WEAK CONVERGENCE THEOREMS FOR EQUILIBRIUM PROBLEMS WITH NONLINEAR OPERATORS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative sequence for finding a common element of the set of fixed points of a nonspreading mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality problem for a monotone and Lipschitz-continuous mapping. We show that the sequence converges weakly to a common element of the above three sets.

Key Words and Phrases: Nonspreading mappings, monotone, Lipschitz-continuous mappings, variational inequalities, fixed points.

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Let f be a bifunction of C × C into R, where R is the set of real numbers. The equilibrium problem for

\[ f(x, y) \geq 0 \text{ for all } y \in C. \] (1.1)

The set of solutions (1.1) is denoted by EP(f). A mapping A of C into H is called monotone if \( \langle Au - Av, u - v \rangle \geq 0 \) for all \( u, v \in C \). The variational inequality problem is to find \( u \in C \) such that \( \langle Au, v - u \rangle \geq 0 \) for all \( v \in C \). The set of solutions of the variational inequality problem is denoted by VI(C, A).

A mapping A of C into H is called \( \alpha \)-inverse-strongly monotone if there exists a positive real number \( \alpha \) such that \( \langle Au - Av, u - v \rangle \geq \alpha \|Au - Av\|^2 \) for all \( u, v \in C \). It is obvious that any \( \alpha \)-inverse-strongly monotone mapping A is monotone and Lipschitz

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continuous; see, for example, [13]. A mapping \( S \) of \( C \) into itself is called nonexpansive if \( \|Su - Sv\| \leq \|u - v\| \) for all \( u, v \in C \). A mapping \( S \) of \( C \) into itself is called nonspreading (see [3, 4]) if
\[
2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2,
\]
for all \( u, v \in C \).

We denote by \( F(S) \) the set of fixed points of \( S \). Recently, in the case when \( S \) is a nonexpansive mapping, Nadezhkina and Takahashi [5] introduced an iterative process for finding a common element of the set \( F(S) \) and the set \( VI(C, A) \) for a monotone and Lipschitz-continuous mapping. On the other hand, Tada and Takahashi [9, 10] and Takahashi and Takahashi [11] obtained weak and strong convergence theorems for finding a common element of the set \( EP(f) \) and the set \( F(S) \) in a Hilbert space.

In this paper, we prove weak convergence theorems for finding a common element of the set \( EP(f) \), the set \( VI(C, A) \) for a monotone and Lipschitz-continuous mapping and the set \( F(S) \) of a nonspreading mapping in a Hilbert space.

## 2. Preliminaries

In this paper, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{R} \) the set of real numbers. Let \( H \) be a real Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). \( x_n \to x \) implies that \( \{x_n\} \) converges strongly to \( x \). \( x_n \rightharpoonup x \) means that \( \{x_n\} \) converges weakly to \( x \). In a real Hilbert space \( H \), we have
\[
\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2
\]
for all \( x, y \in H \) and \( \lambda \in \mathbb{R} \); see [12]. Let \( C \) be a closed convex subset of \( H \). Then, for every point \( x \in H \), there exists a unique nearest point in \( C \), denoted by \( P_Cx \), such that \( \|x - P_Cx\| \leq \|x - y\| \) for all \( y \in C \). \( P_C \) is called the metric projection of \( H \) onto \( C \). We know that \( P_C \) is a nonexpansive mapping of \( H \) onto \( C \). It is also known that \( P_C \) is characterized by the following properties: \( P_Cx \in C \),
\[
\langle x - P_Cx, P_Cx - y \rangle \geq 0 \quad (2.1)
\]
and
\[
\|x - y\|^2 \geq \|x - P_Cx\|^2 + \|y - P_Cx\|^2 \quad (2.2)
\]
for all \( x \in H, y \in C \). Let \( A \) be a monotone mapping of \( C \) into \( H \). In the context of the variational inequality problem, this implies
\[
u \in VI(C, A) \iff u = P_C(u - \lambda Au)
\]
for all \( \lambda > 0 \). It is also known that \( H \) satisfies the Opial condition [6]; i.e., for any sequence \( \{x_n\} \) with \( x_n \rightharpoonup x \), the inequality
\[
\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|
\]
holds for every \( y \in H \) with \( y \neq x \). We also know that \( H \) has the Kadec-Klee property, that is, \( x_n \rightharpoonup x \) and \( \|x_n\| \to \|x\| \) imply \( x_n \to x \). In fact, from
\[
\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x \rangle + \|x\|^2,
\]
we get that a Hilbert space has the Kadec-Klee property. An operator \( A : H \to 2^H \) is said to be monotone if \( \langle x_1 - x_2, y_1 - y_2 \rangle \geq 0 \) whenever \( y_1 \in Ax_1 \) and \( y_2 \in Ax_2 \).

Let \( A \) be a monotone, \( k \)-Lipschitz-continuous mapping of \( C \) into \( H \) and let \( N_{Cv} \) be the normal cone to \( C \) at \( v \in C \); i.e., \( N_{Cv} = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C \} \). Define
\[ T_v = \begin{cases} \mathcal{A}v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \]

Then, \( T \) is maximal monotone and \( 0 \in T_v \) if and only if \( v \in VI(C, A) \); see [7].

For solving the equilibrium problem for a bifunction \( f : C \times C \to \mathbb{R} \), let us assume that \( f \) satisfies the following conditions:

(A1) \( f(x, x) = 0 \) for all \( x \in C \);

(A2) \( f \) is monotone, i.e., \( f(x, y) + f(y, x) \leq 0 \) for all \( x, y \in C \);

(A3) for each \( x, y, z \in C \), \( \lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y) \);

(A4) for each \( x \in C, y \mapsto f(x, y) \) is convex and lower semicontinuous.

We know the following lemmas.

**Lemma 2.1.** The following equality holds in a Hilbert space \( H \): For \( u, v \in H \),

\[ \|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle. \]

**Lemma 2.2.** [1] Let \( C \) be a nonempty closed convex subset of \( H \) and let \( F \) be a bifunction of \( C \times C \) into \( \mathbb{R} \) satisfying (A1)-(A4). Let \( r > 0 \) and \( x \in H \). Then, there exists \( z \in C \) such that

\[ f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \quad \text{for all } y \in C. \]

**Lemma 2.3.** [2] Assume that \( f : C \times C \to \mathbb{R} \) satisfies (A1)–(A4). For \( r > 0 \) and \( x \in H \), define a mapping \( T_r : H \to C \) as follows:

\[ T_r(x) = \{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \} \]

for all \( z \in H \). Then, the following hold:

1. \( T_r \) is single-valued and firmly nonexpansive, i.e.,
   \[ \|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle, \quad \text{for any } x, y \in H; \]

2. \( F(T_r) = EP(f) \);

3. \( EP(f) \) is closed and convex.

**Lemma 2.4.** [8] Let \( H \) be a real Hilbert space, let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 < a \leq \alpha_n \leq b < 1 \) for all \( n \in \{0, 1, 2, \ldots\} \) and let \( \{v_n\} \) and \( \{w_n\} \) be sequences in \( H \). Suppose that there exists \( c \geq 0 \) such that

\[ \limsup_{n \to \infty} \|v_n\| \leq c, \limsup_{n \to \infty} \|w_n\| \leq c \quad \text{and} \quad \lim_{n \to \infty} \|\alpha_n v_n + (1 - \alpha_n)w_n\| = c. \]

Then \( \lim_{n \to \infty} \|v_n - w_n\| = 0. \)

**Lemma 2.5.** [13] Let \( C \) be a nonempty closed convex subset of \( H \). Let \( \{x_n\} \) be a sequence in \( H \). Suppose that

\[ \|x_{n+1} - y\| \leq \|x_n - y\|, \quad \text{for all } y \in C \quad \text{and} \quad n \in \{1, 2, 3, \ldots\}. \]

Then \( \{P_C x_n\} \) converges strongly to some \( z_0 \in C \).
3. Main results

In this section, we prove weak convergence theorems.

**Theorem 3.1.** Let $C$ be a closed convex subset of a Hilbert space $H$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $A$ be a monotone $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspreading mapping of $C$ into itself such that $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by $x_1 = x \in C$ and

$$
\begin{align*}
& \left\{
\begin{array}{l}
  f(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, \forall y \in C, \\
  y_n = P_C(u_n - \lambda_n Au_n), \\
  x_{n+1} = \alpha_n Sx_n + (1 - \alpha_n)P_C(u_n - \lambda_n Ay_n),
\end{array}
\right.
\end{align*}
$$

where $0 < a \leq \lambda_n \leq b < \frac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ and $0 < r \leq r_n$. Then $\{x_n\}$ converges weakly to an element $p \in F(S) \cap VI(C, A) \cap EP(f)$, where $p = \lim_{n \to \infty} P_{F(S) \cap VI(C, A) \cap EP(f)}x_n$.

**Proof.** Put $v_n = P_C(u_n - \lambda_n Ay_n)$ for every $n \in \mathbb{N}$. Let $x^* \in F(S) \cap VI(C, A) \cap EP(f)$. Then $x^* = Sx^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n}x^*$.

From (2.2), we have

$$
\begin{align*}
||v_n - x^*||^2 & \leq ||u_n - \lambda_n Ay_n - x^*||^2 - ||u_n - \lambda_n Ay_n - v_n|| \\
& = ||u_n - x^*||^2 - ||\lambda_n Ay_n||^2 - 2\langle u_n - \lambda_n Ay_n - x^*, \lambda_n Ay_n \rangle \\
& \quad - ||u_n - v_n||^2 + ||\lambda_n Ay_n||^2 + 2\langle u_n - \lambda_n Ay_n - v_n, \lambda_n Ay_n \rangle \\
& = ||u_n - x^*||^2 - ||u_n - v_n||^2 + 2\langle x^* - v_n, \lambda_n Ay_n \rangle \\
& = ||u_n - x^*||^2 - ||u_n - v_n||^2 + 2\lambda_n (A y_n, x^* - v_n) \\
& = ||u_n - x^*||^2 - ||u_n - v_n||^2 + 2\lambda_n (Ax^* - y_n) + \langle Ax^*, x^* - y_n \rangle + \langle Ay_n, y_n - v_n \rangle \\
& \leq ||u_n - x^*||^2 - ||u_n - v_n||^2 + 2\lambda_n (Ay_n, y_n - v_n) \\
& = ||u_n - x^*||^2 - ||u_n - y_n||^2 - 2\langle u_n - y_n, y_n - v_n \rangle - ||y_n - v_n||^2 \\
& \quad + 2\lambda_n (Ay_n, y_n - v_n) \\
& = ||u_n - x^*||^2 - ||u_n - y_n||^2 - ||y_n - v_n||^2 \\
& \quad + 2\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n \rangle.
\end{align*}
$$

From (2.1) and $v_n \in C$, we have

$$
\begin{align*}
\langle u_n - \lambda_n Ay_n - y_n, v_n - y_n \rangle \\
& = \langle u_n - \lambda_n Au_n - y_n, v_n - y_n \rangle + \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n \rangle \\
& \leq \langle \lambda_n Au_n - \lambda_n Ay_n, v_n - y_n \rangle \\
& \leq \lambda_n k ||u_n - y_n|| ||v_n - y_n||.
\end{align*}
$$
Hence, we have
\[||v_n - x^*||^2 \leq ||u_n - x^*||^2 - ||u_n - y_n||^2 - ||y_n - v_n||^2\]
\[+ 2\lambda_n k||u_n - y_n|| ||v_n - y_n||\]
\[\leq ||u_n - x^*||^2 - ||u_n - y_n||^2 - ||y_n - v_n||^2\]
\[+ \lambda_n^2 k^2||u_n - y_n||^2 + ||v_n - y_n||^2\]
\[= ||u_n - x^*||^2 + (\lambda_n^2 k^2 - 1)||u_n - y_n||^2.\]

Then, for (3.1), we obtain also
\[\lim_{n \to \infty} ||u_n - x^*||^2 = \alpha_n(Sx_n - x^*) + (1 - \alpha_n)(v_n - x^*)||^2\]
\[\leq \alpha_n||Sx_n - x^*||^2 + (1 - \alpha_n)||v_n - x^*||^2\]
\[\leq \alpha_n||x_n - x^*||^2 + (1 - \alpha_n)||u_n - x^*||^2\]
\[+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1)||u_n - y_n||^2\]
\[= \alpha_n||x_n - x^*||^2 + (1 - \alpha_n)||T_{\alpha_n}x_n - T_{\alpha_n}x^*||^2\]
\[+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1)||u_n - y_n||^2\]
\[\leq \alpha_n||x_n - x^*||^2 + (1 - \alpha_n)||x_n - x^*||^2\]
\[+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1)||u_n - y_n||^2\]
\[= ||x_n - x^*||^2 + (1 - \alpha_n)(\lambda_n^2 k^2 - 1)||u_n - y_n||^2\] (3.1)
\[\leq ||x_n - x^*||^2.\]

Hence \(||x_n - x^*||\) is bounded and nonincreasing. So, \(\lim_{n \to \infty} ||x_n - x^*||\) exists.

From (3.1), we obtain also
\[||u_n - y_n|| \leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2 k^2)}(||x_n - x^*||^2 - ||x_{n+1} - x^*||^2),\]

and
\[||y_n - v_n||^2 = ||P_C(u_n - \lambda_nAu_n) - P_C(u_n - \lambda_nAy_n)||^2\]
\[\leq ||u_n - \lambda_nAu_n - (u_n - \lambda_nAy_n)||^2\]
\[= ||\lambda_nAy_n - \lambda_nAu_n||^2\]
\[\leq \lambda_n^2 k^2||y_n - u_n||^2.\]

Then \(\lim_{n \to \infty} ||u_n - y_n|| = 0\) and \(\lim_{n \to \infty} ||y_n - v_n|| = 0\).

Consider
\[||u_n - x^*||^2 = ||T_{\alpha_n}x_n - T_{\alpha_n}x^*||^2\]
\[\leq \langle T_{\alpha_n}x_n - T_{\alpha_n}x^*, x_n - x^* \rangle\]
\[= -\langle u_n - x^*, x^* - x_n \rangle\]
\[= \frac{1}{2}(||u_n - x^*||^2 + ||x_n - x^*||^2 - ||x_n - u_n||^2).\]

Then \(||u_n - x^*||^2 \leq ||x_n - x^*||^2 - ||x_n - u_n||^2||.\)
We have
\[ \|x_{n+1} - x^*\|^2 \leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n)\|v_n - x^*\|^2 \]
\[ \leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n)\|u_n - x^*\|^2 \]
\[ \leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 - (1 - \alpha_n)\|x_n - u_n\|^2. \]

So, we have \( \|x_n - u_n\|^2 \leq \frac{1}{1 - \alpha_n} (\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2) \), which implies that
\[ \lim_{n \to \infty} \|x_n - u_n\| = 0. \]
Since \( \|Sv_n - x^*\| \leq \|v_n - x^*\| \leq \|x_n - x^*\| \), we have that
\[ \limsup_{n \to \infty} \|Sv_n - x^*\| \leq \lim_{n \to \infty} \|x_n - x^*\|. \]

Further, we have
\[ \lim_{n \to \infty} \|\alpha_n(Sx_n - x^*) + (1 - \alpha_n)(v_n - x^*)\| = \lim_{n \to \infty} \|x_{n+1} - x^*\|. \]

By Lemma 2.4, we obtain \( \lim_{n \to \infty} \|Sx_n - v_n\| = 0. \) From \( \|Sx_n - x_n\| \leq \|Sx_n - v_n\| + \|v_n - y_n\| + \|y_n - u_n\| + \|u_n - x_n\| \), we get \( \lim_{n \to \infty} \|Sx_n - x_n\| = 0. \)

As \{x_n\} is bounded, there exists a subsequence \{x_{n_i}\} of \{x_n\} such that \( x_{n_i} \to p \) for some \( p \in C. \) Then \( v_{n_i} \to p \) \( Sx_{n_i} \to p. \)

Next, to show \( p \in F(S) \), consider
\[ 2\|Sv_{n_i} - Sp\|^2 \leq \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sp\|^2 \]
\[ = \|Sv_{n_i} - p\|^2 + \|v_{n_i} - Sx_{n_i}\|^2 + 2(v_{n_i} - Sv_{n_i}, v_{n_i} - Sp) + \|Sv_{n_i} - Sp\|^2. \]

Then \( \|Sx_{n_i} - Sp\|^2 \leq \|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sx_{n_i}\|^2 + 2(v_{n_i} - Sx_{n_i}, Sx_{n_i} - Sp). \)

Suppose \( Sp \neq p. \) From Opial’s theorem [6] and \( \lim_{n \to \infty} \|Sx_n - x_n\| = 0, \) we obtain
\[ \liminf_{i \to \infty} \|Sx_{n_i} - p\|^2 < \liminf_{i \to \infty} \|Sx_{n_i} - Sp\|^2 \]
\[ \leq \liminf_{i \to \infty} (\|Sx_{n_i} - p\|^2 + \|x_{n_i} - Sx_{n_i}\|^2 + 2(v_{n_i} - Sx_{n_i}, Sx_{n_i} - Sp)) \]
\[ = \liminf_{i \to \infty} \|Sx_{n_i} - p\|^2. \]

This is a contradiction. Hence \( Sp = p. \)

Next, to show \( p \in VI(C, A), \)
let
\[ Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \]

Then, \( T \) is maximal monotone and \( 0 \in Tv \) if and only if \( v \in VI(C, A). \)
Let \( (v, w) \in G(T). \) Then, we have \( w \in Tv = Av + N_C v \) and hence \( w - Av \in N_C v. \)

So, we have \( (v - u, w - Av) \geq 0 \) for all \( u \in C. \)

On the other hand, from \( v_{n_i} = P_C(u_{n_i} - \lambda_n A y_{n_i}) \) and \( v \in C, \) we have
\[ \langle u_n - \lambda_n A y_n - v_n, v_n - v \rangle \geq 0, \] and hence, \( \langle v - v_n, \frac{v_n - u_n}{\lambda_n} + A y_n \rangle \geq 0. \)
Therefore, from \( w - Av \in NCv \) and \( v_n \in C \), we have
\[
\langle v - v_{n_i}, w \rangle \geq \langle v - v_{n_i}, Av \rangle
\]
\[
\geq \langle v - v_{n_i}, Av \rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i} \rangle
\]
\[
= \langle v - v_{n_i}, Av - Av_{n_i} \rangle + \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle
\]
\[
\geq \langle v - v_{n_i}, Av_{n_i} - Ay_{n_i} \rangle - \langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \rangle.
\]
Since \( x_{n_i} \to p \), \( \lim_{n \to \infty} \|v_n - x_n\| = 0 \), \( \lim_{n \to \infty} \|v_n - u_n\| = 0 \), \( \lim_{n \to \infty} \|y_n - v_n\| = 0 \) and \( A \) is Lipschitz continuous. So we obtain \( \langle v - p, w \rangle \geq 0 \). Since \( T \) is maximal monotone, we have \( p \in T^{-1}0 \) and hence \( p \in VI(C, A) \).

Let us show \( p \in EP(f) \).

Since \( f(u_{n_i}, y) + \frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq 0 \) for all \( y \in C \).

From (A2), we also have
\[
\frac{1}{r_{n_i}} \langle y - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq f(y, u_{n_i})
\]
and hence
\[
\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \rangle \geq f(y, u_{n_i}).
\]

From \( \lim_{n \to \infty} \|u_n - x_n\| = 0 \), we get \( u_{n_i} \to p \). Since \( \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \to 0 \), it follows by (A4) that \( 0 \geq f(y, p) \) for all \( y \in C \). For \( t \) with \( 0 < t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1 - t)p \). Since \( y, p \in C \), we have \( y_t \in C \) and hence \( f(y_t, p) \leq 0 \). So, from (A1) and (A4) we have
\[
0 = f(y_t, y_t) \leq tf(y_t, y) + (1 - t)f(y_t, p) \leq tf(y_t, y)
\]
and hence \( 0 \leq f(y_t, y) \). From (A3), we have \( 0 \leq f(p, y) \) for all \( y \in C \) and hence \( p \in EP(f) \). Thus \( p \in F(S) \cap VI(C, A) \cap EP(f) \).

Let \( \{x_{n_j}\} \) be another subsequence of \( \{x_n\} \) such that \( x_{n_j} \to p^* \).

Then \( p^* \in F(S) \cap VI(C, A) \cap EP(f) \). Assume \( p^* \neq p \). Then we have
\[
\lim_{n \to \infty} \|x_n - p\| = \liminf_{i \to \infty} \|x_{n_i} - p\|
\]
\[
< \liminf_{i \to \infty} \|x_{n_i} - p^*\|
\]
\[
= \lim_{n \to \infty} \|x_n - p^*\|
\]
\[
= \lim_{j \to \infty} \|x_{n_j} - p^*\|
\]
\[
< \lim_{j \to \infty} \|x_{n_j} - p\|
\]
\[
= \lim_{n \to \infty} \|x_n - p\|.
\]

This is a contradiction. Thus \( p = p^* \) and \( x_n \to p \in F(S) \cap VI(C, A) \cap EP(f) \).
Put $p_n = P_{F(S) \cap VI(C, A) \cap EP(f)} x_n$. We show $p = \lim_{n \to \infty} p_n$.

From $p_n = P_{F(S) \cap VI(C, A) \cap EP(f)} x_n$ and $p \in F(S) \cap VI(C, A) \cap EP(f)$, we have

$$|p - p_n| \geq 0.$$  

By Lemma 2.5, $\{p_n\}$ converges strongly to some $p_0 \in F(S) \cap VI(C, A) \cap EP(f)$. Then we have $(p - p_0, p_0 - p) \geq 0$ and hence $p = p_0$. This completes the proof.  

Next, we prove another weak convergence theorem which is different from Theorem 3.1.

**Theorem 3.2.** Let $C$ be a closed convex subset of a Hilbert space $H$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $A$ be a monotone and $k$-Lipschitz continuous mapping of $C$ into $H$ and let $S$ be a nonspraying mapping of $C$ into itself such that $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in $C$ generated by $x_1 = x \in C$ and

$$\begin{cases} f(u_n, y) + \frac{1}{r_n} (y - u_n, u_n - x_n) \geq 0, \forall y \in C, \\ y_n = P_C(u_n - \lambda_n Au_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(u_n - \lambda_n Ay_n), \end{cases}$$  

where $0 < a \leq \lambda_n \leq b < \frac{1}{k}$, $0 < c \leq \alpha_n \leq d < 1$ and $0 < r \leq r_n$. Then $\{x_n\}$ converges weakly to an element $p \in F(S) \cap VI(C, A) \cap EP(f)$, where $p = \lim_{n \to \infty} P_{F(S) \cap VI(C, A) \cap EP(f)} x_n$.

**Proof.** Put $v_n = P_C(u_n - \lambda_n Ay_n)$ for every $n \in \mathbb{N}$. Let $x^* \in F(S) \cap VI(C, A) \cap EP(f)$. Then $x^* = Sx^* = P_C(x^* - \lambda_n Ax^*) = T_{r_n} x^*$.

As in the proof of Theorem 3.1, we have that

$$||v_n - x^*||^2 \leq ||u_n - x^*||^2 + (\lambda_n^2 k^2 - 1) ||u_n - y_n||^2.$$  

Thus

$$||x_{n+1} - x^*||^2 = ||\alpha_n(x_n - x^*) + (1 - \alpha_n)(Sv_n - x^*)||^2$$

$$\leq \alpha_n ||x_n - x^*||^2 + (1 - \alpha_n)||Sv_n - x^*||^2$$

$$\leq \alpha_n ||x_n - x^*||^2 + (1 - \alpha_n)||v_n - x^*||^2$$

$$\leq \alpha_n ||x_n - x^*||^2 + (1 - \alpha_n)||u_n - x^*||^2$$

$$+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1) ||u_n - y_n||^2$$

$$= \alpha_n ||x_n - x^*||^2 + (1 - \alpha_n)||T_{r_n} x_n - T_{r_n} x^*||^2$$

$$+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1) ||u_n - y_n||^2$$

$$\leq \alpha_n ||x_n - x^*||^2 + (1 - \alpha_n)||x_n - x^*||^2$$

$$+ (1 - \alpha_n)(\lambda_n^2 k^2 - 1) ||u_n - y_n||^2$$

$$= ||x_n - x^*||^2 + (1 - \alpha_n)(\lambda_n^2 k^2 - 1) ||u_n - y_n||^2$$

$$\leq ||x_n - x^*||^2.$$  

Hence $\{||x_n - x^*||\}$ is bounded and nonincreasing. So, $\lim_{n \to \infty} ||x_n - x^*||$ exists.
From (3.2), we obtain also
\[
\| u_n - y_n \|^2 \leq \frac{1}{(1 - \alpha_n)(1 - \lambda_n^2k^2)} (\| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2),
\]
and
\[
\| y_n - v_n \|^2 = \| P_C(u_n - \lambda_n Au_n) - P_C(u_n - \lambda_n Ay_n) \|^2
\leq \| u_n - \lambda_n Au_n - (u_n - \lambda_n Ay_n) \|^2
\leq \| \lambda_n Ay_n - \lambda_n Au_n \|^2
\leq \lambda_n^2k^2 \| y_n - u_n \|^2.
\]
Then \( \lim_{n \to \infty} \| u_n - y_n \| = 0 \) and \( \lim_{n \to \infty} \| y_n - v_n \| = 0 \).

Consider
\[
\| u_n - x^* \|^2 = \| T_{\alpha_n}x_n - T_{\alpha_n}x^* \|^2
\leq \{ T_{\alpha_n}x_n - T_{\alpha_n}x^*, x_n - x^* \}
= -\langle u_n - x^*, x^* - x_n \rangle
= \frac{1}{2}(\| u_n - x^* \|^2 + \| x_n - x^* \|^2 - \| x_n - u_n \|^2).
\]
Then \( \| u_n - x^* \|^2 \leq \| x_n - x^* \|^2 - \| x_n - u_n \|^2 \).

We have
\[
\| x_{n+1} - x^* \|^2 \leq \alpha_n \| x_n - x^* \|^2 + (1 - \alpha_n) \| v_n - x^* \|^2
\leq \alpha_n \| x_n - x^* \|^2 + (1 - \alpha_n) \| x_n - x^* \|^2
\leq \alpha_n \| x_n - x^* \|^2 + (1 - \alpha_n) \| x_n - x^* \|^2
- (1 - \alpha_n) \| x_n - u_n \|^2
\leq \| x_n - x^* \|^2 - (1 - \alpha_n) \| x_n - u_n \|^2.
\]
So, we have \( \| x_n - u_n \|^2 \leq \frac{1}{1 - \alpha_n} (\| x_n - x^* \|^2 - \| x_{n+1} - x^* \|^2) \),
which implies that \( \lim_{n \to \infty} \| x_n - u_n \| = 0 \).

Since \( \| Sv_n - x^* \| \leq \| v_n - x^* \| + \| x_n - x^* \| \)
we have \( \limsup_{n \to \infty} \| Sv_n - x^* \| \leq \lim_{n \to \infty} \| x_n - x^* \| \).

Further, we have
\[
\lim_{n \to \infty} \| \alpha_n(x_n - x^*) + (1 - \alpha_n)(Sv_n - x^*) \| = \lim_{n \to \infty} \| x_{n+1} - x^* \|.
\]
By Lemma 2.4, we obtain \( \lim_{n \to \infty} \| Sv_n - x_n \| = 0 \). From \( \| Sv_n - v_n \| \leq \| Sv_n - x_n \| + \| x_n - u_n \| + \| u_n - y_n \| + \| y_n - v_n \| \), we get \( \lim_{n \to \infty} \| Sv_n - v_n \| = 0 \).

As \( \{ x_n \} \) is bounded, there exists a subsequence \( \{ x_{n_i} \} \) of \( \{ x_n \} \) such that \( x_{n_i} \to p \) for some \( p \in C \). Then \( v_{n_i} \to p \) and \( Sv_{n_i} \to p \).
Next, to show that the desired result.

Corollary 4.2. Let \( C \) be a closed convex subset of a Hilbert space \( H \). Let \( A \) be a monotone and \( k \)-Lipschitz continuous mapping of \( C \) into \( H \) and let \( S \) be a nonspreading continuous mapping of \( C \) into itself such that \( F(S) \cap VI(C,A) \neq \emptyset \). Let \( \{x_n\} \) be a sequence in \( C \) generated by

\[
\begin{cases}
  x_1 = x \in C, \\
  y_n = P_C(x_n - \lambda_n Ax_n), \\
  x_{n+1} = \alpha_n Sx_n + (1 - \alpha_n)P_C(x_n - \lambda_n Ay_n),
\end{cases}
\]

where \( 0 < a \leq \lambda_n \leq b < \frac{1}{k}, \ 0 < c \leq \alpha_n \leq d < 1 \). Then \( \{x_n\} \) converges weakly to \( p \in F(S) \cap VI(C,A) \), where \( p = \lim_{n \to \infty} P_{F(S) \cap VI(C,A)}x_n \).

Proof. Putting \( f(x,y) = 0 \) for all \( x,y \in C \) and \( r_n = 1 \) in Theorem 3.1, we obtain the desired result.

Corollary 4.2. Let \( C \) be a closed convex subset of a Hilbert space \( H \). Let \( A \) be a monotone \( k \)-Lipschitz continuous mapping of \( C \) into \( H \) and let \( S \) be a nonspreading continuous mapping of \( C \) into itself such that \( F(S) \cap VI(C,A) \cap EP(f) \neq \emptyset \). Let \( \{x_n\} \) be a sequence in \( C \) generated by

\[
\begin{cases}
  x_1 = x \in C, \\
  y_n = P_C(x_n - \lambda_n Ax_n), \\
  x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ay_n),
\end{cases}
\]
where \(0 < a \leq \lambda_n \leq b < \frac{1}{k}, 0 < c \leq \alpha_n \leq d < 1\). Then \(\{x_n\}\) converges weakly to \(p \in F(S) \cap VI(C,A)\), where \(p = \lim_{n \to \infty} P_{F(S) \cap VI(C,A)}x_n\).

**Proof.** Putting \(f(x,y) = 0\) for all \(x, y \in C\) and \(r_n = 1\) in Theorem 3.2, we obtain the desired result. \(\square\)

**Corollary 4.3.** Let \(C\) be a closed convex subset of a Hilbert space \(H\). Let \(f\) be a bifunction from \(C \times C\) to \(\mathbb{R}\) satisfying (A1)-(A4) and let \(S\) be a nonspreading mapping of \(C\) into itself such that \(F(S) \cap EP(f) \neq \emptyset\). Let \(\{x_n\}\) be a sequence in \(C\) generated by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= \alpha_n Sx_n + (1 - \alpha_n)T_{r_n}x_n,
\end{align*}
\]

where \(0 < c \leq \alpha_n \leq d < 1\) and \(0 < r \leq r_n\). Then \(\{x_n\}\) converges weakly to \(p \in F(S) \cap EP(f)\), where \(p = \lim_{n \to \infty} P_{F(S) \cap EP(f)}x_n\).

**Proof.** Putting \(A = 0\) in Theorem 3.1, we obtain the desired result. \(\square\)

**Corollary 4.4.** Let \(C\) be a closed convex subset of a Hilbert space \(H\). Let \(f\) be a bifunction from \(C \times C\) to \(\mathbb{R}\) satisfying (A1)-(A4) and let \(S\) be a nonspreading mapping of \(C\) into itself such that \(F(S) \cap EP(f) \neq \emptyset\). Let \(\{x_n\}\) be a sequence in \(C\) generated by

\[
\begin{align*}
x_1 &= x \in C, \\
x_{n+1} &= \alpha_n x_n + (1 - \alpha_n)ST_{r_n}x_n,
\end{align*}
\]

where \(0 < c \leq \alpha_n \leq d < 1\) and \(0 < r \leq r_n\). Then \(\{x_n\}\) converges weakly to \(p \in F(S) \cap EP(f)\), where \(p = \lim_{n \to \infty} P_{F(S) \cap EP(f)}x_n\).

**Proof.** Putting \(A = 0\) in Theorem 3.2, we obtain the desired result. \(\square\)

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**References**


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