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# WEAK CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS WITH NONLINEAR OPERATORS IN HILBERT SPACES

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**Abstract.** In this paper, we introduce an iterative sequence for finding a common element of the set of fixed points of a nonspreading mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality problem for a monotone and Lipschitz-continuous mapping. We show that the sequence converges weakly to a common element of the above three sets.

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## 1. Introduction

Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $f$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \text{ for all } y \in C. \tag{1.1}$$

The set of solutions (1.1) is denoted by $EP(f)$. A mapping $A$ of $C$ into $H$ is called monotone if $\langle Au - Av, u - v\rangle \geq 0$ for all $u, v \in C$. The variational inequality problem is to find $u \in C$ such that $\langle Au, v - u\rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

A mapping $A$ of $C$ into $H$ is called $\alpha$-inverse-strongly monotone if there exists a positive real number $\alpha$ such that $\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$ for all $u, v \in C$. It is obvious that any $\alpha$-inverse-strongly monotone mapping $A$ is monotone and Lipschitz

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continuous; see, for example, [13]. A mapping $S$ of $C$ into itself is called nonexpansive if $\|Su - Sv\| \leq \|u - v\|$ for all $u, v \in C$. A mapping $S$ of $C$ into itself is called nonspreading (see [3, 4]) if

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2, \text{ for all } u, v \in C.$$

We denote by $F(S)$ the set of fixed points of $S$. Recently, in the case when $S$ is a nonexpansive mapping, Nadezhkina and Takahashi [5] introduced an iterative process for finding a common element of the set $F(S)$ and the set $VI(C, A)$ for a monotone and Lipschitz-continuous mapping. On the other hand, Tada and Takahashi [9, 10] and Takahashi and Takahashi [11] obtained weak and strong convergence theorems for finding a common element of the set $EP(f)$ and the set $F(S)$ in a Hilbert space. In this paper, we prove weak convergence theorems for finding a common element of the set $EP(f)$, the set $VI(C, A)$ for a monotone and Lipschitz-continuous mapping and the set $F(S)$ of a nonspreading mapping in a Hilbert space.

## 2. Preliminaries

In this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2$$

for all $x, y \in H$ and $\lambda \in \mathbb{R}$; see [12]. Let $C$ be a closed convex subset of $H$. Then, for every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. $P_C$ is called the metric projection of $H$ onto $C$. We know that $P_C$ is a nonexpansive mapping of $H$ onto $C$. It is also known that $P_C$ is characterized by the following properties: $P_C x \in C$,

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \tag{2.1}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.2}$$

for all $x \in H, y \in C$. Let $A$ be a monotone mapping of $C$ into $H$. In the context of the variational inequality problem, this implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda Au)$$

for all $\lambda > 0$. It is also known that $H$ satisfies the Opial condition [6]; i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We also know that $H$ has the Kadec-Klee property, that is, $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$. In fact, from

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x\rangle + \|x\|^2,$$

we get that a Hilbert space has the Kadec.Klee property. An operator $A : H \to 2^H$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$ whenever $y_1 \in Ax_1$ and $y_2 \in Ax_2$. Let $A$ be a monotone, $k$-Lipschitz-continuous mapping of $C$ into $H$ and let $N_C v$ be the normal cone to $C$ at $v \in C$; i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$.
Define

$$Tv = \begin{cases} Av + N_C v, \text{ if } v \in C, \\ \emptyset, \text{ if } v \notin C. \end{cases}$$

Then, T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [7].
For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that $f$ satisfies the following conditions:
(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$, $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;
(A4) for each $x \in C, y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemmas.

**Lemma 2.1.** *The following equality holds in a Hilbert space $H$: For $u, v \in H$,*

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v\rangle.$$

**Lemma 2.2.** [1] *Let $C$ be a nonempty closed convex subset of $H$ and let $F$ be a bifunction of $C \times C$ into $R$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \text{ for all } y \in C.$$

**Lemma 2.3.** [2] *Assume that $f : C \times C \to R$ satisfies (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \{z \in C : f(z, y) + \tfrac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C\}$$

*for all $z \in H$. Then, the following hold:*

1. *$T_r$ is single-valued and firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \text{ for any } x, y \in H;$$

2. *$F(T_r) = EP(f)$ ;*
3. *$EP(f)$ is closed and convex.*

**Lemma 2.4.** [8] *Let $H$ be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \{0, 1, 2, \ldots\}$ and let $\{v_n\}$ and $\{w_n\}$ be sequences in $H$. Suppose that there exists $c \geq 0$ such that*

$$\limsup_{n\to\infty} \|v_n\| \leq c, \limsup_{n\to\infty} \|w_n\| \leq c \text{ and } \lim_{n\to\infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c.$$

*Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.*

**Lemma 2.5.** [13] *Let $C$ be a nonempty closed convex subset of $H$. Let $\{x_n\}$ be a sequence in $H$. Suppose that*

$$\|x_{n+1} - y\| \leq \|x_n - y\|, \text{ for all } y \in C \text{ and } n \in \{1, 2, 3, ...\}.$$

*Then $\{P_C x_n\}$ converges strongly to some $z_0 \in C$.*

## 3. Main results

In this section, we prove weak convergence theorems.

**Theorem 3.1.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $A$ be a monotone $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F(S) \cap VI(C,A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by $x_1 = x \in C$ and*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n S x_n + (1-\alpha_n) P_C(u_n - \lambda_n A y_n), \end{cases}$$

*where $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ and $0 < r \leq r_n$. Then $\{x_n\}$ converges weakly to an element $p \in F(S) \cap VI(C,A) \cap EP(f)$, where $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)\cap EP(f)} x_n$.*

**Proof.** Put $v_n = P_C(u_n - \lambda_n A y_n)$ for every $n \in \mathbb{N}$.
Let $x^* \in F(S) \cap VI(C,A) \cap EP(f)$. Then $x^* = Sx^* = P_C(x^* - \lambda_n A x^*) = T_{r_n} x^*$.
From (2.2), we have

$$\begin{aligned} \|v_n - x^*\|^2 &\leq \|u_n - \lambda_n A y_n - x^*\|^2 - \|u_n - \lambda_n A y_n - v_n\| \\ &= \|u_n - x^*\|^2 - \|\lambda_n A y_n\|^2 - 2\langle u_n - \lambda_n A y_n - x^*, \lambda_n A y_n\rangle \\ &\quad - \|u_n - v_n\|^2 + \|\lambda_n A y_n\|^2 + 2\langle u_n - \lambda_n A y_n - v_n, \lambda_n A y_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\langle x^* - v_n, \lambda_n A y_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, x^* - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n(\langle A y_n - A x^*, x^* - y_n\rangle \\ &\quad + \langle A x^*, x^* - y_n\rangle + \langle A y_n, y_n - v_n\rangle) \\ &\leq \|u_n - x^*\|^2 - \|u_n - v_n\|^2 + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - 2\langle u_n - y_n, y_n - v_n\rangle - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n\langle A y_n, y_n - v_n\rangle \\ &= \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle. \end{aligned}$$

From (2.1) and $v_n \in C$, we have

$$\begin{aligned} &\langle u_n - \lambda_n A y_n - y_n, v_n - y_n\rangle \\ &\quad = \langle u_n - \lambda_n A u_n - y_n, v_n - y_n\rangle + \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\ &\quad \leq \langle \lambda_n A u_n - \lambda_n A y_n, v_n - y_n\rangle \\ &\quad \leq \lambda_n k\|u_n - y_n\|\|v_n - y_n\|. \end{aligned}$$

Hence, we have

$$\begin{aligned}\|v_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + 2\lambda_n k\|u_n - y_n\|\|v_n - y_n\| \\ &\le \|u_n - x^*\|^2 - \|u_n - y_n\|^2 - \|y_n - v_n\|^2 \\ &\quad + \lambda_n^2 k^2\|u_n - y_n\|^2 + \|v_n - y_n\|^2 \\ &= \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.\end{aligned}$$

So, we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(Sx_n - x^*) + (1-\alpha_n)(v_n - x^*)\|^2 \\ &\le \alpha_n\|Sx_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.1) \\ &\le \|x_n - x^*\|^2.\end{aligned}$$

Hence $\{\|x_n - x^*\|\}$ is bounded and nonincreasing. So, $\lim_{n\to\infty}\|x_n - x^*\|$ exists.
From (3.1), we obtain also

$$\|u_n - y_n\|^2 \le \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2),$$

and

$$\begin{aligned}\|y_n - v_n\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u_n - \lambda_n Ay_n)\|^2 \\ &\le \|u_n - \lambda_n Au_n - (u_n - \lambda_n Ay_n)\|^2 \\ &= \|\lambda_n Ay_n - \lambda_n Au_n\|^2 \\ &\le \lambda_n^2 k^2\|y_n - u_n\|^2.\end{aligned}$$

Then $\lim_{n\to\infty}\|u_n - y_n\| = 0$ and $\lim_{n\to\infty}\|y_n - v_n\| = 0$.
Consider

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^*\rangle \\ &= -\langle u_n - x^*, x^* - x_n\rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2).\end{aligned}$$

Then $\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|x_n - u_n\|^2$.

We have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|v_n-x^*\|^2\\ &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|u_n-x^*\|^2\\ &\le \alpha_n\|x_n-x^*\|^2+(1-\alpha_n)\|x_n-x^*\|^2-(1-\alpha_n)\|x_n-u_n\|^2.\end{aligned}$$

So, we have $\|x_n-u_n\|^2 \le \dfrac{1}{1-\alpha_n}(\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2)$, which implies that $\lim_{n\to\infty}\|x_n-u_n\|=0$. Since $\|Sv_n-x^*\| \le \|v_n-x^*\| \le \|x_n-x^*\|$, we have that $\limsup_{n\to\infty}\|Sv_n-x^*\| \le \lim_{n\to\infty}\|x_n-x^*\|$.
Further, we have

$$\lim_{n\to\infty}\|\alpha_n(Sx_n-x^*)+(1-\alpha_n)(v_n-x^*)\| = \lim_{n\to\infty}\|x_{n+1}-x^*\|.$$

By Lemma 2.4 , we obtain $\lim_{n\to\infty}\|Sx_n-v_n\|=0$. From $\|Sx_n-x_n\| \le \|Sx_n-v_n\| + \|v_n-y_n\|+\|y_n-u_n\|+\|u_n-x_n\|$, we get $\lim_{n\to\infty}\|Sx_n-x_n\|=0$.
As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p\in C$. Then $v_{n_i} \rightharpoonup p$ $Sv_{n_i} \rightharpoonup p$. Next, to show $p\in F(S)$, consider

$$\begin{aligned}2\|Sv_{n_i}-Sp\|^2 &\le \|Sv_{n_i}-p\|^2+\|v_{n_i}-Sp\|^2\\ &= \|Sv_{n_i}-p\|^2+\|v_{n_i}-Sx_{n_i}\|^2+2\langle v_{n_i}-Sv_{n_i}, Sv_{n_i}-Sp\rangle\\ &\quad+\|Sv_{n_i}-Sp\|^2.\end{aligned}$$

Then $\|Sx_{n_i}-Sp\|^2 \le \|Sx_{n_i}-p\|^2+\|x_{n_i}-Sx_{n_i}\|^2+2\langle x_{n_i}-Sx_{n_i}, Sx_{n_i}-Sp\rangle$.
Suppose $Sp\ne p$, From Opial's theorem [6] and $\lim_{n\to\infty}\|Sx_n-x_n\|=0$, we obtain

$$\begin{aligned}&\liminf_{i\to\infty}\|Sx_{n_i}-p\|^2\\ &\quad< \liminf_{i\to\infty}\|Sx_{n_i}-Sp\|^2\\ &\quad\le \liminf_{i\to\infty}(\|Sx_{n_i}-p\|^2+\|x_{n_i}-Sx_{n_i}\|^2+2\langle x_{n_i}-Sx_{n_i}, Sx_{n_i}-Sp\rangle)\\ &\quad= \liminf_{i\to\infty}\|Sx_{n_i}-p\|^2.\end{aligned}$$

This is a contradiction. Hence $Sp=p$.
Next, to show $p\in VI(C,A)$,
let

$$Tv=\begin{cases}Av+N_Cv, & \text{if } v\in C,\\ \emptyset, & \text{if } v\notin C.\end{cases}$$

Then, $T$ is maximal monotone and $0\in Tv$ if and only if $v\in VI(C,A)$.
Let $(v,w)\in G(T)$. Then,
we have $w\in Tv=Av+N_Cv$ and hence $w-Av\in N_Cv$.
So, we have $\langle v-u, w-Av\rangle \ge 0$ for all $u\in C$.
On the other hand, from $v_n=P_C(u_n-\lambda_nAy_n)$ and $v\in C$, we have $\langle u_n-\lambda_nAy_n-v_n, v_n-v\rangle \ge 0$, and hence, $\langle v-v_n, \dfrac{v_n-u_n}{\lambda_n}+Ay_n\rangle \ge 0$.

Therefore, from $w - Av \in N_C v$ and $v_n \in C$, we have

$$\begin{aligned}\langle v - v_{n_i}, w\rangle &\geq \langle v - v_{n_i}, Av\rangle \\ &\geq \langle v - v_{n_i}, Av\rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\rangle \\ &= \langle v - v_{n_i}, Av - Av_{n_i}\rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle \\ &\quad - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i}\rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Since $x_{n_i} \rightharpoonup p$, $\lim_{n\to\infty} \|v_n - x_n\| = 0$, $\lim_{n\to\infty} \|v_n - u_n\| = 0$, $\lim_{n\to\infty} \|y_n - v_n\| = 0$ and $A$ is Lipschitz continuous. So we obtain $\langle v - p, w\rangle \geq 0$. Since T is maximal monotone, we have $p \in T^{-1}0$ and hence $p \in VI(C, A)$.

Let us show $p \in EP(f)$.

Since $f(u_{n_i}, y) + \frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq 0$ for all $y \in C$.

From (A2), we also have

$$\frac{1}{r_{n_i}}\langle y - u_{n_i}, u_{n_i} - x_{n_i}\rangle \geq f(y, u_{n_i})$$

and hence

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq f(y, u_{n_i}).$$

From $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we get $u_{n_i} \rightharpoonup p$. Since $\frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, it follows by (A4) that $0 \geq f(y, p)$ for all $y \in C$. For $t$ with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)p$. Since $y, p \in C$, we have $y_t \in C$ and hence $f(y_t, p) \leq 0$. So, from (A1) and (A4) we have

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, p) \leq tf(y_t, y)$$

and hence $0 \leq f(y_t, y)$. From (A3), we have $0 \leq f(p, y)$ for all $y \in C$ and hence $p \in EP(f)$. Thus $p \in F(S) \cap VI(C, A) \cap EP(f)$.

Let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p^*$.

Then $p^* \in F(S) \cap VI(C, A) \cap EP(f)$. Assume $p^* \neq p$. Then we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \liminf_{i\to\infty} \|x_{n_i} - p\| \\ &< \liminf_{i\to\infty} \|x_{n_i} - p^*\| \\ &= \lim_{n\to\infty} \|x_n - p^*\| \\ &= \lim_{j\to\infty} \|x_{n_j} - p^*\| \\ &< \lim_{j\to\infty} \|x_{n_j} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\|.\end{aligned}$$

This is a contradiction. Thus $p = p^*$ and $x_n \rightharpoonup p \in F(S) \cap VI(C, A) \cap EP(f)$.

Put $p_n = P_{F(S)\cap VI(C,A)\cap EP(f)}x_n$. We show $p = \lim_{n\to\infty} p_n$.
From $p_n = P_{F(S)\cap VI(C,A)\cap EP(f)}x_n$ and $p \in F(S) \cap VI(C,A) \cap EP(f)$, we have

$$\langle p - p_n, p_n - x_n\rangle \geq 0.$$

By Lemma 2.5, $\{p_n\}$ converges strongly to some $p_0 \in F(S)\cap VI(C,A)\cap EP(f)$. Then we have $\langle p - p_0, p_0 - p\rangle \geq 0$ and hence $p = p_0$. This completes the proof. □

Next, we prove another weak convergence theorem which is different from Theorem 3.1.

**Theorem 3.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $A$ be a monotone and $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F(S) \cap VI(C,A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by $x_1 = x \in C$ and*

$$\begin{cases} f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ y_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(u_n - \lambda_n A y_n), \end{cases}$$

*where $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ and $0 < r \leq r_n$. Then $\{x_n\}$ converges weakly to an element $p \in F(S) \cap VI(C,A) \cap EP(f)$, where $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)\cap EP(f)}x_n$.*

**Proof.** Put $v_n = P_C(u_n - \lambda_n A y_n)$ for every $n \in \mathbb{N}$. Let $x^* \in F(S) \cap VI(C,A) \cap EP(f)$. Then $x^* = Sx^* = P_C(x^* - \lambda_n A x^*) = T_{r_n}x^*$.
As in the proof of Theorem 3.1, we have that

$$\|v_n - x^*\|^2 \leq \|u_n - x^*\|^2 + (\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2.$$

Thus

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(Sv_n - x^*)\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|Sv_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|T_{r_n}x_n - T_{r_n}x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &\leq \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \\ &= \|x_n - x^*\|^2 + (1-\alpha_n)(\lambda_n^2 k^2 - 1)\|u_n - y_n\|^2 \qquad (3.2) \\ &\leq \|x_n - x^*\|^2. \end{aligned}$$

Hence $\{\|x_n - x^*\|\}$ is bounded and nonincreasing. So, $\lim_{n\to\infty}\|x_n - x^*\|$ exists.

From (3.2), we obtain also

$$\|u_n - y_n\|^2 \leq \frac{1}{(1-\alpha_n)(1-\lambda_n^2 k^2)}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2),$$

and

$$\begin{aligned}\|y_n - v_n\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(u_n - \lambda_n A y_n)\|^2 \\ &\leq \|u_n - \lambda_n A u_n - (u_n - \lambda_n A y_n)\|^2 \\ &= \|\lambda_n A y_n - \lambda_n A u_n\|^2 \\ &\leq \lambda_n^2 k^2 \|y_n - u_n\|^2.\end{aligned}$$

Then $\lim_{n\to\infty} \|u_n - y_n\| = 0$ and $\lim_{n\to\infty} \|y_n - v_n\| = 0$.
Consider

$$\begin{aligned}\|u_n - x^*\|^2 &= \|T_{r_n} x_n - T_{r_n} x^*\|^2 \\ &\leq \langle T_{r_n} x_n - T_{r_n} x^*, x_n - x^* \rangle \\ &= -\langle u_n - x^*, x^* - x_n \rangle \\ &= \frac{1}{2}(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|x_n - u_n\|^2).\end{aligned}$$

Then $\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2$.
We have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|v_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\leq \alpha_n \|x_n - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\ &\quad - (1-\alpha_n)\|x_n - u_n\|^2 \\ &= \|x_n - x^*\|^2 - (1-\alpha_n)\|x_n - u_n\|^2.\end{aligned}$$

So, we have $\|x_n - u_n\|^2 \leq \frac{1}{1-\alpha_n}(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2)$,
which implies that $\lim_{n\to\infty} \|x_n - u_n\| = 0$.
Since $\|Sv_n - x^*\| \leq \|v_n - x^*\| \leq \|x_n - x^*\|$,
we have $\limsup_{n\to\infty} \|Sv_n - x^*\| \leq \lim_{n\to\infty} \|x_n - x^*\|$.
Further, we have

$$\lim_{n\to\infty} \|\alpha_n(x_n - x^*) + (1-\alpha_n)(Sv_n - x^*)\| = \lim_{n\to\infty} \|x_{n+1} - x^*\|.$$

By Lemma 2.4 , we obtain $\lim_{n\to\infty} \|Sv_n - x_n\| = 0$. From $\|Sv_n - v_n\| \leq \|Sv_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - v_n\|$, we get $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$.
As $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$ for some $p \in C$. Then $v_{n_i} \rightharpoonup p$ and $Sv_{n_i} \rightharpoonup p$.

Next, to show $p \in F(S)$,
consider

$$\begin{aligned} 2\|Sv_{n_i} - Sp\|^2 &\le \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sp\|^2 \\ &= \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 - 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle \\ &\quad + \|Sv_{n_i} - Sp\|^2. \end{aligned}$$

Then $\|Sv_{n_i} - Sp\|^2 \le \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 - 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle$.
Suppose $Sp \neq p$, From Opial condition and $\lim_{n\to\infty} \|Sv_n - v_n\| = 0$, we obtain

$$\begin{aligned} &\liminf_{i\to\infty} \|Sv_{n_i} - p\|^2 \\ &\quad < \liminf_{i\to\infty} \|Sv_{n_i} - Sp\|^2 \\ &\quad \le \liminf_{i\to\infty} (\|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sv_{n_i}\|^2 - 2\langle v_{n_i} - Sv_{n_i}, Sv_{n_i} - Sp\rangle) \\ &\quad = \liminf_{i\to\infty} \|Sv_{n_i} - p\|^2. \end{aligned}$$

This is a contradiction. Hence $Sp = p$. We can now follow the proof of Theorem 3.1.□

## 4. Applications

Using Theorems 3.1 and 3.2, we prove four theorems in a real Hilbert space.

**Corollary 4.1.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Let $A$ be a monotone and $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) P_C(x_n - \lambda_n A y_n), \end{cases}$$

*where $0 < a \le \lambda_n \le b < \frac{1}{k}$, $0 < c \le \alpha_n \le d < 1$. Then $\{x_n\}$ converges weakly to $p \in F(S) \cap VI(C, A)$, where $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.*

**Proof.** Putting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.1, we obtain the desired result. □

**Corollary 4.2.** *Let $C$ be a closed convex subset of a Hilbert space $H$. Let $A$ be a monotone $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by*

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A y_n), \end{cases}$$

*where* $0 < a \leq \lambda_n \leq b < \dfrac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$. *Then* $\{x_n\}$ *converges weakly to* $p \in F(S) \cap VI(C, A)$, *where* $p = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

**Proof.** Putting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ in Theorem 3.2, we obtain the desired result. □

**Corollary 4.3.** *Let* $C$ *be a closed convex subset of a Hilbert space* $H$. *Let* $f$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and let* $S$ *be a nonspreading mapping of* $C$ *into itself such that* $F(S) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* $C$ *generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) T_{r_n} x_n, \end{cases}$$

*where* $0 < c \leq \alpha_n \leq d < 1$ *and* $0 < r \leq r_n$. *Then* $\{x_n\}$ *converges weakly to* $p \in F(S) \cap EP(f)$, *where* $p = \lim_{n\to\infty} P_{F(S)\cap EP(f)} x_n$.

**Proof.** Putting $A = 0$ in Theorem 3.1, we obtain the desired result. □

**Corollary 4.4.** *Let* $C$ *be a closed convex subset of a Hilbert space* $H$. *Let* $f$ *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying (A1)-(A4) and let* $S$ *be a nonspreading mapping of* $C$ *into itself such that* $F(S) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* $C$ *generated by*

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S T_{r_n} x_n, \end{cases}$$

*where* $0 < c \leq \alpha_n \leq d < 1$ *and* $0 < r \leq r_n$. *Then* $\{x_n\}$ *converges weakly to* $p \in F(S) \cap EP(f)$, *where* $p = \lim_{n\to\infty} P_{F(S)\cap EP(f)} x_n$.

**Proof.** Putting $A = 0$ in Theorem 3.2, we obtain the desired result. □

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