Fixed Point Theory, 12(2011), No. 2, 401-418
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

DOUBLE PROJECTION NEURAL NETWORK MODEL FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

A. MALEK*, S. EZAZIPOUR, N. HOSSEINIPOUR-MAHANI

Department of Applied Mathematics, Faculty of Mathematical Sciences
Tarbiat Modares University, P.O. Box 14115-134, Tehran, Iran.
*E-mail: mala@modares.ac.ir

Abstract. In this paper we propose a novel double projection recurrent neural network model for solving pseudomonotone variational inequalities based on a technique of updating the state variable and fixed point theorem. This model is stable in the sense of Lyapunov and globally convergent for problems that satisfy Lipschitz continuity and pseudomonotonicity conditions. The global exponential stability of the model under the assumptions of strong pseudomonotonicity and Lipschitz continuity is proved. Numerical simulation to various types of variational inequalities is given to show the applied significance of the results

Key Words and Phrases: Pseudomonotone variational inequalities, Double projection neural network, Fixed point theory, Lyapunov stability, Globally convergence.
2010 Mathematics Subject Classification: 58E35, 47H10, 68T20.

1. Introduction

The variational inequality problem is a general problem formulation that encompasses a plethora of mathematical problems, including among others, nonlinear equations, optimization problems, complementarity problems and fixed point problems [1]. This problem has had a great impact and influence in the development of many branches of pure and applied sciences. On the other hand, the fixed-point theory has played a major role in the development of various numerical algorithms for solving variational inequalities. Using the projection operator technique, one usually establishes an equivalence relation between the variational inequalities and the fixed-point problem, see [1]. This alternative equivalent formulation was used for the first time by Lions and Stampacchia [2] to study the existence of a solution of the variational inequalities. A variety of numerical methods, that use the projection for solving variational inequalities, exist (see, for example, [3-18]). It is well known that in many engineering applications such as signal processing, image processing, filter design, robot control, real time solutions are often desired, see [19-20]. Notice that, very often, these problems may have high dimension and dense structure. Hence, usual numerical methods may not be efficient in such occasions because of stringent requirements on computing time. According to this point, in past two decades, applications of recurrent neural networks have been widely investigated. Recently, Xia, Wang and co-authors [21-26] introduced and developed a new type of projection recurrent neural networks for solving monotone and pseudomonotone variational inequalities.

In this paper we will be concerned with development of a novel efficient algorithm to approximate the solutions of pseudomonotone variational inequalities. By this motivation, a technique of updating the state variable is used to suggest the double projection neural network. This paper is organized as follows. Section 2, provides the necessary mathematical background, which is used to express the novel neural network model. The problem formulation and double projection neural network are presented in Section 3. In Section 4, we discuss the convergence and stability of proposed neural network for solving pseudomonotone variational inequalities. In Section 5, we compare the proposed neural network with the existing neural networks that use for solving pseudomonotone variational inequalities. In Section 6, several examples are solved numerically to evaluate the effectiveness of this recurrent neural network model. Section 7, concludes the paper.

2. Preliminaries

This section provides the necessary mathematical background, which is used to propose the desired neural network and to study its stability and convergence. In what follows, we assume that Ω is a closed convex subset of R^n, x is a vector in R^n and for each fixed time t, $x(t) \in R^n$ is the state variable for the corresponding dynamical system.

Lemma 2.1 ([28]) *For each* $\mathrm{x} \in R^n$*, there is a unique point* $\mathrm{y} \in \Omega$ *such that*

$$||\mathrm{x}-\mathrm{y}|| \;\leq\; ||\mathrm{x}-\mathrm{z}||, \quad \forall \mathrm{z} \in \Omega.$$

The point y *satisfying the above inequality is called the projection of* x *on* Ω *and we write:*

$$\mathrm{y} = P_\Omega(\mathrm{x}) = \arg\min_{\mathrm{z}\in\Omega} \| \mathrm{x}-\mathrm{z} \| .$$

There are some well-known results for the projection operator, and we summarize them in the following Lemma [28].

Lemma 2.2 *For any* $u, v \in R^n$ *and any* $z \in \Omega$

(*i*) $\left\langle (P_\Omega(u)-u)^T, z-P_\Omega(u)\right\rangle \geq 0.$
(*ii*) $\left\langle (P_\Omega(u)-P_\Omega(v))^T, u-v\right\rangle \geq 0.$
(*iii*) $||P_\Omega(u)-P_\Omega(v)|| \;\leq\; ||u-v||.$
(*iv*) $\| u-z \|^2 \geq \| u-P_\Omega(u) \|^2 + \| z-P_\Omega(u) \|^2 .$

Definition 2.1 ([27]) Consider the dynamical system

$$\dot{x}(t) = H(x(t)), \quad x(t_0) = x_0 \in R^n \tag{1}$$

$\bar{x}$ is called an equilibrium point, critical point or steady state of the dynamical system if $H(\bar{x}) = 0$.

Definition 2.2 ([27]) Let $N \subseteq R^n$ be an open neighborhood of $\bar{x}$. A continuously differentiable function $\omega : R^n \to R$ is said to be a Lyapunov function at any equilibrium point $\bar{x}$ over the set N if

$$\begin{cases} \omega(x) \geq 0, \quad \omega(x) = 0 \Leftrightarrow x = \bar{x}, \\ \frac{d\omega(x(t))}{dt} = [\nabla\omega(x(t))]^T H(x(t)) \leq 0, \quad \forall x(t) \in N. \end{cases}$$

Definition 2.3 ([29]) An equilibrium point $\bar{x}$ is Lyapunov stable if for any $x(t_0) = x_0$ and $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\|x_0 - \bar{x}\| < \delta$ then $||x(t) - \bar{x}|| < \varepsilon$ for $t \geq t_0$.

Lemma 2.3 ([27]) *An equilibrium point $\bar{x}$ is Lyapunov stable if there exists a Lyapunov function over some neighborhood of $\bar{x}$.*

Definition 2.4 ([26]) A function $F : R^n \to R^n$ is called Lipschitz continuous with constant $L > 0$ on the set Ω if for every pair of points $\mathrm{x}, \mathrm{y} \in \Omega$

$$\|F(\mathrm{x}) - F(\mathrm{y})\| \leq L \|\mathrm{x} - \mathrm{y}\| .$$

Definition 2.5 ([26]) A function $F : R^n \to R^n$ is called pseudomonotone on the set Ω if for every pair of distinct points $\mathrm{x}, \mathrm{y} \in \Omega$

$$F(\mathrm{x})^T(\mathrm{y} - \mathrm{x}) \geq 0 \Rightarrow F(\mathrm{y})^T(\mathrm{y} - \mathrm{x}) \geq 0,$$

F is called strictly pseudomonotone on the set Ω if for every pair of distinct points $\mathrm{x}, \mathrm{y} \in \Omega$

$$F(\mathrm{x})^T(\mathrm{y} - \mathrm{x}) \geq 0 \Rightarrow F(\mathrm{y})^T(\mathrm{y} - \mathrm{x}) > 0,$$

F is called strongly pseudomonotone on the set Ω if there exists a constant $\gamma > 0$ such that for every pair of distinct points $\mathrm{x}, \mathrm{y} \in \Omega$

$$F(\mathrm{x})^T(\mathrm{y} - \mathrm{x}) \geq 0 \Rightarrow F(\mathrm{y})^T(\mathrm{y} - \mathrm{x}) > \gamma \|\mathrm{y} - \mathrm{x}\|^2 .$$

These different types of pseudomonotonicities are easily seen as listed in order from weak to strong. Moreover the pseudomonotonicity is a generalization of monotonicity. Clearly, monotonicity implies pseudomonotonicity, strict monotonicity implies strict pseudo monotonicity, and strong monotonicity implies strong pseudomonotonicity, but not vice versa ([30]- [15]).

Definition 2.6 ([15]) A function $f : R^n \to R$ is called pseudoconvex on the set Ω if for every pair of distinct points $\mathrm{x}, \mathrm{y} \in \Omega$

$$\nabla f(\mathrm{x})^T(\mathrm{y} - \mathrm{x}) \geq 0 \Rightarrow f(\mathrm{y}) \geq f(\mathrm{x}).$$

3. Double projection neural network model

In this paper, we are concerned with the following variational inequality problem: Find $\mathrm{x}^* \in \Omega$ such that

$$VI(F, \Omega) : \quad F(\mathrm{x}^*)^T(\mathrm{x} - \mathrm{x}^*) \geq 0, \quad \forall \mathrm{x} \in \Omega. \tag{2}$$

Problem (2) is called a variational inequality problem.

Remark 3.1 *For simplicity, from now on, we use x instead of $x(t)$.*

Lemma 3.1 ([28]) $\mathrm{x}^* \in \Omega$ *is a solution of problem (2) if and only if for any* $\alpha \geq 0$, x^* *satisfies the relation*

$$\mathrm{x}^* = P_\Omega(\mathrm{x}^* - \alpha F(\mathrm{x}^*)) \tag{3}$$

i.e. x^* *is a fixed point of the function* $P_\Omega(I - \alpha F) : \Omega \to \Omega$, *where* $I(\mathrm{x}) = \mathrm{x}$.

Lemma 3.1 implies that problems (2) and (3) have the same solution. In view of this relation, we propose the following recurrent neural network model, called double projection neural network for solving (2):

$$\frac{dx}{dt} = -x + P_\Omega(x - F(P_\Omega(x - F(x)))). \tag{4}$$

In this model, a technique of updating state variable is used.

Remark 3.2 ([26]) *In general computing the projection of a point onto a convex set* Ω *is itself a complex optimization problem. However, if* Ω *is a box set or sphere set, the calculation is straightforward. For example, if* $\Omega = \{\mathrm{x} \in R^n | l \leq \mathrm{x} \leq u, \ \forall i = 1, \ldots, n\}$, *then*

$$P_\Omega(\mathrm{x}_i) = \begin{cases} l_i, & \mathrm{x}_i < l_i \\ \mathrm{x}_i, & l_i \leq \mathrm{x}_i \leq u_i \\ u_i, & \mathrm{x}_i > u_i \end{cases}$$

If $\Omega = \{\mathrm{x} \in R^n | \, \|\mathrm{x} - c\| \leq r, \ r > 0\}$ *where* $c \in R^n$ *and* $r \in R$ *are two scalars. Then*

$$P_\Omega(\mathrm{x}) = \begin{cases} \mathrm{x}, & \|\mathrm{x} - c\| \leq r \\ c + r\frac{\mathrm{x}-c}{\|\mathrm{x}-c\|}, & \|\mathrm{x} - c\| > r \end{cases}$$

4. Stability and convergence analysis

In this section, we study some basic properties of the dynamical system (4) and prove its global convergence, Lyapunov stability and exponentially stability.

Theorem 4.1 *Assume that* $F(\mathrm{x})$ *is a locally Lipschitz continuous function in* R^n. *Then:*
(a) For every initial point $x(t_0) = x_0 \in R^n$ *there exists a unique solution* $x(t)$ *for the model (4).*
(b) When $x_0 \notin \Omega$, *the solution* $x(t)$ *will approach* Ω *exponentially*
(c) When $x_0 \in \Omega$, $x(t) \in \Omega$ *for* $t \geq t_0$.

Proof. (a): Since is locally Lipschitz continuous, By Lemma 2.2 we can see that $P_\Omega(x - F(P_\Omega(x - F(x)))) - x$ is also locally Lipschitz continuous. According to the local existence uniqueness theorem of ODEs [29], there exists a unique solution $x(t), \ t \in [t_0, \tau)$ for the double projection neural network model (4) with every initial point.
(b): When $x_0 \notin \Omega$, without loss of generality, we may assume that for every $t \geq t_0, \ x(t) \notin \Omega$. Let

$$\Phi(x) = \|x - P_\Omega(x)\|^2 .$$

Then, $\Phi(x(t))$ is a differentiable function with respect to t [22]. Hence by Lemma 2.2, we have

$$
\begin{aligned}
d\Phi(x(t))/dt &= (d\Phi(x(t))/dx)(dx/dt) = 2\left(x(t) - P_\Omega(x(t))\right)^T (dx/dt) \\
&= 2\left(x(t) - P_\Omega(x(t))\right)^T \left(-x(t) + P_\Omega(x(t) - F(P_\Omega(x(t) - F(x(t)))))\right) \\
&= \left(x(t) - P_\Omega(x(t))\right)^T \left(P_\Omega(x(t) - F(P_\Omega(x(t) - F(x(t))))) - P_\Omega(x(t))\right) \\
&\quad -2\left\|x(t) - P_\Omega(x(t))\right\|^2 \le -2\left\|x(t) - P_\Omega(x(t))\right\|^2 = -2\Phi(x(t)).
\end{aligned}
$$

Thus

$$\|x(t) - P_\Omega(x(t))\| \le \|x(t_0) - P_\Omega(x(t_0))\| \exp(t_0 - t).$$

Hence, any solution of model (4) will approach exponentially to Ω.

(c): We next show that when $x_0 \in \Omega$, $x(t) \in \Omega$for every $t \in [t_0, \tau)$. Otherwise, if there exist $t_1 > t_2$ such that $x(t) \in \Omega$ for $t \in [t_0, t_1]$ and $x(t) \notin \Omega$, for $t \in (t_1, t_2]$, then $\Phi(x(t_1)) = 0$ and $\Phi(x(t_2)) > 0$. By mean value theorem [15], we have

$$\Phi(x(t_2)) - \Phi(x(t_1)) = \left(d\Phi(x(\hat{t}))/dt\right)(t_2 - t_1) > 0,$$

for some $\hat{t} = \lambda t_1 + (1-\lambda)t_2, \lambda \in (0,1)$. Therefore, $d\Phi(x(\hat{t}))/dt > 0$. On the other hand, from (b) we know that

$$d\Phi(x(t))/dt = 2\left(x(t) - P_\Omega(x(t))\right)^T (dx/dt) \le 0$$

This is a contradiction. Thus, $x(t) \in \Omega$ for every $t \in [t_0, \tau)$.

Remark 4.1 *Note that τ is extended to infinity if $F(\mathrm{x})$ satisfies Lipschitz continuity condition on R^n.*

Theorem 4.2 *If $F(\mathrm{x})$ is pseudomonotone on Ω and Lipschitz continuous on R^n with constant $0 \le L \le 1$, then the double projection neural network (4) is stable in the sense of Lyapunov and is globally convergent to a solution of (2). In particular, if $\mathrm{V}I(F,\Omega)$ has a unique solution, the proposed neural network is globally asymptotically stable*

*Proof.*Let $x^* \in \Omega$ be a solution of the problem (2). Since any trajectory $x(t)$ will exponentially approach Ω, when $x_0 \notin \Omega$, and will remain in Ω forever (see Theorem 4.1), it suffices to show the stability of the proposed neural network model (4) with $x_0 \in \Omega$ .Then $x(t) \in \Omega$ for $t \ge t_0$. In the fourth inequality of Lemma 2.2, let $u = x - F(P_\Omega(x - F(x)))$ and $z = x^*$, then we have

$$
\begin{aligned}
\|x^* - P_\Omega(x - F(P_\Omega(x - F(x))))\|^2 &\le \|x - F(P_\Omega(x - F(x))) - x^*\|^2 \\
&\quad - \|x - F(P_\Omega(x - F(x))) - P_\Omega(x - F(P_\Omega(x - F(x))))\|^2 \\
&= \|x - x^*\|^2 - \|x - P_\Omega(x - F(P_\Omega(x - F(x))))\|^2 \\
&\quad +2\left\langle (x^* - P_\Omega(x - F(P_\Omega(x - F(x)))))^T, F(P_\Omega(x - F(x)))\right\rangle.
\end{aligned}
$$

Since $F(x)$ is pseudomonotone, we have

$$
\begin{aligned}
&\langle F(x^*)\ ,\ P_\Omega(x - F(x)) - x^*\rangle \ge 0 \Rightarrow \\
&\qquad\qquad \langle F(P_\Omega(x - F(x)))\ ,\ P_\Omega(x - F(x)) - x^*\rangle \ge 0,
\end{aligned}
$$

and consequently

$$\langle F(P_\Omega(x - F(x))), x^* - P_\Omega(x - F(P_\Omega(x - F(x))))\rangle =$$

$$\begin{aligned}&\langle F(P_\Omega(x-F(x))), x^*-P_\Omega(x-F(x))\rangle + \\ &\qquad \langle F(P_\Omega(x-F(x))), P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\rangle \\ &\leq \langle F(P_\Omega(x-F(x))), P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\rangle\end{aligned}$$

Hence

$$\begin{aligned}&\|x^*-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\quad\leq \|x-x^*\|^2-\|x-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\quad+2\langle F(P_\Omega(x-F(x))), P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\rangle \\ &\quad= \|x-x^*\|^2-\|x-P_\Omega(x-F(x))\|^2 \\ &\quad-\|P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\quad-2\langle x-P_\Omega(x-F(x)), P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\rangle \\ &\quad+2\langle F(P_\Omega(x-F(x))), P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\rangle \\ &\quad= \|x-x^*\|^2-\|x-P_\Omega(x-F(x))\|^2 \\ &\quad-\|P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\quad+2(P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x)))^T. \\ &\qquad\qquad (x-F(P_\Omega(x-F(x)))-P_\Omega(x-F(x)))\end{aligned}$$

By applying Lemma 2.2 and the Cauchy-Schwartz inequality, we observe that

$$\begin{aligned}&(P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x)))^T. \\ &\qquad\qquad (x-F(P_\Omega(x-F(x)))-P_\Omega(x-F(x))) \\ &= \langle P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x)), x-F(x)-P_\Omega(x-F(x))\rangle \\ &+\langle P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x)), F(x)-F(P_\Omega(x-F(x)))\rangle \\ &\leq \langle P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x)), F(x)-F(P_\Omega(x-F(x)))\rangle \\ &\leq \|P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x))\|\,\|F(x)-F(P_\Omega(x-F(x)))\|.\end{aligned}$$

Thus

$$\begin{aligned}&\|x^*-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \leq \|x-x^*\|^2-\|x-P_\Omega(x-F(x))\|^2 \\ &\quad-\|P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\quad+2L\|P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x))\|\,\|x-P_\Omega(x-F(x))\| \\ &\quad\leq \|x-x^*\|^2-\|x-P_\Omega(x-F(x))\|^2 \\ &\qquad-\|P_\Omega(x-F(x))-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \\ &\qquad+L^2\|x-P_\Omega(x-F(x))\|^2 \\ &\qquad+\|P_\Omega(x-F(P_\Omega(x-F(x))))-P_\Omega(x-F(x))\|^2.\end{aligned}$$

Therefore,

$$\|x^*-P_\Omega(x-F(P_\Omega(x-F(x))))\|^2 \leq \|x-x^*\|^2-(1-L^2)\|x-P_\Omega(x-F(x))\|^2$$

Since $0 \leq L \leq 1$, it follows that

$$\|x^*-P_\Omega(x-F(P_\Omega(x-F(x))))\| \leq \|x-x^*\|.$$

Consider the function

$$V(x(t)) = \frac{1}{2}\|x(t)-x^*\|^2, \quad \forall x(t)\in\Omega.$$

Then

$$\begin{aligned}\frac{dV}{dt} &= (x(t)-x^*)^T\frac{dx}{dt} = (x(t)-x^*)^T(P_\Omega(x(t)-F(P_\Omega(x(t)-F(x(t)))))-x(t))\\ &= (x(t)-x^*)^T(P_\Omega(x(t)-F(P_\Omega(x(t)-F(x(t)))))-x^*)-\|x(t)-x^*\|^2\\ &\le \|x(t)-x^*\|\,\|P_\Omega(x(t)-F(P_\Omega(x(t)-F(x(t)))))-x^*\|-\|x(t)-x^*\|^2\le 0\end{aligned}$$

Hence, the double projection neural network is stable in the sense of Lyapunov. Consider a sequence $\{x(t_n)\}_{n=1}^{\infty}\subseteq\Omega$ such that $\lim\limits_{n\to\infty} x(t_n)\to\infty$. According to the definition of $V(x(t))$, we have $\lim\limits_{n\to\infty} V(x(t_n))\to\infty$. Therefore, any level set of $V(x(t))$ is bounded. Thus, for any initial point $x(t_0)\in\Omega$, there exist a convergent subsequence $\{x(t_k)\}$ such that

$$\lim_{k\to\infty} x(t_k)=\hat{x}.$$

Define the following function

$$\hat{V}(x)=\|x-P_\Omega(x-F(x))\|^2 .$$

$$\begin{aligned}d\hat{V}(x(t))\Big/dt &= \left(d\hat{V}(x(t))\Big/dx\right)(dx/dt)\\ &= 2\left(x(t)-P_\Omega(x(t)-F(x(t)))\right)^T(dx/dt)\\ &= 2\left(x(t)-P_\Omega(x(t)-F(x(t)))\right)^T\cdot\\ &\qquad\qquad(-x(t)+P_\Omega(x(t)-F(P_\Omega(x(t)-F(x(t))))))\\ &\le -2\left\|x-P_\Omega(x(t)-F(x(t)))\right\|^2\\ &\quad+\left(x(t)-P_\Omega(x(t)-F(x(t)))\right)^T\cdot\\ &\qquad(P_\Omega(x(t)-F(P_\Omega(x(t)-F(x(t)))))-P_\Omega(x(t)-F(x(t))))\\ &\le -2\left\|x-P_\Omega(x(t))\right\|^2=-2\hat{V}(x(t))\end{aligned}$$

Thus

$$\|x(t)-P_\Omega(x(t)-F(x(t)))\|\le\|x(t_0)-P_\Omega(x(t_0)-F(x(t_0)))\|\exp(t_0-t).$$

For, convergent subsequence $\{x(t_k)\}$, we have

$$\|x(t_k)-P_\Omega(x(t_k)-F(x(t_k)))\|\le\|x(t_0)-P_\Omega(x(t_0)-F(x(t_0)))\|\exp(t_0-t_k).$$

Therefore, when $k\to\infty$ we obtain

$$\|\hat{x}-P_\Omega(\hat{x}-F(\hat{x}))\|=0.$$

Hence, $\hat{x}$ is a solution of $VI(F,\Omega)$.
Finally, define a new Lyapunov function

$$\bar{V}(x(t))=\frac{1}{2}\|x(t)-\hat{x}\|^2,\quad \forall x(t)\in\Omega.$$

It is easy to see that $\bar{V}(x(t))$ decreases along the trajectory of $VI(F,\Omega)$ and satisfies $\bar{V}(\hat{x})=0$. Therefore for any $\varepsilon>0$, there exists $q>0$ such that, for all $t>t_q$

$$\bar{V}(x(t))=\frac{1}{2}\|x(t)-\hat{x}\|^2\le\bar{V}(x(t_q))<\varepsilon$$

Thus, $\lim_{t\to\infty} x(t) = \hat{x}$. It follows that the double projection neural network (4) is globally convergent to a solution of $VI(F,\Omega)$. In particular, if $VI(F,\Omega)$ has a unique solution, the proposed neural network is globally asymptotically stable.

Theorem 4.3 *Let F(x) be strongly pseudomonotone on Ω with constant $\gamma > 0$ and Lipschitz continuous in R^n with constant $L > 0$, if $\gamma > 2L$ the double projection neural network (4) is globally exponentially stable and consequently, globally convergent to a solution of (2)*

Proof. Let $x^* \in \Omega$ be a solution of (2). Similar to the argument stated in the beginning of the proof of Theorem 4.2, it suffices to show the exponential stability of the proposed neural network model (4) with $x_0 \in \Omega$.Then, $x(t) \in \Omega$ for $t > t_0$. By the definition of $VI(F,\Omega)$, we have

$$\langle F(x^*)\ ,\ P_\Omega(x - F(P_\Omega(x - F(x)))) - x^*\rangle \geq 0$$

Since F(x) is strongly pseudomonotone on Ω, we obtain

$$\langle F(P_\Omega(x - F(P_\Omega(x - F(x)))))\ ,\ P_\Omega(x - F(P_\Omega(x - F(x)))) - x^*\rangle \geq \\ \gamma \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2 . \qquad (5)$$

In the first inequality of the Lemma 2.2, let $u = x - F(P_\Omega(x - F(x)))$ and $z = x^*$, then we have

$$(P_\Omega(x - F(P_\Omega(x - F(x)))) - x + F(P_\Omega(x - F(x))))^T. \\ (x^* - P_\Omega(x - F(P_\Omega(x - F(x))))) \geq 0$$

Adding this inequality with (5) implies

$$(P_\Omega(x - F(P_\Omega(x - F(x)))) - x + F(P_\Omega(x - F(x))) - F(P_\Omega(x - F(P_\Omega(x - F(x))))))^T \\ .(x^* - P_\Omega(x - F(P_\Omega(x - F(x))))) \geq \gamma \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2$$

Therefore,

$$(x^* - x + F(P_\Omega(x - F(x))) - F(P_\Omega(x - F(P_\Omega(x - F(x))))))^T. \\ (x^* - P_\Omega(x - F(P_\Omega(x - F(x))))) \geq (1+\gamma) \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2$$

By Cauchy-Schwarz inequality, Lipschitz continuity of the operator and the third inequality in Lemma 2.1, we have

$$(\|F(P_\Omega(x - F(x))) - F(x^*) + F(x^*) - F(P_\Omega(x - F(P_\Omega(x - F(x)))))\|) \\ +(\|x^* - x\|).\,\|\mathrm{x}^* - P_\Omega(x - F(P_\Omega(x - F(x))))\| \\ \geq\ (1+\gamma) \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2 ,$$

which implies

$$(\|x^* - x\| + \|F(P_\Omega(x - F(x))) - F(x^*)\| \\ + \|F(x^*) - F(P_\Omega(x - F(P_\Omega(x - F(x)))))\|).\,\|\mathrm{x}^* - P_\Omega(x - F(P_\Omega(x - F(x))))\| \\ \geq (1+\gamma) \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2$$

and thus

$$(\|x^* - x\| + L \left\| P_\Omega(x - F(x)) - x^* \right\| + L \left\| x^* - P_\Omega(x - F(P_\Omega(x - F(x)))) \right\|) \\ \|\mathrm{x}^* - P_\Omega(x - F(P_\Omega(x - F(x))))\| \geq (1+\gamma) \left\| P_\Omega(x - F(P_\Omega(x - F(x)))) - x^* \right\|^2$$

Now, if we substitute $P_\Omega(x - F(x))$ in (2) and if we let $u = x - F(x)$ and $z = x^*$, in the first inequality of the Lemma 2.2, by the same process, we obtain,

$$\|P_\Omega(x - F(x)) - x^*\| \ \leq \ \frac{1+L}{1+\gamma-L}\,\|x^* - x\|\,.$$

Therefore,

$$\left(\|x^* - x\| + \frac{L(1+L)}{1+\gamma-L}\,\|x^* - x\| + L\,\|x^* - P_\Omega(x - F(P_\Omega(x - F(x))))\|\right).$$

$$\|\mathrm{x}^* - P_\Omega(x - F(P_\Omega(x - F(x))))\| \ \geq (1+\gamma)\,\|P_\Omega(x - F(P_\Omega(x - F(x)))) - x^*\|^2$$

So,

$$\left((\frac{1+\gamma+L^2}{1+\gamma-L})\,\|x^* - x\| + L\,\|x^* - P_\Omega(x - F(P_\Omega(x - F(x))))\|\right).$$

$$\|\mathrm{x}^* - P_\Omega(x - F(P_\Omega(x - F(x))))\| \geq (1+\gamma)\,\|P_\Omega(x - F(P_\Omega(x - F(x)))) - x^*\|^2$$

By noting that $(1 + \gamma - L) > 0$, we have

$$(\frac{1+\gamma+L^2}{(1+\gamma-L)^2})\,\|x^* - x\| \geq \|x^* - P_\Omega(x - F(P_\Omega(x - F(x))))\|\,.$$

Consider the function

$$V(x(t)) = \frac{1}{2}\,\|x(t) - x^*\|^2\,, \quad \forall x(t) \in \Omega.$$

Then

$$\begin{aligned}
\frac{dv}{dt} &= (x(t) - x^*)^T\frac{dx}{dt} = (x(t) - x^*)^T(P_\Omega(x(t) - F(P_\Omega(x(t) - F(x(t))))) - x(t)) \\
&= (x(t) - x^*)^T(P_\Omega(x(t) - F(P_\Omega(x(t) - F(x(t))))) - x^*) - \|x(t) - x^*\|^2 \\
&\leq \|x(t) - x^*\|\,\|P_\Omega(x(t) - F(P_\Omega(x(t) - F(x(t))))) - x^*\| - \|x(t) - x^*\|^2 \\
&\leq (\frac{1+\gamma+L^2}{(1+\gamma-L)^2})\,\|x(t) - x^*\|^2 - \|x(t) - x^*\|^2 \leq -\beta\,\|x(t) - x^*\|^2\,,
\end{aligned}$$

where $\beta = \frac{(1+\gamma-L)^2-(1+\gamma+L^2)}{(1+\gamma-L)^2} > 0$. Hence,

$$\|x(t) - x^*\| \ \leq \ \|x_0 - x^*\|\ e^{-\beta(t-t_0)}, \quad \forall t > t_0.$$

The double projection neural network is globally exponentially stable.

Result 1. For Nonlinear Programming With General Constraints

Consider the following optimization problem:

$$Min\ f(\mathrm{x}) \ \ subject\ to\ \ g(\mathrm{x}) \leq 0, \ \ h(\mathrm{x}) = 0 \tag{6}$$

where $f(x)$ is continuously differentiable and pseudoconvex and $g : R^n \to R^m$,and $h : R^n \to R^r$ be continuously differentiable vector-valued functions. From now on we make the assumptions: g and h are convex and linear functions respectively. The following well-known result reveals the relationship between optimization problems and variational inequalities.

Lemma 4.1 ([28]) *Let S be a closed convex subset of R^n and $f : R^n \to R$ be differentiable and pseudoconvex on S. Then $x^* \in S$ satisfies relation $\nabla f(\mathrm{x}^*)^T(\mathrm{x} - \mathrm{x}^*) \geq 0, \quad \forall \mathrm{x} \in S$ if and only if x^* is a minimum of $f(x)$ in S.*

According to Lemma 4.1, the optimization problem (6) transfers to the following variational inequality.

$$\langle \nabla f(\mathrm{x}^*) \ , \ \mathrm{x} - \mathrm{x}^* \rangle \geq 0, \quad \forall \mathrm{x} \in \Omega,$$

where $\Omega = \{\mathrm{x} \in R^n | g(\mathrm{x}) \leq 0, h(\mathrm{x}) = 0\}$. thus this problem can be solve by proposed double projection neural network with $F(\mathrm{x}) = \nabla f(\mathrm{x})$.

Result 2. Nonlinear Complementarity Problems (NCP)

Consider the following nonlinear complementarity problem :
find a vector $x \in R^n$ such that

$$\mathrm{x}^T U(\mathrm{x}) = 0, \quad U(\mathrm{x}) \geq 0, \ \ \mathrm{x} \geq 0. \tag{7}$$

where $U(x)$ is a differentiable vector valued function from R^n into R^n.

Lemma 4.2 ([26]) *x^* is a solution for NCP if and only if x^* be a solution of $VI(U, R^n_+)$.*

By Lemma 4.2, nonlinear complementarity problem (7) can be solved by proposed double projection neural network (4) with $F(x) = U(x)$ and $\Omega = R^n_+$.

5. Comparison

In this section we will compare the proposed model (4) and the projection neural network model

$$\frac{dx}{dt} = \lambda \left\{ -x + P_\Omega(x - \alpha F(x)) \right\}. \tag{8}$$

That first introduced by Wang and his co-authors [21-27]. The model (8) is developed by Wang and Hu to solve pseudomonotone variational inequality problems [26]. For simplicity, we summarize the stability and convergence conditions of two model neural networks (4) and (8) in the table 1.

When $\nabla F(\mathrm{x})$ is asymmetric, the double projection neural network (4) needs only pseudomonotonicity of $F(x)$ whiles projection neural network (8) needs strongly pseudomonotonicity of $F(x)$. Hence, in this case, model (4) is more suitable in application. When $\nabla F(\mathrm{x})$ is symmetric, the projection neural network (8) is more applicable since, model (8) needs locally Lipschitz continuous condition rather than Lipschitz continuous condition in model (4)(See table 1). However double projection has more computational cost.

6. Numerical examples

In order to demonstrate the effectiveness and performance of the double projection neural network model (4) in solving pseudomonotone variational inequalities we give several illustrative examples. All the simulations conducted in Matlab 7.1. The 4th order of Runge-Kutta technique is used.

Example 1. Consider the following linear variational inequality problem

$$F(x) = Mx + q$$

TABLE 1. Comparison between two models (4) and (8) for stability and convergence conditions.

Neural Network Model	Projection neural network		double Projection neural network	
Stability and Convergence Condition	Symmetric of ∇F	Asymmetric of ∇F	Symmetric of ∇F	Asymmetric of ∇F
	Pseudo-monotonicity	Strongly Pseudo-monotonicity with constant γ	Pseudo-monotonicity	Pseudo-monotonicity
	Locally Lipschitz continuous	Lipschitz continuous on Ω with constant L and $\gamma > 2L$	Lipschitz continuous with constant $0 \leq L \leq 1$	Lipschitz continuous with constant $0 \leq L \leq 1$

where

$$M = \begin{pmatrix} 0.1 & 0.1 & -0.5 \\ 0.1 & 0.1 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ 1 \\ -0.5 \end{pmatrix}$$

and $\Omega = \{x \in R^3 | -10 \leq x_i \leq 10\}$. Obviously $F(x)$ is Lipschitz continuous with constant $L = 0.5$ and $\nabla F(x)$ is asymmetric.$F(x)$ is monotone and consequently pseudomonotone in R^n. It is not strongly pseudomonotone. We solve this problem with neural network models (4) and (8). All simulation results show that the double projection neural network (4) is stable and globally convergence to the solution of this problem, whiles model (8) is not. In Figure 1. and Figure 2. we display the output trajectories for model (8) with $\lambda = \alpha = 1$ and model (4)respectively, using the initial point $x_0 = (0.5, -0.48, -2.5)^T$. Figure 3. and Figure 4. transient behavior of neural network models (8) and (4) respectively, with initial point $x_0 = (-10, 10, -10)^T$ for $\lambda = \alpha = 1$.

Example 2. Let us consider the nonlinear variational inequality, $VI(F, \Omega)$, with

$$F(x) = \begin{pmatrix} (x_1^2 + (x_2 - 1)^2)(1 + x_2) \\ -x_1^3 - x_1(x_2 - 1)^2 \end{pmatrix}$$

and $\Omega = \{x \in R^2 | -10 \leq x_i \leq 10, i = 1, 2\}$. This problem has unique solution $x^* = (0, 1)^T$. It is easy to see that $F(x)$ is not a monotone map on Ω. However it is not easy to verify that it is pseudomonotone on Ω. In general, it is very difficult task to check the pseudomonotonicity of a mapping in practice. In such occasions, researchers use the Monte Carlo approach [26]. By this approach we are confident that $F(x)$ is pseudomonotone on Ω (one million point is tested). Clearly $\nabla F(x)$is asymmetric. We solve this problem with neural network models (4) and (8). All simulation results show that the double projection neural network (4) is stable and globally convergence to the solution of this problem, whiles model (8) is not. For

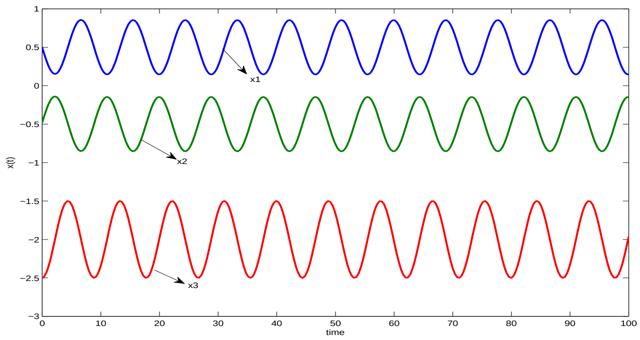

FIGURE 1. Transient behavior of the neural network models (8) for the Example 1 with $x_0 = (0.5, -0.48, -2.5)$

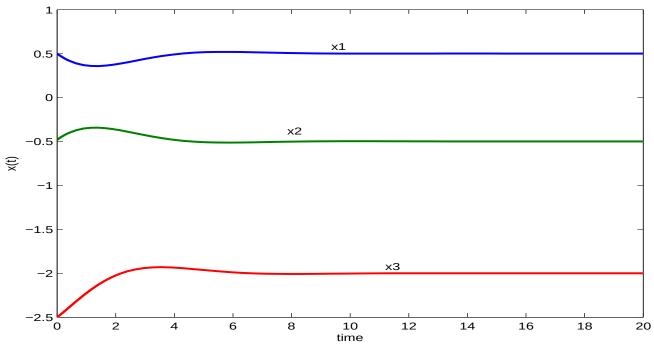

FIGURE 2. Transient behavior of the neural network models (4) for the Example 1 with $x_0 = (0.5, -0.48, -2.5)$

instance Figure 5. and Figure 6. display the output trajectories of (a) model (8) with and (b) model (4) using initial point $x_0 = (-1, 2)^T$.

Example 3. ([26]) Consider the following two-dimensional VI where

$$F(x) = \begin{pmatrix} 0.5x_1x_2 - 2x_2 - 10^7 \\ -4x_1 + 0.1x_2^2 - 10^7 \end{pmatrix}$$

and $\Omega = \left\{x \in R^2 | (x_1 - 2)^2 + (x_2 - 2)^2 \leq 1\right\}$. It is easy to see that $F(x)$ is not a monotone map on Ω. $F(x)$ is strongly pseudomonotone on Ω with constant $\gamma = 11$, and it is Lipschitz continuous with constant $L = 5$ [26]. Thus, the condition in theorem 4.3 is satisfied. Then we use the neural network model (4) to solve aforementioned example. Also, we solve this problem with model (8) for comparison. Figure 7. and Figure 8. show the trajectories of model (4) and model (8)respectively, with the initial

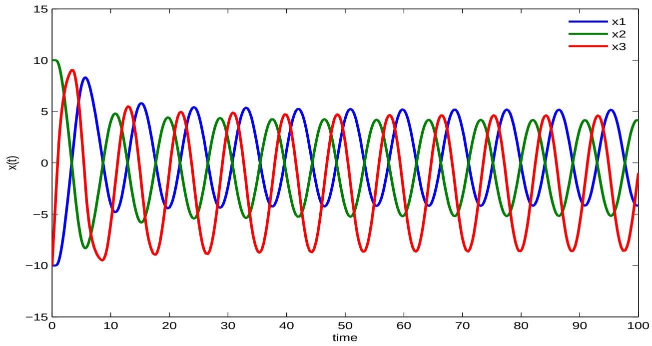

FIGURE 3. Transient behavior of the neural network models (8) for the Example 1 with $x_0 = (-10, 10, -10)$

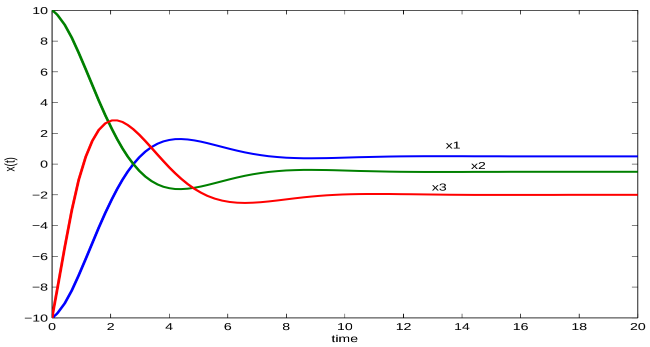

FIGURE 4. Transient behavior of the neural network models (4) for the Example 1 with $x_0 = (-10, 10, -10)$

point $\mathrm{x}_0 = (0, 4)^T$ Figure 9. shows the trajectories of the model (4) with six different initial points $p_1 = (0, 0)^T, p_2 = (4, 0)^T, p_3 = (4, 4)^T, p_4 = (0, 4)^T, p_5 = (1, 2)^T$ and $p_6 = (2, 1)^T$ among which the last two points are located in Ω and the others are not. In this problem both models give the same attitude in converging to the correct solution.

Example 4. ([26]) We now use the double projection neural network to solve a pseudoconvex optimization problem. Consider the following fractional programming problem

$$\min f(x) = \frac{x^T Qx + a^T x + a_0 x}{b^T x + b_0},$$
$$\textit{subject to} \;\; x \in X = \left\{ x \in R^n | b^T x + b_0 > 0 \right\}$$

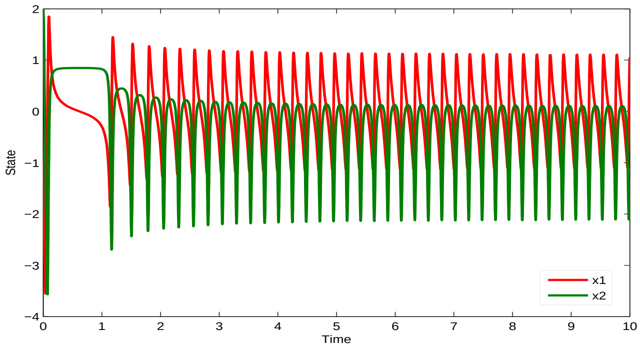

FIGURE 5. Transient behavior of the neural network model (8) for the Example 2, using the initial point $x_0 = (-1, 2)^T$ with $\lambda = \alpha = 1$

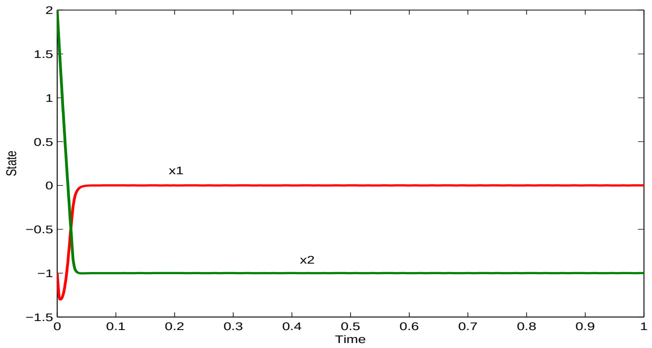

FIGURE 6. Transient behavior of the neural network model (4) for the Example 2, using the initial point $x_0 = (-1, 2)^T$ with $\lambda = \alpha = 1$

where

$$Q = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 3 & 0 & 5 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

It is easy to verify that Q is symmetric and positive definite in R^4 and consequently f is pseudoconvex on $X = \{x \in R^n | b^T x + b_0 > 0\}$. We minimize f over $\Omega = \{x \in R^4 | -10 \leq x_i \leq 10, i = 1, \dots, 4\} \subset X$ by using neural network model (4) with $F(x) = \nabla f(x)$. This problem has a unique solution $x^* = (1, 1, 1, 1)^T \in \Omega$. All simulations show that double projection neural network model (4) is globally convergent to the unique optimum solution. For instance, Figure 10. shows the trajectories of model (4) with five random initial points.

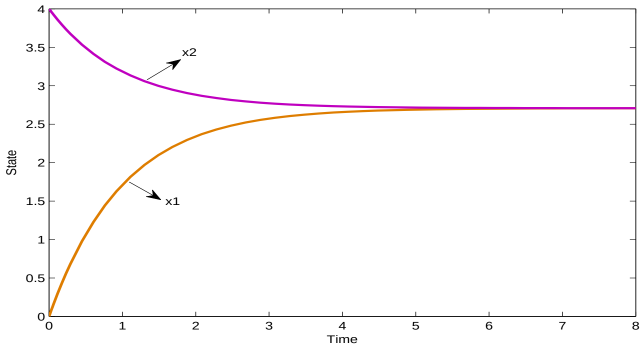

FIGURE 7. Transient behavior of the neural network model (4) for the Example 3, using the initial point $x_0 = (0, 4)^T$ with $\lambda = \alpha = 1$

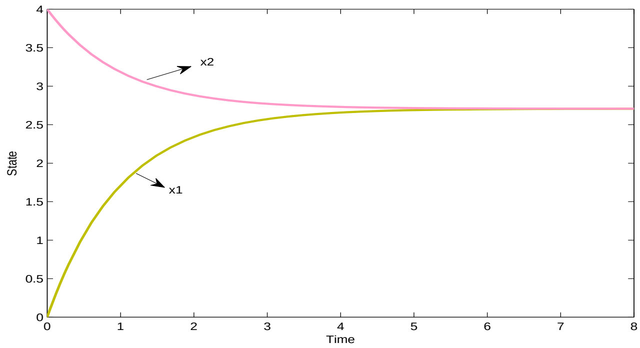

FIGURE 8. Transient behavior of the neural network model (8) for the Example 3, using the initial point $x_0 = (0, 4)^T$ with $\lambda = \alpha = 1$

Example 5. ([30]) Let us consider the following nonlinear complementarity problem with

$$U(x) = \begin{pmatrix} x_1 + x_2x_3x_4x_5/50 \\ x_2 + x_1x_3x_4x_5/50 - 3 \\ x_3 + x_1x_2x_4x_5/50 - 1 \\ x_4 + x_1x_2x_3x_5/50 + 0.5 \\ x_5 + x_1x_2x_3x_4/50 \end{pmatrix}$$

This problem has a unique solution $x^* = (0, 3, 1, 0, 0)^T$. All simulations show that double projection neural network model (4) is globally convergent to the unique optimum solution. For instance, Figure 11. shows the trajectories of model (4) with the initial point$x_0 = (1, -1, 2, -2, 5)$.

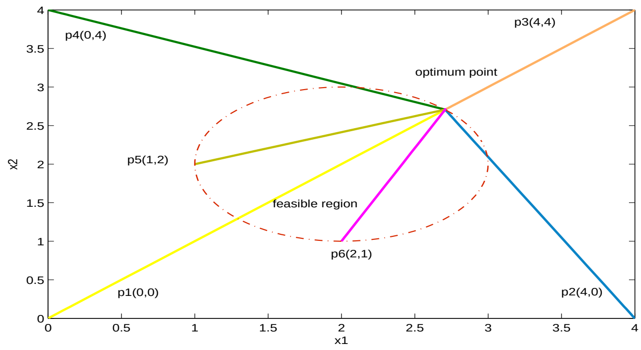

FIGURE 9. Transient behavior of the double projection neural network model (4) with six different initial points in Example 3

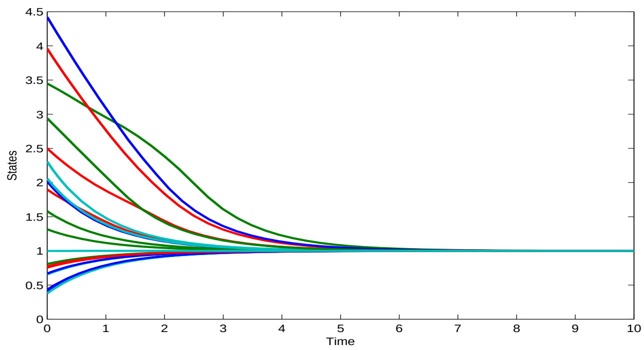

FIGURE 10. Transient behavior of the double projection neural network model (4) with five random initial points in Example 4.

7. Conclusions

In this paper, a novel double projection recurrent neural network model for solving pseudomonotone variational inequalities and related problems is proposed. In the case of pseudomonotoncity condition, we proved that the proposed neural network is globally convergent, stable in the sense of Lyapunov and in the case of strongly pseudomonotonicity condition and other conditions; we proved that the double projection neural network is globally exponentially stable. Moreover, the simulation results have demonstrated the global convergence behavior and characteristics of the proposed neural network for solving different types of variational inequality problems.

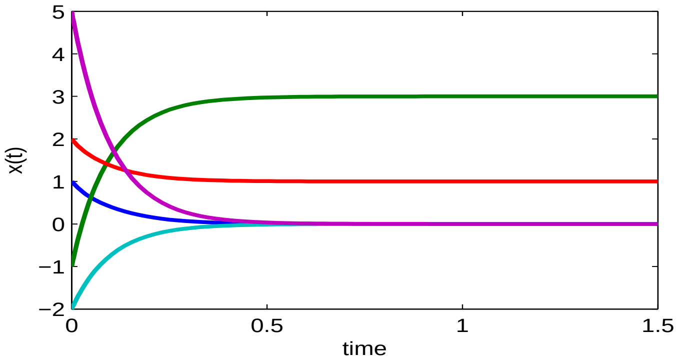

FIGURE 11. Transient behavior of the double projection neural network model (4) with the initial point $x_0 = (1, -1, 2, -2, 5)$ in Example 5.

REFERENCES

[1] A. Nagurney, *Network Economics: A Variational Inequality Approach*, Kluwer Acad. Publ., Boston, 1999.

[2] J.L. Lions, G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math., **20**(1967), 493-512.

[3] C. Baiocchi and A. Capelo, *Variational and Quasi-Variational Inequalities*, J. Wiley and Sons, New York, 1984.

[4] D.P. Bertsekas and J. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods,* Prentice Hall, Englewood Cliffs, 1989.

[5] R.W. Cottle, F. Giannessi, J.L. Lions, *Variational Inequalities and Complementarity Problems: Theory and Applications*, J. Wiley and Sons, New York, 1980.

[6] D. Gabay, *Applications of the method of multipliers to variational inequalities*, in: Augmented Lagrangian Methods: Applications to the Solution of Boundary Value Problems, (M. Fortin and R. Glowinski-Eds.), North Holland, Amsterdam, 1983, 299-331.

[7] F. Giannessi and A. Maugeri, *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York, 1995.

[8] R. Glowinski, J.L. Lions, R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North Holland, Amsterdam, 1981.

[9] B. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim., **35**(1997), 69-76.

[10] M.A. Noor, *A modified projection method for monotone variational inequalities*, Appl. Math. Lett., **12**(1999), no. 5, 83-87.

[11] M.A. Noor, *Some recent advances in variational inequalities, Part I: Basic concepts*, New Zealand J. Math., **26**(1997), 53-80.

[12] M.A. Noor,*Some recent advances in variational inequalities, Part II: Other concepts*, New Zealand J. Math., **26**(1997), 229-255.

[13] M.A. Noor, *A modified extragradient method for general monotone variational inequalities*, Computer Math. Appl., **38**(1999), no. 3, 19-24.

[14] M.A. Noor, *Some algorithms for general monotone mixed variational inequalities*, Math. Comput. Modelling, **19**(1999), no. 7, 1-9.

[15] M. Sibony, *Methodes iteratives pour les equations et inequations aux derivees partielles nonlineaires de type monotone*, Calcolo, **7**(1970), 65-183.

[16] M.V. Solodov and P. Tseng, *Modified projection-type methods for monotone variational inequalities*, SIAM J. Control Optim., **34**(1996), 1814-1830.

[17] G. Stampacchia, *Formes bilineaires coercivities sur les ensembles convexes*, C.R. Acad. Sci. Paris, **258**(1964), 4413-4416.

[18] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim., **38**(2000), 431-446.

[19] T. Yoshikawa, *Foundations of Robotics: Analysis and Control*, MIT Press, Cambridge, 1990.

[20] B. Kosko,*Neural Networks for Signal Processing* Englewood Cliffs, Prentice-Hall, 1992.

[21] Y. Xia and J. Wang, *A recurrent neural network for solving linear projection equations*, Neural Netw., **13**(2000), no. 3, 337-350.

[22] Y. Xia and J. Wang, *On the stability of globally projected dynamical systems*, J. Optim. Theory Appl., **106**(2000), no. 1, 129-150.

[23] X. Liang and J. Si, *Global exponential stability of neural networks with globally Lipschitz continuous activations and its application tolinear variational inequality problem*, IEEE Trans. Neural Netw., **12**(2001), no.2, 349-359.

[24] Y. Xia, H. Leung, J. Wang, *A projection neural network and its application to constrained optimization problem*, IEEE Trans. Circuits Syst. I, Reg. Papers, **49**(2002), no. 4, 447-458.

[25] Y. Xia, *Further results on global convergence and stability of globally projected dynamical systems*, J. Optim. Theory Appl., **122**(2004), no. 3, 627-649.

[26] X.Hu and J. Wang, *Solving pseudomonotone variational inequalities and pseudo-convex optimization problems using the projection neural network*, IEEE Trans. Neural Netw., **17**(2006), no. 6, 1487-1499.

[27] M. Yashtini, A. Malek, *Solving complementarity and variational inequalities problems using neural networks*, Appl. Math. Comput., **190**(2007), 216-230.

[28] D. Kinderlehrer and G. Stampcchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press New York, 1980.

[29] R.K. Miller, A.N. Michel, *Ordinary Differential Equations*, Academic Press New York, 1982.

[30] Y. Xia, H. Leung, J. Wang, *Projection neural network and its application to constrained optimization problems*, IEEE Trans. Circuits Syst. I, Reg. Papers, **49**(2002), no. 4, 447-458.

[31] M.S. Bazaraa, C.M. Shetty, *Nonlinear Programming Theory and Algorithms*, John Wiley and Sons, New York, 1990.

Received: August 1, 2009; Accepted: November 5, 2009.