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FIXED POINT THEOREMS IN KASAHARA SPACES WITH RESPECT TO AN OPERATOR AND APPLICATIONS

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Abstract. The purpose of this paper is to introduce the concept of Kasahara space with respect to an operator and to prove in this setting some fixed point theorems. As applications, integral and differential equations are considered.
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1. Introduction

The following theorem was given by Maia in 1968 (see [4]):

Theorem 1.1. *Let X be a nonempty subset, d and ρ be two metrics on X and $f : X \to X$ be a mapping. Suppose that:*

(i) *$\rho(x, y) \leq d(x, y)$, for all $x, y \in X$;*
(ii) *(X, ρ) is a complete metric space;*
(iii) *$f : (X, \rho) \to (X, \rho)$ is continuous;*
(iv) *$f : (X, d) \to (X, d)$ is an α-contraction, i.e., there exists $\alpha \in [0, 1[$ such that $d(f(x), f(y)) \leq \alpha \cdot d(x, y)$, for all $x, y \in X$.*

Then

(1) *$F_f = \{x^*\}$;*
(2) *$(f^n(x_0))_{n\in\mathbb{N}}$ converges in (X, ρ) to x^*, for all $x_0 \in X$.*

In applications we usually consider a variant of Maia's Theorem given by I.A. Rus in [10] (see also [11]). More precisely, we have:

Remark 1.1. *Theorem 1.1 remains true if condition (i) is replaced by*

(i') *there exists $c > 0$ such that $\rho(f(x), f(y)) \leq c \cdot d(x, y)$, for all $x, y \in X$;*

For other Maia type results see [11], [10], [6], [7], [8]. Fixed point theorems in Kasahara spaces are natural generalizations of Maia type theorems.

We recall first the notion of L-space, introduced by M. Fréchet, see [1].

Definition 1.1. *Let X be a nonempty set. Let*

$$s(X) := \{(x_n)_{n\in\mathbb{N}} \mid x_n \in X,\ n \in \mathbb{N}\}.$$

Let $c(X)$ be a subset of $s(x)$ and $Lim : c(X) \to X$ be an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space (denoted by $(X, \to)$) if the following conditions are satisfied:

(i) *if $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.*

(ii) *if $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences $(x_{n_i})_{i\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and*

$$Lim(x_{n_i})_{i\in\mathbb{N}} = x.$$

By following S. Kasahara (see [3]) and M.G. Maia (see [4]), the notion of Kasahara space was introduced by I.A. Rus in [13] as follows:

Definition 1.2. *Let $(X, \to)$ be an L-space and $d : X \times X \to \mathbb{R}_+$ be a functional. The triple $(X, \to, d)$ is a Kasahara space if and only if the following compatibility condition between $\to$ and d holds:*

$$x_n \in X,\ \sum_{n\in\mathbb{N}} d(x_n, x_{n+1}) < +\infty\ \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges in } (X, \to).$$

The purpose of this paper is to introduce the concept of Kasahara space with respect to an operator and to prove in this setting some fixed point theorems. As applications, integral and differential equations are considered.

2. Fixed point theorems in Kasahara spaces with respect to an operator

Let $(X, \to)$ be an L-space. Let $f : X \to X$ be an operator. Then we denote by $f^0 := 1_X$, $f^1 = f$, $f^{n+1} = f \circ f^n$ for all $n \in \mathbb{N}$, the iterates of f.

Let $x \in X$. The sequence $(x_n)_{n\in\mathbb{N}} \subset X$, defined by $x_n := f^n(x)$ for all $n \in \mathbb{N}$ is called the sequence of successive approximations for f starting from x. In the sequel, we shall denote this sequence by $(f^n(x))_{n\in\mathbb{N}}$.

The notion of Kasahara space with respect to an operator is introduced as follows.

Definition 2.1. *Let $(X, \to)$ be an L-space, $d : X \times X \to \mathbb{R}_+$ be a functional and $f : X \to X$ be an operator. The triple $(X, \to, d)$ is a Kasahara space with respect to the operator f if and only if*

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty, \text{ for all } x \in X$$

implies that

$$(f^n(x))_{n\in\mathbb{N}} \text{ is convergent in } (X, \to), \text{ for all } x \in X.$$

Recall first a very useful tool for proving the uniqueness of a fixed point in a Kasahara space (see S. Kasahara [3], I.A. Rus [13]).

Lemma 2.1. *(Kasahara's lemma) Let $(X, \to, d)$ be a Kasahara space. Then:*

$$x, y \in X,\ d(x, y) = d(y, x) = 0 \implies x = y.$$

Notice that, in a Kasahara space with respect to an operator the above implication need not to be satisfied. Notice also that a Kasahara space is a Kasahara space with respect to an operator, but the reverse implication is false.

Example 2.1. *Let X be a nonempty set, $f : X \to X$ be an operator and $d, \rho : X \times X \to \mathbb{R}_+$ be two functionals. We suppose:*

(i) (X, ρ) is a complete metric space;

(ii) there exists $c > 0$ such that $\rho(f(x), f(y)) \leq cd(x, y)$, for all $x, y \in X$.

Then $(X, \xrightarrow{\rho}, d)$ is a Kasahara space with respect to f.

Indeed, let $x \in X$ be such that $\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) < +\infty$. Then, for $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$, we can write

$$\rho(f^n(x), f^{n+p}(x)) \leq \sum_{k=n-1}^{n+p-2} \rho(f^{k+1}(x), f^{k+2}(x)) \leq c \sum_{k=n-1}^{n+p-2} d(f^k(x), f^{k+1}(x)) \to 0$$

as $n \to +\infty$. Thus, since (X, ρ) is a complete metric space, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in (X, ρ). This completes the proof.

Example 2.2. *Let $X = C(\overline{\Omega}) := \{x : \overline{\Omega} \to \mathbb{R} \mid x \text{ is a continuous function }\}$, where $\Omega \subseteq \mathbb{R}^m$ is a bounded domain.*

Let $\xrightarrow{\rho}$ be the convergence structure induced by $\rho : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$, where

$$\rho(x, y) := \|x - y\|_\infty := \sup_{t\in\overline{\Omega}} |x(t) - y(t)|, \text{ for all } x, y \in C(\overline{\Omega}).$$

Let $d : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R}_+$ be the functional defined by

$$d(x, y) := \|x - y\|_{L^2(\Omega)} := \left(\int_\Omega |x(t) - y(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } x, y \in C(\overline{\Omega}).$$

We consider the operator $f : C(\overline{\Omega}) \to C(\overline{\Omega})$, defined by

$$f(x)(t) := \int_\Omega K(t, s, x(s))ds$$

where $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$.

We assume that there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that

$$|K(t, s, u) - K(t, s, v)| \leq L(t, s)|u - v|,$$

for all $t, s \in \overline{\Omega}$ and $u, v \in \mathbb{R}$.

Then the triple $(X, \xrightarrow{\rho}, d)$, i.e., $\left(C(\overline{\Omega}), \xrightarrow{\|\cdot\|_\infty}, \|\cdot\|_{L^2(\Omega)}\right)$ is a Kasahara space with respect to the operator f.

Indeed, since

$$\rho(f(x), f(y)) \leq \sup_{t\in\overline{\Omega}} \left(\int_\Omega L(t, s)^2 ds \right)^{\frac{1}{2}} \cdot d(x, y),$$

we are in the conditions of Example 2.1 and the conclusion follows.

Now we will present some remarks concerning operators on Kasahara spaces or Kasahara spaces with respect to an operator.

Definition 2.2. *Let $(X, \to, d)$ be a Kasahara space and $f : X \to X$ be an operator. Then, by definition:*

(i) *f is a Picard operator if and only if f is Picard in $(X, \to)$, i.e., $F_f = \{x^*\}$ and $f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$;*
(ii) *f is a weakly Picard operator if and only if f is weakly Picard in $(X, \to)$, i.e., $f^n(x) \to x^*(x) \in F_f$ as $n \to \infty$, for all $x \in X$;*
(iii) *if f is a weakly Picard operator, then we define the operator*
$$f^\infty : X \to X \text{ by } f^\infty(x) := Lim(f^n(x))_{n\in\mathbb{N}};$$
(iv) *f is with closed graph if and only if f has closed graph in $(X, \to)$, i.e.,*
$$x_n \to x^* \text{ and } f(x_n) \to y^* \Rightarrow f(x^*) = y^*;$$
(v) *f is continuous if and only if f is continuous in $(X, \to)$;*
(vi) *f is k-Lipschitz if and only if f is k-Lipschitz in (X, d);*
(vii) *f is k-contraction if and only if f is k-contraction in (X, d).*

For other considerations on Picard operators and weakly Picard operators see I.A. Rus [12], [11], I.A. Rus, A. Petruşel and M.A. Şerban [14].

Theorem 2.1. *Let X be a nonempty set and $f : X \to X$ be an operator. Suppose that $(X, \to, d)$ is a Kasahara space with respect to f. We assume that:*

(i) *$f : (X, \to) \to (X, \to)$ has closed graph;*
(ii) *(ii) $f : (X, d) \to (X, d)$ is an α-contraction, i.e., there exists $\alpha \in [0, 1[$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$, for all $x, y \in X$;*
(iii) *$d(x, y) = d(y, x) = 0 \;\Rightarrow\; x = y$.*

Then

(1) *$F_f = F_{f^n} = \{x^*\}$ for all $n \in \mathbb{N}^*$ and $d(x^*, x^*) = 0$.*
(2) *$f^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$, i.e., f is a Picard operator.*
(3) *We have:*
  (3_a) *$d(f^n(x), x^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$, for all $x \in X$;*
  (3_b) *$d(x^*, f^n(x)) \overset{\mathbb{R}}{\to} 0$, as $n \to \infty$, for all $x \in X$.*
(4) *If d is a quasimetric (i.e., $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ for all $x, y \in X$ and d satisfies the triangle inequality), then:*
  (4_a) *$d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x))$, for all $x \in X$;*
  (4_b) *$d(x^*, x) \leq \frac{1}{1-\alpha} d(f(x), x)$, for all $x \in X$;*
  (4_c) *$d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x))$, for all $x \in X$ and all $n \in \mathbb{N}$;*
  (4_d) *$d(x^*, f^n(x)) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$, for all $x \in X$ and all $n \in \mathbb{N}$;*
  (4_e) *if $(z_n)_{n\in\mathbb{N}} \subset X$ is such that $d(z_n, f(z_n)) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$ then $d(z_n, x^*) \overset{\mathbb{R}}{\to} 0$ as $n \to \infty$, i.e., the fixed point problem for the operator f is well-posed with respect to d;*

(4_f) *if* $(z_n)_{n\in\mathbb{N}} \subset X$ *is such that* $d(z_{n+1}, f(z_n)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$ *then* $d(z_{n+1}, f^{n+1}(z)) \xrightarrow{\mathbb{R}} 0$ *as* $n \to \infty$*, for all* $z \in X$*, i.e., the operator* f *has the limit shadowing property with respect to* d*;*

(4_g) *If* $g : X \to X$ *is an operator such that*

$$d(f(x), g(x)) \leq \eta, \text{ for all } x \in X,$$

then

$$d(x^*, y^*) \leq \frac{\eta}{1-\alpha}, \text{ for all } y^* \in F_g.$$

Proof. (1) & (2). Let $x \in X$ and $(f^n(x))_{n\in\mathbb{N}}$ be the sequence of successive approximations of f starting from x.

By (ii) and by induction after $n \in \mathbb{N}^*$ we have that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha^n d(x, f(x)). \tag{2.1}$$

It follows that

$$\sum_{n\in\mathbb{N}} d(f^n(x), f^{n+1}(x)) \leq \sum_{n\in\mathbb{N}} \alpha^n d(x, f(x)) = \frac{1}{1-\alpha} d(x, f(x)) < \infty.$$

Since $(X, \to, d)$ is a Kasahara space with respect to the operator f, we get that the sequence $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. Hence, there exists an element $x^* \in X$ such that $f^n(x) \to x^*$ as $n \to \infty$.

By (i) we obtain that $x^* \in F_f$. Since $x^* = f(x^*) = f(f(x^*)) = \ldots = f^n(x^*)$ we also conclude that $x^* \in F_{f^n}$.

Next, we show the uniqueness of the fixed point x^*.

Let $y^* \in X$ be another fixed point for the operator f such that $x^* \neq y^*$. Then

$$\begin{aligned} d(x^*, y^*) &= d(f^n(x^*), f^n(y^*)) \leq \alpha d(f^{n-1}(x^*), f^{n-1}(y^*)) \\ &\leq \ldots \leq \alpha^n d(x^*, y^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \end{aligned} \tag{2.2}$$

Similarly, we get that $d(y^*, x^*) = 0$. By (iii), we conclude that $x^* = y^*$. Hence f is a Picard operator.

Finally, if $x^* \in F_f$ then we can show that $d(x^*, x^*) = 0$.

Indeed, by (2.2), we have

$$d(x^*, x^*) \leq \alpha^n d(x^*, x^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty.$$

(3_a). Let $x \in X$. Then by (ii) we have

$$\begin{aligned} d(f^n(x), x^*) = d(f^n(x), f^n(x^*)) &\leq \alpha d(f^{n-1}(x), f^{n-1}(x^*)) \\ &\leq \ldots \leq \alpha^n d(x, x^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty, \end{aligned}$$

so (3_a) holds. By a similar approach we obtain (3_b).

(4_a). Let $x \in X$. Since the functional d satisfies the triangle inequality, we have $d(x, x^*) \leq d(x, f(x)) + d(f(x), f(x^*)) \leq d(x, f(x)) + \alpha d(x, x^*)$ and hence

$$d(x, x^*) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4_a) holds. Similarly we get that (4_b) holds.

(4_c). Using the property (4_a), we have for each $n \in \mathbb{N}$ the following estimation

$$d(f^n(x), x^*) \leq \frac{1}{1-\alpha} d(f^n(x), f^{n+1}(x)), \text{ for all } x \in X \tag{2.3}$$

By (2.3) and (2.1) we obtain

$$d(f^n(x), x^*) \leq \frac{\alpha^n}{1-\alpha} d(x, f(x)), \text{ for all } x \in X,$$

so (4_c) holds. By a similar procedure we obtain (4_d).

We prove next (4_e). Let $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$. By (4_a) we have

$$d(z_n, x^*) \leq \frac{1}{1-\alpha} d(z_n, f(z_n)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty$$

so (4_e) holds.

(4_f). Let $z \in X$ and $(z_n)_{n\in\mathbb{N}} \subset X$ such that $d(z_{n+1}, f(z_n)) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$. Since $x^* \in F_f$, by (ii) and (3_b) we have that

$$d(x^*, f^{n+1}(z)) = d(f(x^*), f^{n+1}(z)) \leq \alpha d(x^*, f^n(z)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \tag{2.4}$$

We need to prove that $d(z_{n+1}, x^*) \xrightarrow{\mathbb{R}} 0$ as $n \to \infty$.

We have

$$\begin{aligned} d(z_{n+1}, x^*) &\leq d(z_{n+1}, f(z_n)) + d(f(z_n), x^*) \leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, x^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \alpha^2 d(z_{n-1}, x^*) \\ &\leq d(z_{n+1}, f(z_n)) + \alpha d(z_n, f(z_{n-1})) + \ldots + \alpha^{n+1} d(z_0, x^*). \end{aligned}$$

From a Cauchy lemma (see the references in [11], [12] or [15]) we have that

$$d(z_{n+1}, x^*) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$d(z_{n+1}, f^{n+1}(z)) \leq d(z_{n+1}, x^*) + d(x^*, f^{n+1}(z)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty.$$

Finally, we show (4_g). Let $y^* \in F_g$. By (4_b) we have that

$$d(x^*, y^*) \leq \frac{1}{1-\alpha} d(f(y^*), y^*) = \frac{1}{1-\alpha} d(f(y^*), g(y^*)) \leq \frac{\eta}{1-\alpha}.$$

□

Theorem 2.2. *Let X be a nonempty set and $f : X \to X$ be an operator. Suppose that $(X, \to, d)$ is a Kasahara space with respect to f. We assume that:*

- *(i) $f : (X, \to) \to (X, \to)$ has closed graph;*
- *(ii) $f : (X, d) \to (X, d)$ is an α-graphic contraction, i.e., there exists $\alpha \in [0, 1[$ such that $d(f(x), f^2(x)) \leq \alpha d(x, f(x))$, for all $x \in X$.*

Then the following statements hold:

- *(1) $F_f \neq \emptyset$.*
- *(2) $f^n(x) \to f^\infty(x) \in F_f$ as $n \to \infty$, for all $x \in X$, i.e., $f : (X, \to) \to (X, \to)$ is a weakly Picard operator.*

(3) *$d(x^*, x^*) = 0$, for all $x^* \in F_f$.*
(4) *if d satisfies the triangle inequality and d is continuous with respect to $\to$, then*
  (4_a) *$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x))$, for all $x \in X$,*
  (4_b) *Let $g : X \to X$ be an operator. If there exists $c > 0$ such that*

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)), \text{ for all } x \in X \tag{2.6}$$

and for each $x \in X$, there exists $\eta > 0$ such that

$$\max\{d(g(x), f(x)), d(f(x), g(x))\} \leq \eta, \tag{2.7}$$

then

$$H_d(F_f, F_g) \leq \max\left\{\frac{1}{1-\alpha}, c\right\}\eta,$$

where H_d stands for the Pompeiu-Hausdorff functional generated by d (see [2]).

Proof. (1) & (2). Let $x \in X$ and consider the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from x. Since f is an α-graphic contraction, we deduce that

$$d(f^n(x), f^{n+1}(x)) \leq \alpha d(f^{n-1}(x), f^n(x)) \leq \ldots \leq \alpha^n d(x, f(x)), \text{ for all } n \in \mathbb{N}.$$

By the proof of Theorem 2.1 we get that $(f^n(x))_{n\in\mathbb{N}}$ is convergent in $(X, \to)$. By (i) it follows that its limit is a fixed point of f. So $F_f \neq \emptyset$.

(3). Let $x^* \in F_f$. Then by (ii) we have

$$\begin{aligned} d(x^*, x^*) &= d(f^n(x^*), f^{n+1}(x^*)) \leq \alpha d(f^{n-1}(x^*), f^n(x^*)) \\ &\leq \alpha^2 d(f^{n-2}(x^*), f^{n-1}(x^*)) \leq \ldots \leq \alpha^n d(x^*, f(x^*)) \xrightarrow{\mathbb{R}} 0 \text{ as } n \to \infty. \end{aligned}$$

(4). Let $x \in X$. Then

$$\begin{aligned} d(x, f^\infty(x)) &\leq d(x, f^n(x)) + d(f^n(x), f^\infty(x)) \\ &\leq d(x, f(x)) + d(f(x), f^2(x)) + \ldots + d(f^{n-1}(x), f^n(x)) \\ &\quad + d(f^n(x), f^\infty(x)) \\ &\leq (1 + \alpha + \ldots + \alpha^{n-1}) d(x, f(x)) + d(f^n(x), f^\infty(x)) \\ &\leq \frac{1}{1-\alpha} d(x, f(x)) + d(f^n(x), f^\infty(x)), \text{ for all } n \in \mathbb{N}. \end{aligned}$$

By letting $n \to \infty$ and by using (3), we obtain

$$d(x, f^\infty(x)) \leq \frac{1}{1-\alpha} d(x, f(x)), \text{ for each } x \in X,$$

so (4_a) holds.

We show next (4_b).

Let $x \in F_f$ and $y \in F_g$. Since g satisfies (2.6) and (2.7), we have

$$d(x, g^\infty(x)) \leq c \cdot d(x, g(x)) = c \cdot d(f(x), g(x)) \leq c\eta.$$

Since $g^\infty(x) \in F_g$ we have

$$\inf_{y \in F_g} d(x, y) \leq d(x, g^\infty(x)) \leq c\eta$$

and by taking the supremum over $x \in F_f$, we obtain

$$\sup_{x \in F_f} \inf_{y \in F_g} d(x, y) \leq c\eta. \tag{2.8}$$

On the other hand, since f satisfies (4_a), we have

$$d(y, f^\infty(y)) \leq \frac{1}{1-\alpha} d(y, f(y)) = \frac{1}{1-\alpha} d(g(y), f(y)) \leq \frac{\eta}{1-\alpha}.$$

Since $f^\infty(y) \in F_f$ we have

$$\inf_{x \in F_f} d(y, x) \leq d(y, f^\infty(y)) \leq \frac{\eta}{1-\alpha}$$

and by taking the supremum over $y \in F_g$, we obtain

$$\sup_{y \in F_g} \inf_{x \in F_f} d(y, x) \leq \frac{\eta}{1-\alpha}. \tag{2.9}$$

By (2.8) and (2.9) we get

$$H_d(F_f, F_g) := \max\left\{ \sup_{x \in F_f} \inf_{y \in F_g} d(x, y), \sup_{y \in F_g} \inf_{x \in F_f} d(y, x) \right\} \leq \max\left\{ \frac{1}{1-\alpha}, c \right\} \eta.$$

□

3. Existence and uniqueness for integral equations and boundary value problems

We will present now some applications of the abstract results given in Section 2.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $K \in C(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R})$ and $g \in C(\overline{\Omega})$. We suppose that:*

(i) *$K(t, s, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing, for all $t, s \in \overline{\Omega}$.*

(ii) *there exists $L \in C(\overline{\Omega} \times \overline{\Omega})$ such that*

$$|K(t, s, u) - K(t, s, v)| \leq L(t, s)|u - v|,$$

for all $t, s \in \overline{\Omega}$ and $u, v \in \mathbb{R}$.

(iii) $\displaystyle\int_{\Omega \times \Omega} L(t, s)^2 ds dt < 1.$

Then the integral equation

$$x(t) = \int_\Omega K(t, s, x(s)) ds + g(t), \ \ t \in \Omega \tag{3.1}$$

has a unique solution $x^* \in C(\overline{\Omega})$.

Proof. Let $X = C(\overline{\Omega})$ and $\to := \xrightarrow{\|\cdot\|_\infty}$ be the convergence induced by $\|\cdot\|_\infty$ on X, where $\|x\|_\infty = \sup\limits_{t\in\overline{\Omega}}|x(t)|$, for all $x \in C(\overline{\Omega})$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x,y) = \|x-y\|_{L^2(\Omega)} = \left(\int_\Omega |x(t)-y(t)|^2 dt\right)^{\frac{1}{2}}, \text{ for all } x, y \in X.$$

We consider the operator $A : X \to X$, $x \mapsto Ax$, defined by

$$Ax(t) = \int_\Omega K(t,s,x(s))ds + g(t), \text{ for all } t \in \overline{\Omega}.$$

Then the integral equation (3.1) is equivalent with the fixed point problem $x = Ax$.

Notice that, since A is a continuous operator on $(X, \xrightarrow{\|\cdot\|_\infty})$, we get that A has closed graph in $(X, \xrightarrow{\|\cdot\|_\infty})$.

On the other hand, A is a contraction in (X,d). Indeed, by the definition of d we have

$$\begin{aligned} d(Ax,Ay) &= \left(\int_\Omega |Ax(t)-Ay(t)|^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_\Omega \left|\int_\Omega \big[K(t,s,x(s)) - K(t,s,y(s))\big]ds\right|^2 dt\right)^{\frac{1}{2}}. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} &\left|\int_\Omega \big[K(t,s,x(s)) - K(t,s,y(s))\big]ds\right| \leq \int_\Omega \big|K(t,s,x(s)) - K(t,s,y(s))\big|ds \\ &\leq \int_\Omega L(t,s)|x(s)-y(s)|ds \overset{\text{Hölder}}{\leq} \left(\int_\Omega L(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_\Omega |x(s)-y(s)|^2 ds\right)^{\frac{1}{2}}. \end{aligned}$$

Hence, for all $x, y \in X$ we have

$$d(Ax,Ay) \leq \left(\int_\Omega \left(\int_\Omega L(t,s)^2 ds\right) d(x,y)^2 dt\right)^{\frac{1}{2}} = \left(\int_\Omega \int_\Omega L(t,s)^2 ds dt\right)^{\frac{1}{2}} d(x,y)$$

and by (iii), we get that A is a contraction in (X,d).

Thus, the triple $(C(\Omega), \xrightarrow{\|\cdot\|_\infty}, d)$ is a Kasahara space with respect to the operator A (see also Example 2.2). Applying Theorem 2.1 the conclusion follows. □

We consider next the following boundary value problem

$$\begin{cases} y''(t) = f(t,y(t)), \ \textit{for all } t \in [a,b] \\ a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b) = 0 \\ b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b) = 0 \end{cases} \tag{3.2}$$

where $a_i, b_i \in \mathbb{R}$, $i = \overline{1,4}$ and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We consider also the following linear mappings:

(1) $L : C^2([a,b]) \to C([a,b])$, $L(y) = y''(t)$;
(2) $l_1 : C^2([a,b]) \to \mathbb{R}$, $l_1(y) = a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b)$
(3) $l_2 : C^2([a,b]) \to \mathbb{R}$, $l_2(y) = b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b)$

Then the boundary value problem (3.2) can be written as follows:

$$L(y) = f(\cdot, y),\ l_1(y) = 0,\ l_2(y) = 0. \tag{3.3}$$

We recall that the Green's function associated to the boundary value problem (3.3) is the mapping

$$G : [a, b] \times [a, b] \to \mathbb{R};\ (t, s) \mapsto G(t, s)$$

which satisfies the following conditions:

(i) $G \in C([a, b] \times [a, b])$;

(ii) For any $s \in [a, b]$, $G(\cdot, s) \in C^2([a, s[\cup]s, b])$ and

$$\frac{\partial}{\partial t}G(s + 0, s) - \frac{\partial}{\partial t}G(s - 0, s) = -\frac{1}{p(s)},$$

where $p \in C([a, b])$ and $p(s) \neq 0$ for any $s \in [a, b]$;

(iii) $G(\cdot, s)$ is a solution for $L(y) = 0$ on $[a, b] \setminus \{s\}$ and satisfies the boundary conditions $l_1(y) = l_2(y) = 0$.

We have the following result.

Theorem 3.2. *Let $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and consider the boundary value problem* (3.3)*. We assume that:*

(i) *there exists $L_f > 0$ such that*

$$|f(s, u) - f(s, v)| \leq L_f |u - v|,$$

for all $s \in [a, b]$ and $u, v \in \mathbb{R}$;

(ii) $\displaystyle\int_a^b \int_a^b G(t, s)^2 ds dt < 1$*, where G is the Green's function associated to the boundary value problem* (3.3)*.*

If the homogeneous boundary value problem

$$\begin{cases} L(y) = 0 \\ l_1(y) = l_2(y) = 0 \end{cases} \tag{3.4}$$

admits only the trivial solution $y \equiv 0$, then the boundary value problem (3.3) *has a unique solution in $C([a, b])$.*

Proof. Since the problem (3.4) admits only the trivial solution $y \equiv 0$, there exists a unique Green function G, associated to the problem (3.3). Moreover, (see for example P. Pavel and I.A. Rus [5], p.160) the boundary value problem (3.3) is equivalent with the Fredholm type integral equation

$$y(t) = -\int_a^b G(t, s) f(s, y(s)) ds, \text{ for all } t \in [a, b]. \tag{3.5}$$

Let $X = C([a, b])$, $\to := \xrightarrow{\|\cdot\|_\infty}$ be the convergence structure on X, where $\|x\|_\infty = \sup_{t \in [a,b]} |x(t)|$, for all $x \in C([a, b])$. Let $d : X \times X \to \mathbb{R}_+$ be defined by

$$d(x, y) = \|x - y\|_{L^2([a,b])} = \left(\int_a^b |x(t) - y(t)|^2 dt \right)^{\frac{1}{2}}, \text{ for all } x, y \in X.$$

We consider the operator $A : X \to X$, $x \mapsto Ax$, defined by

$$Ax(t) = -\int_a^b G(t,s)f(s,x(s))ds, \text{ for all } t \in [a,b].$$

Then the integral equation (3.5) is equivalent with the fixed point problem $y = Ay$.

Notice that, since A is a continuous operator on $(X, \xrightarrow{\|\cdot\|_\infty})$, we have that A has closed graph in $(X, \xrightarrow{\|\cdot\|_\infty})$.

On the other hand, A is a contraction in (X, d). Indeed, by the definition of d we have

$$\begin{aligned} d(Ax, Ay) &= \left(\int_a^b |Ax(t) - Ay(t)|^2 dt \right)^{\frac{1}{2}} \\ &= \left(\int_a^b \left| \int_a^b G(t,s)\big[f(s,x(s)) - f(s,y(s))\big] ds \right|^2 dt \right)^{\frac{1}{2}}. \end{aligned}$$

Using Hölder's inequality, we get

$$\begin{aligned} &\left| \int_a^b G(t,s)\big[f(s,x(s)) - f(s,y(s))\big] ds \right| \\ &\quad \le \int_a^b G(t,s)\big|f(s,x(s)) - f(s,y(s))\big| ds \le \int_a^b G(t,s) L_f |x(s) - y(s)| ds \\ &\quad \overset{\text{Hölder}}{\le} L_f \left(\int_a^b G(t,s)^2 ds \right)^{\frac{1}{2}} \left(\int_a^b |x(s) - y(s)|^2 ds \right)^{\frac{1}{2}}. \end{aligned}$$

Hence, for all $x, y \in X$ we have

$$\begin{aligned} d(Ax, Ay) &\le \left(\int_a^b L_f^2 \left(\int_a^b G(t,s)^2 ds \right) d(x,y)^2 dt \right)^{\frac{1}{2}} \\ &= L_f \left(\int_a^b \int_a^b G(t,s)^2 ds dt \right)^{\frac{1}{2}} d(x,y). \end{aligned} \tag{3.6}$$

and by (ii), we get that A is a contraction in (X, d).

The triple $(C([a,b]), \xrightarrow{\|\cdot\|_\infty}, d)$ is a Kasahara space with respect to the operator A (see Example 2.2 and take $\overline{\Omega} = [a,b]$). By Theorem 2.1 the conclusion follows. $\square$

Let us consider now the following particular form of a boundary value problem.

$$\begin{cases} y''(t) = f(t, y(t)), \ \textit{for all } t \in [a,b] \\ y(a) = y(b) = 0 \end{cases} \tag{3.7}$$

In this case, the boundary value problem (3.7) is equivalent with the Fredholm type integral equation (3.5) where the Green's function $G : [a,b] \times [a,b] \to \mathbb{R}$ is defined by

$$G(t,s) = \begin{cases} \frac{(b-t)(s-a)}{b-a}, & s \le t \\ \frac{(b-s)(t-a)}{b-a}, & s > t \end{cases}$$

Notice that the Green function G is symmetric, continuous, positive on $[a,b]^2$ and

$$G(t,s) \leq \frac{b-a}{4}, \text{ for all } t,s \in [a,b]. \tag{3.8}$$

By Theorem 3.2 we get the following result.

Theorem 3.3. *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function. We assume that:*

(i) *there exists $L_f > 0$ such that*

$$|f(s,u) - f(s,v)| \leq L_f |u - v|, \textit{ for all } s \in [a,b], \textit{ and } u,v \in \mathbb{R}.$$

(ii) $L_f \frac{(b-a)^2}{4} < 1$.

Then, the boundary value problem (3.7) *has a unique solution in* $C([a,b])$.

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