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QUASI-FIXED POLYNOMIAL FOR VECTOR-VALUED POLYNOMIAL FUNCTIONS ON $\mathbb{R}^n \times \mathbb{R}$

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Abstract. Let $F : \mathbb{R}^n \times \mathbb{R} \longrightarrow \mathbb{R}^k$ be a vector-valued polynomial function:

$$F(\overline{x}, y) = \big(F_1, F_2, ..., F_k\big)(\overline{x}, y), \qquad \overline{x} \in \mathbb{R}^n, \quad y \in \mathbb{R}.$$

Each component F_i of F is a real-valued polynomial function, the degree of y of F_i is $\deg_y F_i = s_i$, and is represented by:

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} f_{i,j}(\overline{x}) y^j, \quad i = 1, 2, \ldots, k,$$

where $f_{i,j}(\overline{x}) \in \mathbb{R}[\overline{x}]$.

In this paper, for each F_i, we give an irreducible polynomial $p_i^{m_i}(\overline{x})$ of m_i-power and consider a real-valued quasi-fixed point problem as the form:

$$F_i(\overline{x}, y) = a_i p_i^{m_i}(\overline{x}), \quad i = 1, 2, \cdots, k.$$

We aim to find a polynomial function $y = y(\overline{x})$, $\overline{x} \in \mathbb{R}^n$ to satisfy the following vector-valued polynomial equation:

$$(\circledast) \quad F(\overline{x}, y(\overline{x})) = \big(a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x}), \cdots, a_n p_k^{m_k}(\overline{x})\big),$$

where $(a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$ is a constant vector depending on the solution $y(\overline{x})$. We will investigate the solution sets of $(\circledast)$ and containing either (i) of finitely many or (ii) of infinitely many quasi-fixed (point) solutions. In case of (i), the number of solutions do not exceed

$$\max_{1 \le i \le k} \{s_i + 2\}.$$

While the case (ii), all solutions are represented as the form

$$\{-f_{s_i - 1}(\overline{x})/s_i f_{s_i}(\overline{x}) + \lambda p^t(\overline{x}) : \text{ for all } \lambda \in \mathbb{R}\}$$

where $t \le m_i/s_i$ for any i, $1 \le i \le k$.
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1. Introduction

This paper is a continuous work of Lai and Chen [3] from real-valued polynomial function extends to vector one. Part of the results are anounced in Chen and Lai [4]. The concept and sprit are based on Lenstra [1] and Tung [2]. In [1], Lenstra considered a real-valued polynomial function $F(x, y) : F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ to intend to find a polynomial function $y = y(x)$ so that it turn to reduce to a fixed point for $F(x, y)$ as the form:

$$F(x, y(x)) = x. \tag{1.1}$$

Recently, Tung [2] extended this fixed point concept to search a polynomial function $y = y(x)$ such that

$$F(x, y) = cx^m \quad \text{for given } m \in \mathbb{N}, \tag{1.2}$$

where $\mathbb{N}$ is the set of all natural numbers, and c is a constant depending on the solution $y = y(x)$. Based on the concept of (1.1) and (1.2), Lai and Chen [3] investigated the real-valued polynomial function $y = y(\overline{x})$ with an irreducible polynomial $p(\overline{x})$ to satisfy the equation:

$$F(\overline{x}, y) = cp^m(\overline{x}) \tag{1.3}$$

where $x \in \mathbb{R}$ in (1.1) is replaced by $\overline{x} \in \mathbb{R}^n$ and x^m in (1.2) is replaced by a polynomial $p^m(\overline{x})$. In the present paper, we consider a quasi-fixed point problem for the vector-valued polynomial function (c.f. Lai and Chen [3]) as the form:

$$F(\overline{x}, y) = (F_1, F_2, \cdots, F_k)(\overline{x}, y) \quad \text{and}$$
$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} f_{i,j}(\overline{x}) y^j, \quad 1 \le i \le k,$$

where the degree of y in F_i is denoted by

$$\deg_y F_i = s_i \ge 1, \quad 1 \le i \le k.$$

Thus we consider the vector-valued quasi-fixed point problem as the form:

$$F(\overline{x}, y) = (a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x}), \ldots, a_k p_k^{m_k}(\overline{x})), \tag{1.4}$$

where $p_i(\overline{x})$, $1 \le i \le k$ are given irreducible polynomials.

The main purpose of this paper is to establish some conditions so that the equation (1.4) is solvable. Moreover, as the equation (1.4) is solvable, we will establish the solution set S of (1.4). It may be either (i) finitely many solutions in which the number of solutions is bounded, and is actually not exceed the number depending on the degrees of y in each component F_i:

$$\ell = \max_{1 \le i \le k} \{s_i + 2\},$$

or (ii) if the number of all solutions is infinitely many, then the solutions must be of the types as the form:

$$\{-f_{s_i-1}(\overline{x})/s_i f_{s_i}(\overline{x}) + \lambda p^t(\overline{x}) : \ \lambda \in \mathbb{R}\},$$

where $t \le m_i/s_i$ for any i, $1 \le i \le k$.

2. Preliminary and Some Lemmas

For convenience, let S be the solution set of equation (1.4), and S_i the solution set of i-th component:

$$F_i(\overline{x}, y(\overline{x})) = a_i p_i^{m_i}(\overline{x}),$$

where the irreducible polynomials $p_i(\overline{x})$ and m_i are given, $1 \leq i \leq k$.

Let $S = \{y(\overline{x}) : y(\overline{x})$ satisfies equation (1.4)$\}$ and $S_i = \{y(\overline{x}) : F_i(\overline{x}, y(\overline{x})) = a_i p^{m_i}(\overline{x})\}$. Then it is clear that $S = \bigcap_{1 \leq i \leq k} S_i$.

In this paper, the cardinal number $\mid S \mid$ may be either infinitely many or finitely many, or not solvable. If $\mid S \mid$ is infinite, then all solutions in S are represented by a fixed form. If $\mid S \mid$ is finite, we would find the upper bound of cardinal number $\mid S \mid$. For convenience, we explain some interesting properties of quasi-fixed point solutions as the following lemmas. At first we describe the relationship of any two quasi-fixed solutions corresponding to distinct quasi-fixed vectors.

Lemma 2.1. *The expression of the difference for two quasi-fixed solutions corresponding to different quasi-fixed vectors is a power of $p_i(\overline{x})$ up to a constant for some $i \in \{1, 2, \ldots, k\}$.*

Proof. Let $y_1(\overline{x})$ and $y_2(\overline{x})$ be two quasi-fixed solutions of $F(\overline{x}, y)$ with two distinct quasi-fixed vectors $(a_1, a_2, \ldots, a_k)$ and $(b_1, b_2, \ldots, b_k)$ in $\mathbb{R}^k$, respectively. Thus, there exist $a_i \neq b_i$ for some i, $1 \leq i \leq k$. Without lose of generality, we may assume $a_1 \neq b_1$, and consider the first component as

$$F_1(\overline{x}, y_1(\overline{x}) = a_1 p_1^{m_1}(\overline{x})$$
$$F_1(\overline{x}, y_2(\overline{x})) = b_1 p_1^{m_1}(\overline{x}).$$

Subtracting the above two equations, it yields

$$F_1(\overline{x}, y_1(\overline{x})) - F_1(\overline{x}, y_2(\overline{x})) = (a_1 - b_1) p_1^{m_1}(\overline{x}). \tag{2.1}$$

The left hand side of the above equality

$$\begin{aligned}
&= f_{1,s_1}(\overline{x})[y_1^{s_1}(\overline{x}) - y_2^{s_1}(\overline{x})] + f_{1,s_1-1}(\overline{x})[y_1^{s_1-1}(\overline{x}) - y_2^{s_1-1}(\overline{x})] + \cdots + f_{1,1}(\overline{x})[y_1(\overline{x}) - y_2(\overline{x})] \\
&= [y_1(\overline{x}) - y_2(\overline{x})][f_{1,s_1}(\overline{x}) G_{s_1}(y_1(\overline{x}), y_2(\overline{x}))] + \cdots + [y_1(\overline{x}) - y_2(\overline{x})][f_{1,1}(\overline{x})] \\
&= [y_1(\overline{x}) - y_2(\overline{x})][f_{1,s_1}(\overline{x}) G_{s_1}(y_1(\overline{x}), y_2(\overline{x})) + f_{1,s_1-1}(\overline{x}) G_{s_1-1}(y_1(\overline{x}), y_2(\overline{x})) + \cdots + f_{1,1}(\overline{x})] \\
&= [y_1(\overline{x}) - y_2(\overline{x})] Q(\overline{x}, y_1(\overline{x}), y_2(\overline{x})),
\end{aligned} \tag{2.2}$$

where $G_j(y_1(\overline{x}), y_2(\overline{x})) = y_1^{j-1}(\overline{x}) + y_1^{j-2}(\overline{x}) y_2(\overline{x}) + \cdots + y_2^{j-1}(\overline{x})$ for $j = 1, 2, \ldots, s_1$ and $Q(\overline{x}, y_1(\overline{x}), y_2(\overline{x})) = f_{1,s_1}(\overline{x}) G_{s_1}(y_1(\overline{x}), y_2(\overline{x})) + f_{1,s_1-1}(\overline{x}) G_{s_1-1}(y_1(\overline{x}), y_2(\overline{x})) + \cdots + f_{1,1}(\overline{x})$.

By (2.1) and (2.2), we see that $y_1(\overline{x}) - y_2(\overline{x})$ is divisible the term $(a_1 - b_1) p^{m_1}(\overline{x})$. Since $a_1 \neq b_1$, we get

$$y_1(\overline{x}) - y_2(\overline{x}) = c p_1^t(\overline{x}) \quad \text{for some } c \in \mathbb{R} \text{ and } t \leq m_1 \text{ in } \mathbb{N}.$$

Note that $p_1^{m_1}(\overline{x})$ can be $p_i^{m_i}(\overline{x})$, and each $p_i(\overline{x})$, $i = 2, \ldots, k$, can be replaced by $p_1(\overline{x})$. So each component in equation (1.4) reduced to irreducible polynomial $p_i(\overline{x})$

are of the same type up to a constant vector. □ Let a real-valued polynomial function

$$G(\overline{x}, y) = \sum_{i=0}^{s} g_i(\overline{x}) y^i,$$

be regarded as a component in the vector-valued quasi-fixed problem. If there exists a polynomial function $y(\overline{x}) \in \mathbb{R}[\overline{x}]$ with a constant $a \in \mathbb{R}$ such that

$$G(\overline{x}, y(\overline{x})) = a p^m(\overline{x}), \tag{2.3}$$

then $y(\overline{x})$ is a quasi-fixed solution corresponding to a quasi-fixed value a. By Lai and Chen [3 Theorem 3.2], we have

Theorem 2.2. *The following three conditions are equivalent:*

(i) *The equality* (2.3) *has at least* $s+3$ *quasi-fixed solutions,*

(ii) *the polynomial function* $G(\overline{x}, y)$ *is expressed by the series*

$$G(\overline{x}, y) = \sum_{i=0}^{s} c_i \Big(y - y(\overline{x})\Big)^i (p(\overline{x}))^{m-it}, \quad \textit{for some } y(\overline{x}) \in \mathbb{R}[\overline{x}],\ t \in \mathbb{N}$$

and $c_i \in \mathbb{R}$, $i = 0, 1, \cdots, s$.

(iii) *the polynomial function* $G(\overline{x}, y)$ *in* (2.3) *has infinitely many quasi-fixed solutions.*

Remark. *It is remarkable that from Theorem 2.2 (ii), any quasi-fixed solution* $h(\overline{x})$ *in equation* (2.3) *is represented by:*

$$h(\overline{x}) = y(\overline{x}) + d p^t(\overline{x}), \qquad \textit{for some } d \in \mathbb{R} \textit{ and } t \in \mathbb{N}.$$

3. Characterization for vector polynomial function

Let a vector-valued polynomial function be

$$F(\overline{x}, y) = (F_1(\overline{x}, y), F_2(\overline{x}, y), \cdots, F_k(\overline{x}, y))$$

with $F_i(\overline{x}, y) = \sum_{j=0}^{s_i} f_{i,j}(\overline{x}) y^j$, $\deg_y F_i = s_i \geq 1$ for $i = 1, 2, \ldots, k$.

Lemma 3.1. *If some component* $F_i(\overline{x}, y)$ *of* $F(\overline{x}, y)$ *is represented as the form:*

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y_i(\overline{x})\Big)^j (p_i(\overline{x}))^{m_i - j t_i}$$

for a real-valued polynomial functions $y_i(\overline{x})$ *with a irreducible polynomial* $p_i(\overline{x})$ *and* $m_i, t_i \in \mathbb{N}$ *for some* $i = 1, 2, \ldots, k$*, then a quasi-fixed solution* $y(\overline{x})$ *of problem* (1.4) *is represented by*

$$y(\overline{x}) = y_i(\overline{x}) + d_i (p_i(\overline{x}))^{t_i} \qquad \textit{for some } d_i \in \mathbb{R}.$$

Proof. Let $y(\overline{x})$ be a quasi-fixed solution of $F(\overline{x}, y)$ in (1.4). Then $y(\overline{x})$ is a quasi-fixed solution of $F_i(\overline{x}, y)$, for each $i = 1, 2, \cdots, k$. By assumption, there exists an integer $i \in \{1, 2, \ldots, k\}$ such that

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y_i(\overline{x})\Big)^j (p_i(\overline{x}))^{m_i - j t_i}.$$

It follows from the Remark in last section, we get $y(\overline{x}) = y_i(\overline{x}) + d_i(p_i(\overline{x}))^{t_i}$ for some $d_i \in \mathbb{R}$. □

Note that form of $F_i(\overline{x}, y)$ in Lemma 3.1 is not seldom. For example, it will be happened in Theorem 2.2.

By definition of the solution sets S and S_i, we have $S = \bigcap_{1\leq i\leq k} S_i$ provided the cardinal number $| S | \geq \aleph_0$, thus $S = S_i$ for any $i = 1, 2, \ldots, k$. If all quasi-fixed solutions have infinitely many, we will show this result as follows.

Theorem 3.2. *Suppose that the cardinal number* $| S | \geq \aleph_0$, *then* $S = S_i$ *for any* $i = 1, 2, \ldots, k$.

Proof. We claim that $S_i \subseteq S_j$ for any $i \neq j \in \{1, 2, \ldots, k\}$.

Since $S = \bigcap_{1\leq i\leq k} S_i$ and the cardinal number $| S |$ is infinite, then the cardinal number $| S_i |$ is also infinite for any $i = 1, 2, \ldots, k$. By Theorem 2.2, we know that the polynomial function $F(\overline{x}, y)$ can be expanded to the power series of the form:

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij}\Big(y - y_i(\overline{x})\Big)^j (p_i(\overline{x}))^{m_i - jt_i},$$

for some $y_i(\overline{x}) \in \mathbb{R}[\overline{x}]$ and $c_{ij} \in \mathbb{R}$, $j = 0, 1, \cdots, s_i$.
Let $y(\overline{x})$ and $h(\overline{x})$ be two distinct solutions in S. Then by Lemma 3.1, $y(\overline{x})$ and $h(\overline{x})$ can be represented by

$$y(\overline{x}) = y_i(\overline{x}) + d_y^i p_i^{t_i}(\overline{x}) \quad \text{and} \quad h(\overline{x}) = y_i(\overline{x}) + d_h^i p_i^{t_i}(\overline{x})$$

for some $d_y^i,\ d_h^i \in \mathbb{R}$, $i = 1, 2, \cdots, k$. Hence

$$\begin{aligned} y(\overline{x}) - h(\overline{x}) &= (d_y^i - d_h^i) p_i^{t_i}(\overline{x}) \\ &= \beta_i p_i^{t_i}(\overline{x}) \quad \text{where } \beta_i = d_y^i - d_h^i \in \mathbb{R}. \end{aligned}$$

Since $p_i^{t_i}(\overline{x})$ for $i = 1, 2, \ldots, k$ have no common factor, it follows that

$$\beta_1 p_1^{t_1}(\overline{x}) = \beta_2 p_2^{t_2}(\overline{x}) = \cdots = \beta_k p_k^{t_k}(\overline{x}).$$

Since $p_i(\overline{x})$ is irreducible for each $i = 1, 2, \ldots, k$, the above identities reduce

$$p_1(\overline{x}) = p_2(\overline{x}) = \cdots = p_k(\overline{x}) = p(\overline{x}), \text{ say} \quad \text{and} \quad t_1 = t_2 = \cdots = t_k = t, \text{ say}. \tag{3.1}$$

If $i \neq j \in \{1, 2, \ldots, k\}$ and $y(\overline{x}) \in S$ implies that $y(\overline{x}) \in S_i \bigcap S_j$, by Lemma 3.1,

$$y(\overline{x}) = y_i(\overline{x}) + d_y^i p_i^{t_i}(\overline{x}) \quad \text{and} \quad y(\overline{x}) = y_j(\overline{x}) + d_y^j p_j^{t_j}(\overline{x})$$

for some $d_y^i,\ d_y^j \in \mathbb{R}$. Hence

$$\begin{aligned} y_i(\overline{x}) - y_j(\overline{x}) &= (y(\overline{x}) - d_y^i p_i^{t_i}(\overline{x})) - (y(\overline{x}) - d_y^j p_j^{t_j}(\overline{x})) \\ &= (d_y^j p_j^{t_j}(\overline{x}) - d_y^i p_i^{t_i}(\overline{x})) \\ \text{by (3.1)} \quad &= (d_y^j - d_y^i) p^t(\overline{x}) \\ &= d_{ij} p^t(\overline{x}) \quad \text{where } d_{ij} = d_y^j - d_y^i. \end{aligned} \tag{3.2}$$

If any $i \neq j$ in $\{1, 2, \ldots, k\}$, and any $h_i(\overline{x}) \in S_i$, by Theorem 2.2, we get

$$\begin{aligned} h_i(\overline{x}) &= y_i(\overline{x}) + d_y^i p_i^{t_i}(\overline{x}) \\ \text{by (3.1)} \quad &= y_i(\overline{x}) + d_y^i p^t(\overline{x}) \\ \text{by (3.2)} \quad &= (y_j(\overline{x}) + d_{ij} p^t(\overline{x})) + d_y^i p^t(\overline{x}) \\ \text{by (3.1)} \quad &= y_j(\overline{x}) + (d_{ij} + d_y^i) p_j^t(\overline{x}) \in S_j \ \text{(by Remark)}. \end{aligned}$$

This proves $S_i \subseteq S_j$. It is the same for $S_j \subseteq S_i$, thus $S_i = S_j = S$. Hence the proof is completed. □

This theorem shows that if the quasi-fixed solutions have infinitely many, then $S = S_i$ for $i = 1, 2, \ldots, k$ and by the result of Theorem 2.2, we have

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y_i(\overline{x})\Big)^j (p_i(\overline{x}))^{m_i - jt_i}.$$

It follows that $p_i(\overline{x})$ and t_i are independent to the index " i " and $F_i(\overline{x}, y)$ can be written as the form :

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y_i(\overline{x})\Big)^j (p(\overline{x}))^{m_i - jt}.$$

But a question rises that if $| S | \geq \max_{1 \leq i \leq k}\{s_i + 3\}$, we will show that $F_i(\overline{x}, y)$ has the expression:

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y(\overline{x})\Big)^j (p(\overline{x}))^{m_i - jt}.$$

Here $y(\overline{x})$, $p(\overline{x})$ and t are independent to the index " i ". Precisely, we state it as the following theorem.

Theorem 3.3. *Suppose that the number of all quasi-fixed solutions in $F(\overline{x}, y)$ is at least $\max_{1 \leq i \leq k}\{s_i + 3\}$, then for any $i = 1, 2, \ldots, k$, we have*

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} \Big(y - y(\overline{x})\Big)^j (p(\overline{x}))^{m_i - jt} \quad \textit{for some } c_{ij} \in \mathbb{R},\ 0 \leq j \leq s_i$$

in the above expression, the polynomial functions $y(\overline{x})$, $p(\overline{x}) \in \mathbb{R}[\overline{x}]$ and $t \in \mathbb{N}$ are independent of " i ".

Proof. If $F(\overline{x}, y)$ has $\max_{1 \leq i \leq k}\{s_i + 3\}$ quasi-fixed solutions, then each $F_i(\overline{x}, y)$ has $\max_{1 \leq i \leq k}\{s_i + 3\}$ quasi-fixed solutions, $i = 1, 2, \ldots, k$. By the equivalent relation in Theorem 2.2, $F_i(\overline{x}, y)$ has infinitely many quasi-fixed solutions for any $i = 1, 2, \ldots, k$, and by (3.1), we have

$$p_1(\overline{x}) = p_2(\overline{x}) = \cdots = p_k(\overline{x}) = p(\overline{x}) \quad \text{and} \quad t_1 = t_2 = \cdots = t_k = t.$$

Moreover from Theorem 2.2, we have

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij} (y - y_i(\overline{x}))^j p^{m_i - jt}(\overline{x}),$$

for some $y_i(\overline{x}) \in \mathbb{R}[\overline{x}]$, $c_{ij} \in \mathbb{R}$, $j = 0, 1, \ldots, s_i$ and $t \in \mathbb{N}$.
By Lemma 3.1, any quasi-fixed solution $y(\overline{x})$ of $F(\overline{x}, y)$ can be represented by

$$y(\overline{x}) = y_i(\overline{x}) + d_i p^t(\overline{x}) \quad \text{for some } d_i \in \mathbb{R}$$

whence $y(\overline{x}) = y_i(\overline{x}) + d_i p^t(\overline{x}) = y_1(\overline{x}) + d_1 p^t(\overline{x})$. It follows that

$$\begin{aligned} y_i(x) &= y_1(x) + (d_1 - d_i)p^t(\overline{x}) \\ &= y_1(x) + \rho_i p^t(\overline{x}) \quad \text{where } \rho_i = d_1 - d_i. \end{aligned}$$

Substituting $y_i(\overline{x})$ by $y_1(x) + \rho_i p^t(\overline{x})$ in $F_i(\overline{x}, y)$, for $i = 1, 2, \cdots, k$, we then obtain

$$\begin{aligned} F_i(\overline{x}, y) &= \sum_{j=0}^{s_i} c_{ij}(y - y_i(\overline{x}))^j p_i^{m_i - jt_i}(\overline{x}) \\ &= \sum_{j=0}^{s_i} c_{ij}(y - y_1(\overline{x}) - \rho_i p^t(\overline{x}))^j p^{m_i - jt}(\overline{x}) \\ &= \sum_{j=0}^{s_i} c_{ij}\Big(\sum_{r=0}^{j} e_r (y - y_1(\overline{x}))^r (\rho_i p^t(\overline{x}))^{j-r}\Big) p^{m_i - jt}(\overline{x}) \\ &= \sum_{j=0}^{s_i}\sum_{r=0}^{j} c_{ij} e_r \rho_i^{j-r} (y - y_1(\overline{x}))^r (p^t(\overline{x}))^{j-r} p^{m_i - jt}(\overline{x}) \\ &= \sum_{r=0}^{s_i}\big(\sum_{j=0}^{r} c_{ij} e_r \rho_i^{j-r}\big)(y - y_1(\overline{x}))^r p^{m_i - rt}(\overline{x}) \\ &= \sum_{r=0}^{s_i} d_{ir}(y - y_1(\overline{x}))^r p^{m_i - rt}(\overline{x}) \quad \text{where } d_{ir} = \textstyle\sum_{j=0}^{r} c_{ij} e_r \rho_i^{j-r}, \end{aligned}$$

with $r = 1, 2, \ldots, s_i$, and the proof is completed. □

4. Main Theorems

If the cardinal number $\mid S \mid \geq \aleph_0$, then any quasi-fixed solution will be formated as the following theorem.

Theorem 4.1. *Suppose that $\mid S \mid \geq \aleph_0$, then for any quasi-fixed point solution $y(\overline{x})$ in S must be of the form*

$$-\frac{f_{i,s_i-1}(\overline{x})}{s_i f_{i,s_i}(\overline{x})} + \lambda p^t(\overline{x}) \quad \textit{where } t = (m_i - k_i)/s_i,$$

for any $\lambda \in \mathbb{R}$ and $i \in \{1, 2, \ldots, k\}$.

Proof. Since $F(\overline{x}, y)$ has infinitely many quasi-fixed solutions, each $F_i(\overline{x}, y)$ has infinitely many quasi-fixed solutions for $i = 1, 2, \ldots, k$. By Theorem 2.2 and Theorem

3.2, the polynomial function $F_i(\overline{x}, y)$ can be represented by

$$\begin{aligned} F_i(\overline{x}, y) &= f_{i,s_i}(\overline{x})y^{s_i} + f_{i,s_i-1}(\overline{x})y^{s_i-1} + \cdots + f_{i,0}(\overline{x}) \\ &= \sum_{j=0}^{s_i} c_{ij}\Big(y - y_i(\overline{x})\Big)^j (p(\overline{x}))^{m_i-jt} \end{aligned}$$

for some $c_{i,j} \in \mathbb{R}$, and $t \leq m_i/s_i$. Comparing both sides of the coefficient of y^{s_i} and y^{s_i-1} in the above expression for $F_i(\overline{x}, y)$, we get

$$\begin{aligned} &f_{i,s_i}(\overline{x}) = c_{is_i}p^{r_i}(\overline{x}) \quad \text{where } r_i = m_i - s_i t \in \mathbb{N}, \\ \text{and} \quad &f_{i,s_i-1}(\overline{x}) = -s_i c_{i,s_i} p^{m_i-s_i t}(\overline{x}) y_i(\overline{x}) + c_{i,s_i-1}p^{m_i-(s_i-1)t}(\overline{x}). \end{aligned}$$

It follows that

$$\begin{aligned} y_i(\overline{x}) &= \frac{f_{i,s_i-1}(\overline{x}) - c_{i,s_i-1}p^{m_i-(s_i-1)t}(\overline{x})}{-s_i c_{i,s_i}p^{m_i-st}(\overline{x})} \\ &= -\frac{f_{i,s_i-1}(\overline{x})}{s_i c_{i,s_i}p^{m_i-st}(\overline{x})} - \frac{c_{i,s_i-1}p^{m_i-(s_i-1)t}(\overline{x})}{-s_i c_{i,s_i}p^{m_i-st}(\overline{x})} \\ &= -\frac{f_{i,s_i-1}(\overline{x})}{s_i f_{i,s_i}(\overline{x})} - \frac{c_{i,s_i-1}}{s_i c_{i,s_i}}p^t(\overline{x}) \\ &= -\frac{f_{i,s_i-1}(\overline{x})}{s_i f_{i,s_i}(\overline{x})} - \lambda_i p^t(\overline{x}) \quad \text{where } \lambda_i = c_{i,s_i-1}/s_i c_{i,s_i}. \end{aligned} \tag{4.1}$$

By Theorem 3.1, any quasi-fixed solution $y(\overline{x})$ in S can be represented by

$$\begin{aligned} y(\overline{x}) &= y_i(\overline{x}) + d_i p^t(\overline{x}), \quad \text{for some } d_i \in \mathbb{R} \\ \text{by (4.1)} &= -\frac{f_{i,s_i-1}(\overline{x})}{s_i f_{i,s_i}(\overline{x})} - d_i p^t(\overline{x}) + \lambda_i p^t(\overline{x}) \\ &= -\frac{f_{i,s_i-1}(\overline{x})}{s_i f_{i,s_i}(\overline{x})} + \lambda p^t(\overline{x}) \quad \text{where } \lambda = -d_i + \lambda_i. \end{aligned}$$

This completes the proof. □

Next we are curious if the cardinal number $\mid S \mid \neq \infty$, then how about the upper bound of the number $\mid S \mid$? The result will be given in the following theorem.

Theorem 4.2. *Suppose that the number of all quasi-fixed solutions for $F(\overline{x}, y)$ is finite. Then the number of all quasi-fixed solutions is at most* $\max_{1 \leq i \leq k}\{s_i + 2\}$.

Proof. Suppose on the contrary that the number of all quasi-fixed solutions were at least $\max_{1 \leq i \leq k}\{s_i + 3\}$. By Theorem 3.3, for each i, $1 \leq i \leq k$, the component function $F_i(\overline{x}, y)$ can be represented by

$$F_i(\overline{x}, y) = \sum_{j=0}^{s_i} c_{ij}(y - y(\overline{x}))^j p^{m_i-jt}(\overline{x})$$

for some $y(\overline{x}) \in \mathbb{R}[\overline{x}]$, $c_{ij} \in \mathbb{R}$, $j = 0, 1, \ldots, s_i$ and $t \in \mathbb{N}$.
Now consider $y = y(\overline{x}) + \lambda p^t(\overline{x})$ for $\lambda \in \mathbb{R}$, we want to show that for any $\lambda \in \mathbb{R}$, y is

also a quasi-fixed solution of $F(\overline{x}, y)$. Thus for each i, $1 \le i \le k$,

$$F_i(\overline{x}, y(\overline{x}) + \lambda p^t(\overline{x})) = \sum_{j=0}^{s_i} c_{ij}(\lambda p^t(\overline{x}))^j p^{m_i - jt}(\overline{x})$$
$$= (\sum_{j=0}^{s_i} c_{ij}\lambda^j) p^{m_i}(\overline{x}).$$

That is to say, $y(\overline{x}) + \lambda p^t(\overline{x})$ is a quasi-fixed solution of $F_i(\overline{x}, y)$, $1 \le i \le k$. For arbitrary $\lambda \in \mathbb{R}$, it follows that $y(\overline{x}) + \lambda p^t(\overline{x})$ is also a quasi-fixed solution of $F(\overline{x}, y)$. This means that the number of all quasi-fixed solutions for $F(\overline{x}, y)$ is infinitely many (in fact, $\sharp(\mathbb{R})$). This is a contradiction, and the theorem is proved. □

The following example shows that not any vector polynomial function is solvable !

Example 1. Let $\overline{x} = (x_1, x_2)$ and

$$F(\overline{x}, y) = \big((x_1 + x_2)y^2, (x_1 + x_2)(y + 1)\big).$$

Suppose that $p_1(\overline{x}) = x_1$, $p_2(\overline{x}) = x_2$, $m_1 = 1$, $m_2 = 1$. Then

$$F(\overline{x}, y) = \big(a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x})\big)$$

is not solvable for $y = y(\overline{x})$.

Proof. If there exists an quasi-fixed solution $y(\overline{x})$ of $F(\overline{x}, y)$, then

$$(x_1 + x_2)y^2(\overline{x}) = a_1 x_1$$
$$(x_1 + x_2)\big(y(\overline{x}) + 1\big) = a_2 x_2.$$

From the last equation, we have $a_2 = 0$ and $y(\overline{x}) + 1 = 0$, this implies $y(\overline{x}) = -1$. Substituting $y(\overline{x}) = -1$ to $(x_1 + x_2)y^2(\overline{x}) = a_1 x_1$, we have $x_1 + x_2 = a_1 x_1$, this means x_2 depends on x_1, but it is impossible. This shows that

$$F(\overline{x}, y) = \big(a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x})\big)$$

is not solvable for $y = y(\overline{x})$.

The following example will be shown that if $\mid S \mid \ge \aleph_0$, then any $y(\overline{x}) \in S$ can be obtained by the formula in Theorem 4.1.

Example 2. Let $\overline{x} = (x_1, x_2)$, and

$$F(\overline{x}, y) = (F_1(\overline{x}, y), F_2(\overline{x}, y))$$

where

$$F_1(\overline{x}, y) = f_{1,2}(\overline{x})y^2 + f_{1,1}(\overline{x})y + f_{1,0}(\overline{x})$$
$$= y^2 - (2x_1x_2 - x_1 - x_2)y + (x_1^2x_2^2 - x_1^2x_2 - x_1x_2^2 + x_1^2 + 2x_1x_2 + x_2^2)$$
$$F_2(\overline{x}, y) = f_{1,1}(\overline{x})y + f_{1,0}(\overline{x})$$
$$= y - x_1x_2 + x_1 + x_2$$

and assume that $p_1(\overline{x}) = p_2(\overline{x}) = x_1 + x_2$, $m_1 = 2$ and $m_2 = 1$. Prove that the number of all quasi-fixed solutions of

$$F(\overline{x}, y) = \big(a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x})\big)$$

is infinitely many and how to represent all quasi-fixed solutions.

Proof. Assume that there exist infinitely many quasi-fixed solutions in $F(\overline{x}, y)$, by Theorem 4.1, $f_{1,2}(\overline{x}) = c_2 p_1^{r_1}(\overline{x}) = 1$, we have $r_1 = 0$ and $t = \frac{m_1 - r_1}{s_1} = \frac{2-0}{2} = 1$. Then by Theorem 4.1, any quasi-fixed solution $y(\overline{x})$ must be the form

$$\begin{aligned} y(\overline{x}) &= -f_{1,2}(\overline{x})/s_1 f_{1,1}(\overline{x}) + \lambda p^t(\overline{x}) \qquad \text{for any } \lambda \in \mathbb{R} \\ &= (2x_1x_2 - x_1 - x_2)/2 + \lambda p^t(\overline{x}) \\ &= (2x_1x_2 - x_1 - x_2)/2 + \lambda p(\overline{x}) \\ &= x_1x_2 + (\lambda - 1/2)p(\overline{x}) \\ &= x_1x_2 + \widetilde{\lambda} p(\overline{x}), \qquad \widetilde{\lambda} = \lambda - 1/2 \text{ is arbitrary} \\ &= x_1x_2 + \widetilde{\lambda}(x_1 + x_2). \end{aligned}$$

Substituting $y = y(\overline{x}) = x_1x_2 + \widetilde{\lambda}(x_1 + x_2)$ in $F(\overline{x}, y)$, we obtain

$$F(\overline{x}, y(\overline{x})) = \big((\widetilde{\lambda}^2 - \widetilde{\lambda} + 1)p^2(\overline{x}), (1 - \widetilde{\lambda})p(\overline{x})\big)$$

and each $\widetilde{\lambda} \in \mathbb{R}$ yields a quasi-fixed solution, $y(\overline{x})$. So the number of all quasi-fixed solutions of $F(\overline{x}, y) = \big(a_1 p_1^{m_1}(\overline{x}), a_2 p_2^{m_2}(\overline{x})\big)$ has the cardinal $\sharp(\mathbb{R}) \geq \aleph_0$.

For examples of finitely many quasi-fixed solutions we refer [3].

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