$, we obtain

$$\|x^* - \tilde{x}\|^2 \le \langle \tilde{x}, J(\tilde{x} - x^*)\rangle. \tag{7}$$

Adding up (6) and (7), we see that $2\|x^* - \tilde{x}\|^2 \le \|x^* - \tilde{x}\|^2$, which implies that $\tilde{x} = x^*$. This completes the proof of Theorem 3.1. □

A similar theorem has recently been proved in [8] under stronger assumptions on J and b. Another way to achieve the strong convergence of the curve $\{x_t : 0 \leq t < 1\}$ defined by (1) to a fixed point of T, without assuming that the duality mapping is weakly sequentially continuous at zero, is to use the following theorem.

Theorem 3.2. [5]. *Let K be a closed and convex subset of a uniformly smooth Banach space E, and let $S : K \to K$ be a nonexpansive mapping. For $x \in K$ and $t \in [0, 1)$, let $G_t x$ be the unique fixed point of the strict contraction $g_x : K \to K$ defined by $g_x y := (1-t)x + tSy$ for $y \in K$. In other words,*

$$G_t x = (1-t)x + tSG_t x,\ x \in K.$$

If S has a fixed point, then for each $x \in K$, the strong $\lim_{t\to 1^-} G_t x$ exists and is a fixed point of S.

Theorem 3.3. *Let E be a real uniformly smooth Banach space, C a convex nonexpansive retract of E, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For each $t \in (0, 1)$, let the net $\{x_t\}$ be defined by (1). Then, as $t \to 1^-$, the net $\{x_t\}$ converges strongly to a fixed point of T.*

Proof. We apply Theorem 3.2 with $S := TR_C : E \to C$. Pick $x = 0$, and denote

$$z_t = G_t 0 = tSz_t. \tag{8}$$

It is clear that $\mathrm{Fix}(T) = \mathrm{Fix}(S)$. Therefore we may invoke Theorem 3.2 to conclude that $\{z_t\}$ converges strongly as $t \to 1^-$ to a fixed point $x^* \in C$ of T. Note that by setting $x_t = \frac{1}{t} z_t$, we obtain (1) from (8). Since the strong convergence of $\{z_t\}$ to x^* also implies the strong convergence of $\{x_t\}$ to x^*, this completes the proof of Theorem 3.3. $\square$

So far we have considered the implicit continuous scheme defined by (1). Now we turn to an analogous explicit discrete method.

Theorem 3.4. *Let E be a real uniformly smooth Banach space. Let C be a convex nonexpansive retract of E, and let $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $(0, 1)$. Given an arbitrary $x_0 \in C$, let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} y_n = R_C[(1-\alpha_n)x_n], \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \quad n \geq 0. \end{cases} \tag{9}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (9) converges strongly to a fixed point of T.

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Take $u \in \mathrm{Fix}(T)$. From (9), we have

$$\begin{aligned}\|x_{n+1} - u\| &= \|(1-\beta_n)(x_n - u) + \beta_n(Ty_n - u)\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n\|y_n - u\| \\ &\le (1-\beta_n)\|x_n - u\| + \beta_n[(1-\alpha_n)\|x_n - u\| + \alpha_n\|u\|] \\ &= (1-\alpha_n\beta_n)\|x_n - u\| + \alpha_n\beta_n\|u\| \\ &\le \max\{\|x_n - u\|, \|u\|\}.\end{aligned}$$

Hence $\{x_n\}$ is bounded and so is $\{Tx_n\}$. Set $z_n = Ty_n$, $n \ge 0$. It follows that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|Ty_{n+1} - Ty_n\| \\ &\le \|y_{n+1} - y_n\| \\ &\le \|(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n\| \\ &\le \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\|.\end{aligned}$$

Hence $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. This inequality, when combined with Lemma 2.2, implies that $\lim_{n\to\infty}\|z_n - x_n\| = 0$.
Therefore $\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}\beta_n\|x_n - z_n\| = 0$. Now we observe that

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - Tx_n\| + \beta_n\|Ty_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - Tx_n\| + \beta_n\|y_n - x_n\| \\ &\le \|x_n - x_{n+1}\| + (1-\beta_n)\|x_n - Tx_n\| + \alpha_n\|x_n\|.\end{aligned}$$

That is, $\|x_n - Tx_n\| \le \frac{1}{\beta_n}\{\|x_{n+1} - x_n\| + \alpha_n\|x_n\|\} \to 0$. Let the net $\{x_t\}$ be defined by (1). By Theorem 3.3, we know that $x_t \to x^*$ as $t \to 1^-$. We assert that $\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \le 0$. Indeed,

$$\begin{aligned}\|x_t - x_n\|^2 &= \|x_t - Tx_n + Tx_n - x_n\|^2 \\ &\le \|x_t - Tx_n\|^2 + 2\langle Tx_n - x_n, J(x_t - Tx_n)\rangle \\ &\quad + \max\{1, \|x_t - Tx_n\|\}\|Tx_n - x_n\|b(\|Tx_n - x_n\|) \\ &\le \|x_t - Tx_n\|^2 + M\|Tx_n - x_n\| \\ &\le \|(x_t - x_n) - (1-t)x_t\|^2 + M\|Tx_n - x_n\| \\ &\le \|x_t - x_n\|^2 - 2(1-t)\langle x_t, J(x_t - x_n)\rangle \\ &\quad + \max\{1, \|x_t - x_n\|\}(1-t)\|x_t\|b((1-t)\|x_t\|) + M\|Tx_n - x_n\| \\ &\le \|x_t - x_n\|^2 - 2(1-t)\langle x_t, J(x_t - x_n)\rangle \\ &\quad + (1-t)Mb((1-t)M) + M\|Tx_n - x_n\|,\end{aligned}$$

where $M := \sup_{t\in(0,1), n\ge 0}\{2\|x_t - Tx_n\| + \max\{1, \|x_t - Tx_n\|\}b(\|Tx_n - x_n\|), \max\{1, \|x_t - x_n\|\}\|x_t\|\}$.

It follows that

$$\langle x_t, J(x_t - x_n)\rangle \le \frac{M}{2}b((1-t)M) + \frac{M}{2(1-t)}\|Tx_n - x_n\|$$

and therefore $\limsup_{t\to 1^-}\limsup_{n\to\infty}\langle x_t, J(x_t - x_n)\rangle \le 0$. Next, we consider

$$\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \le \limsup_{n\to\infty}\langle x_t, J(x_t - x_n)\rangle$$
$$+\limsup_{n\to\infty}\langle x^* - x_t, J(x_t - x_n)\rangle + \limsup_{n\to\infty}\langle x^*, J(x^* - x_n) - J(x_t - x_n)\rangle.$$

Taking $\limsup_{t\to 1^-}$ on both sides, and using the fact that the duality mapping is uniformly continuous on bounded sets [6], we obtain $\limsup_{n\to\infty}\langle x^*, J(x^* - x_n)\rangle \le 0$, as asserted.

Finally, we show that $x_n \to x^*$. From (9), we have

$$\|x_{n+1} - x^*\|^2 \le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|y_n - x^*\|^2$$
$$\le (1-\beta_n)\|x_n - x^*\|^2 + \beta_n[(1-\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n(1-\alpha_n)\langle x^*, J(x_n - x^*)\rangle$$
$$+\max\{1, (1-\alpha_n)\|x_n - x^*\|\}\alpha_n\|x^*\|b(\alpha_n\|x^*\|)]$$
$$\le (1-\alpha_n\beta_n)\|x_n - x^*\|^2 + \alpha_n\beta_n[2(1-\alpha_n)\langle x^*, J(x^* - x_n)\rangle + M'b(\alpha_n M')],$$

where $M' := \sup_{n\ge 0}\{\max\{1, (1-\alpha_n)\|x_n - x^*\|\}\|x^*\|\}$.

It is not difficult to check that all the assumptions of Lemma 2.3 are satisfied. Therefore $x_n \to x^*$. This completes the proof of Theorem 3.4. □

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