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ITERATIVE METHODS FOR GENERALIZED EQUILIBRIUM PROBLEMS, SYSTEMS OF GENERAL GENERALIZED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS FOR NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we introduce a system of general generalized equilibrium problems and propose an iterative scheme for finding the approximate solutions of a generalized equilibrium problem, a system of general generalized equilibrium problems and a fixed point problem of a nonexpansive mapping in a Hilbert space. We establish a strong convergence theorem for a sequence generated by our proposed iterative scheme to a common solution of these three problems. Utilizing this result, we prove three new strong convergence theorems for sequences generated by iterative schemes for fixed point problems, variational inequalities, equilibrium problems and systems of general generalized equilibrium problems.
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1. Introduction

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle ., . \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $S : C \to C$ be a mapping. We denote by $F(S)$ the set of all fixed points of S.

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Very recently, Takahashi and Takahashi [15] introduced and considered the following *generalized equilibrium problem*: Find $\bar{x} \in C$ such that

$$F(\bar{x}, y) + \langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction and $A : C \to H$ is a nonlinear mapping. The set of solutions of generalized equilibrium problem is denoted by EP, that is,

$$EP = \{z \in C : F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C\}.$$

Whenever $A \equiv 0$, generalized equilibrium problem reduces to the *equilibrium problem* of finding $\bar{x} \in C$ such that

$$F(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{1.1a}$$

In this case, EP is denoted by $EP(F)$.

Whenever $F \equiv 0$, problem (1.1) reduces to the *classical variational inequality*, denoted by $\mathrm{VI}(A, C)$, which is to find an $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.1b}$$

In this case, EP is denoted by $VI(C, A)$, that is, the set of all solutions of $\mathrm{VI}(A, C)$. The solution methods for computing the approximate solutions of variational inequalities have been widely studied in the literature; See, for example, [4, 3, 10, 11, 12, 13, 14, 16, 24] and the references therein. The problem (1.1) is an unified frame of several problems, namely, optimization problems, saddle point problems, complementarity problems, fixed point problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, etc; See, for example, [20, 21].

In the recent past, much attention has been paid by several researchers to study the iterative methods for finding an element of $EP(F) \cap F(S)$; See, for example, [7, 8, 18, 19, 23, 26] and references therein. Moudafi [22] introduced an iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is an inverse-strongly monotone mapping and proved a weak convergence theorem. Motivated by Moudafi [22], Takahashi and Takahashi [15] introduced another iterative method for finding an element of $EP \cap F(S)$, where $A : C \to H$ is also an inverse-strongly monotone mapping and then obtained a strong convergence theorem.

On the other hand, let C be a nonempty closed convex subset of a real Hilbert space H. Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions and $B_1, B_2 : C \to H$ be two nonlinear mappings. Consider the following problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} G_1(\bar{x}, x) + \langle B_1\bar{y}, x - \bar{x}\rangle + \frac{1}{\mu_1}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \ \forall x \in C, \\ G_2(\bar{y}, y) + \langle B_2\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu_2}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \ \forall y \in C. \end{cases} \tag{1.2}$$

It is called *general system of generalized equilibrium problems* where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

Special Cases. (1) If $G_1 \equiv G_2 \equiv F$ and $B_1 \equiv B_2 \equiv A$, then problem (1.2) reduces to the following problem of finding $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} F(\bar{x}, x) + \langle A\bar{y}, x - \bar{x}\rangle + \frac{1}{\mu_1}\langle \bar{x} - \bar{y}, x - \bar{x}\rangle \geq 0, \ \forall x \in C, \\ F(\bar{y}, y) + \langle A\bar{x}, y - \bar{y}\rangle + \frac{1}{\mu_2}\langle \bar{y} - \bar{x}, y - \bar{y}\rangle \geq 0, \ \forall y \in C. \end{cases} \tag{1.3}$$

It is called a *new system of generalized equilibrium problems* where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants.

(2) If $G_1 \equiv G_2 \equiv F$, $B_1 \equiv B_2 \equiv A$, and $\bar{x} \equiv \bar{y}$, then problem (1.2) reduces to problem (1.1).

(3) If $G_1 \equiv G_2 \equiv 0$, then problem (1.2) reduces to the following *general system of variational inequalities*: Find $(\bar{x}, \bar{y}) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 \bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, \ \forall x \in C, \\ \langle \mu_2 B_2 \bar{x} + \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, \ \forall y \in C, \end{cases} \tag{1.4}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are two constants. This problem is introduced and considered by Ceng et. al. [16]. They proposed a relaxed extragradient method for finding the solutions of problem (1.4), and derived a strong convergence theorem for problem (1.4). The problem (1.4) was introduced and studied by Ansari and Yao [25] for an infinite number of inequalities. They proved the existence of a solution of such problem. They also considered a more general system of generalized variational inequalities and proved the existence of its solution. By using such existence result for a solution, they provided the existence of a solution of Nash equilibrium problem for nondifferentiable and nonconvex functions.

(4) If $B_1 = B_2 = A$ in (1.4), then problem (1.4) reduces to the following *new system of variational inequalities*:

$$\begin{cases} \langle \mu_1 A \bar{y} + \bar{x} - \bar{y}, x - \bar{x} \rangle \geq 0, \ \forall x \in C, \\ \langle \mu_2 A \bar{x} + \bar{y} - \bar{x}, y - \bar{y} \rangle \geq 0, \ \forall y \in C, \end{cases} \tag{1.5}$$

which is considered and studied by Verma [3].

(5) If $\bar{x} = \bar{y}$ in (1.5), then problem (1.5) reduces to the *classical variational inequality* (1.1b).

Inspired by the work of Takahashi and Takahashi [15] and Ceng et. al. [16], we introduce a modified iterative method for problem (1.1), problem (1.2) and fixed point problem for S, where $A, B_1, B_2 : C \to H$ are inverse-strongly monotone mappings. We establish a strong convergence theorem for a sequence generated by proposed iterative scheme to a solution of problem (1.2). Utilizing this theorem, we derived three new strong convergence results for (i) problem (1.1a), problem (1.2) and the fixed point problem of S; (ii) problem (1.1b), problem (1.2) and the fixed point problem of S; and (iii) problem (1.1), problem (1.2) and the fixed point problem of S, where $A \equiv I - T$ and $T : C \to C$ is a strictly pseudocontractive mapping and I is the identity mapping on C.

2. Preliminaries

Throughout the paper, unless otherwise specified, C is a nonempty closed convex subset of a real Hilbert space H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. If $\{x_n\}$ converges strongly to x, we denote it by $x_n \to x$. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The operator $P_C : H \to C$ is called the *metric projection* of H onto C. It is well known that P_C is a firmly nonexpansive mapping of H onto C, that is, $\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2, \quad \forall x,y\in H$. Recall that, P_Cx is characterized, for all $x\in H$ and $y\in C$, by the following properties:

$$P_Cx\in C,\ \langle x-P_Cx, y-P_Cx\rangle \leq 0 \text{ and } \|x-y\|^2 \geq \|x-P_Cx\|^2+\|P_Cx-y\|^2, \quad (2.1)$$

For further detail, we refer to Goebel and Kirk [2].

A mapping $S: C\to C$ is called *nonexpansive* if $\|Sx-Sy\|\leq\|x-y\|, \quad \forall x,y\in C$. It is well known that the set $F(S)$ of fixed points of S is closed and convex if the mapping S is nonexpansive. Further, if C is bounded, closed and convex, then $F(S)$ is nonempty. A mapping $A: C\to H$ is called *inverse-strongly monotone* if there exists $\alpha>0$ such that $\langle x-y, Ax-Ay\rangle\geq\alpha\|Ax-Ay\|^2, \quad \forall x,y\in C$. It is well known that $A\equiv I-S$ is inverse-strongly monotone with constant $\frac{1}{2}$ if $S: C\to C$ is nonexpansive and I is the identity mapping on C; See, for example, [9] for further details.

We need the following propositions and lemmas for the proof of our main result.

Lemma 2.1. Let $T: C\to H$ be a firmly nonexpansive mapping. Then,

$$\|(x-y)-(Tx-Ty)\|^2\leq\|x-y\|^2-\|Tx-Ty\|^2, \quad \forall x,y\in C.$$

Let $F: C\times C\to\mathbb{R}$ be a bifunction.

Condition A.

(A1) $F(x,x)=0$ for all $x\in C$;

(A2) F is monotone, that is, $F(x,y)+F(y,x)\leq 0$ for all $x,y\in C$;

(A3) $\lim\limits_{t\to 0^+} F(tz+(1-t)x,y)\leq F(x,y)$ for all $x,y,z\in C$;

(A4) For each fixed $x\in C$, $y\mapsto F(x,y)$ is a convex and lower semicontinuous function.

Lemma 2.2. [Lemma 2.2 in [15]] *Let $F: C\times C\to\mathbb{R}$ be a bifunction such that Condition A holds. Then, for any $r>0$ and $x\in H$, there exists $z\in C$ such that*

$$F(z,y)+\frac{1}{r}\langle y-z, z-x\rangle\geq 0, \quad \forall y\in C.$$

Furthermore, if $T_r^F x=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\}$, then,

(1) T_r^F is a single-valued map;

(2) T_r^F is firmly nonexpansive, that is,

$$\left\|T_r^Fx-T_r^Fy\right\|^2\leq\langle T_r^Fx-T_r^Fy, x-y\rangle, \quad \forall x,y\in H;$$

(3) $F(T_r^F)=EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 2.3. [Lemma 2.3 in [15]] Let F and T_r^Fx be the same as in Lemma 2.2. Then,

$$\left\|T_s^Fx-T_t^Fx\right\|^2\leq\frac{s-t}{s}\left\langle T_s^Fx-T_t^Fx, T_s^Fx-x\right\rangle, \quad \forall s,t>0 \text{ and } \forall x\in H.$$

Proposition 2.1. *Let $G_1, G_2: C\times C\to\mathbb{R}$ be two bifunctions such that Condition A holds. Then $(\bar{x},\bar{y})\in C\times C$ is a solution of problem (1.2) if and only if $\bar{x}$ is a fixed point of the mapping $\Gamma: C\to C$ defined by*

$$\Gamma(x)=T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(x-\mu_2B_2x)-\mu_1B_1T_{\mu_2}^{G_2}(x-\mu_2B_2x)\right] \text{ and } \bar{y}=T_{\mu_2}^{G_2}(\bar{x}-\mu_2B_2\bar{x}).$$

Corollary 2.1. [Lemma 2.1 in [16]] *The point $(\bar{x}, \bar{y}) \in C \times C$ is a solution of problem (1.4) if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\left[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\right] \text{ and } \bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x}).$$

Proposition 2.2. [6] *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.4. [Lemma 2.1 in [10]] *Let $\{\gamma_n\} \subseteq (0,1)$ and $\{\delta_n\}$ be sequences such that the following conditions hold:*

(i) $\displaystyle\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\displaystyle\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\displaystyle\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \le (1-\gamma_n)a_n + \delta_n, \quad \forall n \ge 1.$$

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. (Demi-closedness Principle, see [2]) *Let $T : C \to C$ be a nonexpansive mapping. If T has a fixed point, then $I - T$ is demi-closed, that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$, and the sequence $\{(I-T)x_n\}$ converges strongly to some y, it follows that $(I-T)x = y$, where I is the identity mapping on H.*

Lemma 2.6. *In a real Hilbert space H, the following inequality holds:*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

3. Main Results

We present an iterative scheme to find the approximate solutions of (1.2). We prove the strong convergence of a sequence generated by our iterative scheme to a solution of (1.2).

Theorem 3.1. *Let $F, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions such that Condition A holds. Let the mappings $A, B_1, B_2 : C \to H$ be inverse-strongly monotone with constants α, β_1, β_2, respectively, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap EP \cap \mho \ne \emptyset$, where $\mho$ is the set of all fixed points of the mapping $\Gamma : C \to C$ defined as $\Gamma(x) = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(x - \mu_2 B_2 x) - \mu_1 B_1 T_{\mu_2}^{G_2}(x - \mu_2 B_2 x)]$. Let*

$u \in C$, $x_1 \in C$ and let $\{x_n\} \subset C$ be a sequence generated by the following iterative scheme:

$$\begin{cases} z_n = T_{\lambda_n}^F(x_n - \lambda_n A x_n), \\ y_n = T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S\left[\alpha_n u + (1-\alpha_n) y_n\right], \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$, and $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions $0 < c \le \beta_n \le d < 1$, $\quad 0 < a \le \lambda_n \le b < 2\alpha$, $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$, $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP\cap \mho} u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.2), where $\bar{y} = T_{\mu_2}^{G_2}(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. Let $z \in F(S) \cap EP \cap \mho$ be an arbitrary point. Since $z = T_{\lambda_n}^F(z - \lambda_n A z)$, A is α-inverse-strongly monotone and $0 \le \lambda_n \le 2\alpha$, we have, for any $n \in \mathbb{N}$,

$$\|z_n - z\|^2 = \left\|T_{\lambda_n}^F(x_n - \lambda_n A x_n) - T_{\lambda_n}^F(z - \lambda_n A z)\right\|^2 \le \|(x_n - \lambda_n A x_n) - (z - \lambda_n A z)\|^2$$

$$= \|(x_n - z) - \lambda_n(Ax_n - Az)\|^2 = \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + \lambda_n^2\|Ax_n - Az\|^2$$

$$\le \|x_n - z\|^2 - 2\lambda_n\alpha\|Ax_n - Az\|^2 + \lambda_n^2\|Ax_n - Az\|^2 = \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Az\|^2$$

$$\le \|x_n - z\|^2. \tag{3.2}$$

Also, since $z = T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(z - \mu_2 B_2 z) - \mu_1 B_1 T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right]$, B_1 and B_2 are inverse-strongly monotone with constant β_1 and β_2, respectively, $0 \le \mu_1 \le 2\beta_1$ and $0 \le \mu_2 \le 2\beta_2$, we have, for any $n \in \mathbb{N}$,

$$\begin{aligned} \|y_n - z\|^2 &= \Big\|T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)\right] \\ &\quad - T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(z - \mu_2 B_2 z) - \mu_1 B_1 T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right]\Big\|^2 \\ &\le \Big\|\left[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)\right] \\ &\quad - \left[T_{\mu_2}^{G_2}(z - \mu_2 B_2 z) - \mu_1 B_1 T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right]\Big\|^2 \\ &= \Big\|\left[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right] \\ &\quad - \mu_1\left[B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right]\Big\|^2 \\ &\le \left\|T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right\|^2 \\ &\quad + \mu_1(\mu_1 - 2\beta_1)\left\|B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - B_1 T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right\|^2 \\ &\le \left\|T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\right\|^2 \end{aligned}$$

Thus,

$$\begin{aligned} \|y_n - z\|^2 &\le \|(z_n - \mu_2 B_2 z_n) - (z - \mu_2 B_2 z)\|^2 = \|(z_n - z) - \mu_2(B_2 z_n - B_2 z)\|^2 \\ &\le \|z_n - z\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 z_n - B_2 z\|^2 \le \|z_n - z\|^2. \end{aligned} \tag{3.3}$$

Let $t_n = \alpha_n u + (1-\alpha_n) y_n$, then by using (3.2) and (3.3), we obtain

$$\|t_n - z\| = \|\alpha_n(u - z) + (1-\alpha_n)(y_n - z)\| \le \alpha_n\|u - z\| + (1-\alpha_n)\|y_n - z\|$$

$$\le \alpha_n\|u - z\| + (1-\alpha_n)\|z_n - z\| \le \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|.$$

So, we have

$$\begin{aligned}\|x_{n+1}-z\| &= \|\beta_n(x_n-z)+(1-\beta_n)(St_n-z)\|\\ &\le \beta_n\|x_n-z\|+(1-\beta_n)\|t_n-z\|\\ &\le \beta_n\|x_n-z\|+(1-\beta_n)\left(\alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|\right)\\ &= (1-\alpha_n(1-\beta_n))\|x_n-z\|+\alpha_n(1-\beta_n)\|u-z\|.\end{aligned}$$

Letting $K=\max\{\|x_1-z\|,\|u-z\|\}$, we have $\|x_n-z\|\le K$ for all $n\in\mathbb{N}$.

Indeed, it is obvious that $\|x_1-z\|\le K$. Suppose that $\|x_k-z\|\le K$ for some $k\in\mathbb{N}$. Then, we have

$$\begin{aligned}\|x_{k+1}-z\| &\le (1-\alpha_k(1-\beta_k))\|x_k-z\|+\alpha_k(1-\beta_k)\|u-z\|\\ &\le (1-\alpha_k(1-\beta_k))K+\alpha_k(1-\beta_k)K=K.\end{aligned}$$

By induction, we obtain $\|x_n-z\|\le K \quad$ for all $n\in\mathbb{N}$. So, $\{x_n\}$ is bounded. Hence, $\{Ax_n\}$, $\{y_n\}$, $\{z_n\}$, $\{t_n\}$ and $\{St_n\}$ are also bounded. Let $u_n=T^{G_2}_{\mu_2}(z_n-\mu_2B_2z_n)$. Then, we obtain

$$t_{n+1}-t_n=\alpha_{n+1}u+(1-\alpha_{n+1})y_{n+1}-(\alpha_nu+(1-\alpha_n)y_n)$$

$$=(\alpha_{n+1}-\alpha_n)u+(1-\alpha_{n+1})T^{G_1}_{\mu_1}(u_{n+1}-\mu_1B_1u_{n+1})-(1-\alpha_n)T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)$$

$$=(\alpha_{n+1}-\alpha_n)u+(1-\alpha_{n+1})\left[T^{G_1}_{\mu_1}(u_{n+1}-\mu_1B_1u_{n+1})-T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\right]$$

$$+(\alpha_n-\alpha_{n+1})T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n). \tag{3.4}$$

$$\begin{aligned}&\left\|T^{G_1}_{\mu_1}(u_{n+1}-\mu_1B_1u_{n+1})-T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\right\|^2\\ &\le \left\|(u_{n+1}-\mu_1B_1u_{n+1})-(u_n-\mu_1B_1u_n)\right\|^2\\ &= \left\|(u_{n+1}-u_n)-\mu_1(B_1u_{n+1}-B_1u_n)\right\|^2\\ &\le \|u_{n+1}-u_n\|^2+\mu_1(\mu_1-2\beta_1)\left\|B_1u_{n+1}-B_1u_n\right\|^2\\ &\le \|u_{n+1}-u_n\|^2\\ &= \left\|T^{G_2}_{\mu_2}(z_{n+1}-\mu_2B_2z_{n+1})-T^{G_2}_{\mu_2}(z_n-\mu_2B_2z_n)\right\|^2\\ &\le \left\|(z_{n+1}-\mu_2B_2z_{n+1})-(z_n-\mu_2B_2z_n)\right\|^2\\ &= \left\|(z_{n+1}-z_n)-\mu_2(B_2z_{n+1}-B_2z_n)\right\|^2\\ &\le \|z_{n+1}-z_n\|^2+\mu_2(\mu_2-2\beta_2)\left\|B_2z_{n+1}-B_2z_n\right\|^2\\ &\le \|z_{n+1}-z_n\|^2,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&\|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_nAx_n)\|\\ &= \|x_{n+1}-x_n-\lambda_{n+1}(Ax_{n+1}-Ax_n)+(\lambda_n-\lambda_{n+1})Ax_n\|\\ &\le \|x_{n+1}-x_n-\lambda_{n+1}(Ax_{n+1}-Ax_n)\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|\\ &\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|z_{n+1}-z_n\| &= \left\|T^F_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}Ax_{n+1})-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\| \\ &= \left\|T^F_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}Ax_{n+1})-T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)\right. \\ &\quad \left.+T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\| \\ &\le \left\|T^F_{\lambda_{n+1}}(x_{n+1}-\lambda_{n+1}Ax_{n+1})-T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)\right\| \\ &\quad +\left\|T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\| \\ &\le \|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_nAx_n)\| \\ &\quad +\left\|T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\| \\ &\le \|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &\quad +\left\|T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\|.\end{aligned} \tag{3.7}$$

From (3.4)–(3.7), it follows that

$$\begin{aligned}\|t_{n+1}-t_n\| &\le |\alpha_{n+1}-\alpha_n|\,\|u\|+\left\|T^{G_1}_{\mu_1}(u_{n+1}-\mu_1B_1u_{n+1})-T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\right\| \\ &\quad +|\alpha_n-\alpha_{n+1}|\,\left\|T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\right\| \\ &\le |\alpha_{n+1}-\alpha_n|\,\|u\|+\|z_{n+1}-z_n\|+|\alpha_n-\alpha_{n+1}|\,\left\|T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\right\| \\ &\le |\alpha_{n+1}-\alpha_n|\,\|u\|+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\| \\ &\quad +|\alpha_{n+1}-\alpha_n|\|T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\| \\ &\quad +\left\|T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\|.\end{aligned}$$

Therefore, we have:
$\|St_{n+1}-St_n\| \le \|t_{n+1}-t_n\| \le |\alpha_{n+1}-\alpha_n|\|u\|+\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|+|\alpha_{n+1}-\alpha_n|\|T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\|+\left\|T^F_{\lambda_{n+1}}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(x_n-\lambda_nAx_n)\right\|.$

Since $\{z_n\}$ is bounded, B_1 and B_2 are Lipschitz continuous, and $T^{G_1}_{\mu_1}$ and $T^{G_2}_{\mu_2}$ are firmly nonexpansive, we conclude that $\{u_n\}$ is bounded and so is $\{T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\}$. It follows from Lemma 2.3 that $\limsup_{n\to\infty}(\|St_{n+1}-St_n\|-\|x_{n+1}-x_n\|)\le 0$. From Lemma 2.1, we get

$$\|St_n-x_n\|\to 0. \tag{3.8}$$

Consequently, we obtain $\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|St_n-x_n\|=0$. Using (3.2) and (3.3), we get

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(St_n-z)\|^2 \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|St_n-z\|^2 \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|t_n-z\|^2 \\ &= \beta_n\|x_n-z\|^2+(1-\beta_n)\|\alpha_n(u-z)+(1-\alpha_n)(y_n-z)\|^2 \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)(\alpha_n\|u-z\|^2+(1-\alpha_n)\|y_n-z\|^2) \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)(\alpha_n\|u-z\|^2+(1-\alpha_n)\|z_n-z\|^2) \\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)[\alpha_n\|u-z\|^2+(1-\alpha_n)(\|x_n-z\|^2 \\ &\quad +\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2)] \\ &\le \|x_n-z\|^2+(1-\beta_n)\alpha_n\|u-z\|^2 \\ &\quad +(1-\beta_n)(1-\alpha_n)\lambda_n(\lambda_n-2\alpha)\|Ax_n-Az\|^2\end{aligned} \tag{3.9}$$

and hence

$$\begin{aligned}
&(1-d)(1-\alpha_n)a(2\alpha-b)\|Ax_n-Az\|^2\\
&\le (1-\beta_n)(1-\alpha_n)\lambda_n(2\alpha-\lambda_n)\|Ax_n-Az\|^2\\
&\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+(1-\beta_n)\alpha_n\|u-z\|^2\\
&= (\|x_n-z\|-\|x_{n+1}-z\|)(\|x_n-z\|+\|x_{n+1}-z\|)+(1-\beta_n)\alpha_n\|u-z\|^2\\
&\le \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+(1-\beta_n)\alpha_n\|u-z\|^2.
\end{aligned}$$

Since $0<c\le\beta_n\le d<1$, $\alpha_n\to 0$ and $\|x_n-x_{n+1}\|\to 0$, we have

$$\|Ax_n-Az\|\to 0. \tag{3.10}$$

Using Lemma 2.2 and (3.3), we obtain

$$\begin{aligned}
\|z_n-z\|^2 &= \left\|T^F_{\lambda_n}(x_n-\lambda_nAx_n)-T^F_{\lambda_n}(z-\lambda_nAz)\right\|^2\\
&\le \langle (x_n-\lambda_nAx_n)-(z-\lambda_nAz), z_n-z\rangle\\
&= \tfrac{1}{2}(\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)\|^2+\|z_n-z\|^2\\
&\quad -\|(x_n-\lambda_nAx_n)-(z-\lambda_nAz)-(z_n-z)\|^2)\\
&\le \tfrac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|(x_n-z_n)-\lambda_n(Ax_n-Az)\|^2)\\
&= \tfrac{1}{2}(\|x_n-z\|^2+\|z_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\langle x_n-z_n, Ax_n-Az\rangle\\
&\quad -\lambda_n^2\|Ax_n-Az\|^2).
\end{aligned}$$

So, we have

$$\|z_n-z\|^2\le\|x_n-z\|^2-\|x_n-z_n\|^2+2\lambda_n\langle x_n-z_n, Ax_n-Az\rangle-\lambda_n^2\|Ax_n-Az\|^2. \tag{3.11}$$

From (3.3), (3.9) and (3.11), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\beta_n(x_n-z)+(1-\beta_n)(St_n-z)\|^2\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)(\alpha_n\|u-z\|^2+(1-\alpha_n)\|y_n-z\|^2)\\
&\le \beta_n\|x_n-z\|^2+(1-\beta_n)(\alpha_n\|u-z\|^2+(1-\alpha_n)\|z_n-z\|^2)\\
&\le \beta_n\|x_n-z\|^2+\alpha_n\|u-z\|^2+(1-\beta_n)\|z_n-z\|^2\\
&\le \beta_n\|x_n-z\|^2+\alpha_n\|u-z\|^2+(1-\beta_n)(\|x_n-z\|^2-\|x_n-z_n\|^2\\
&\quad +2\lambda_n\langle x_n-z_n, Ax_n-Az\rangle-\lambda_n^2\|Ax_n-Az\|^2)\\
&\le \|x_n-z\|^2+\alpha_n\|u-z\|^2-(1-\beta_n)\|x_n-z_n\|^2\\
&\quad +2(1-\beta_n)\lambda_n\|x_n-z_n\|\|Ax_n-Az\|
\end{aligned}$$

and hence

$$\begin{aligned}
(1-d)\|x_n-z_n\|^2 &\le (1-\beta_n)\|x_n-z_n\|^2\\
&\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|u-z\|^2\\
&\quad +2(1-\beta_n)\lambda_n\|x_n-z_n\|\|Ax_n-Az\|\\
&\le \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+\alpha_n\|u-z\|^2\\
&\quad +2(1-\beta_n)\lambda_n\|x_n-z_n\|\|Ax_n-Az\|.
\end{aligned}$$

Since $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, by using (3.10), we obtain

$$\|x_n-z_n\|\to 0. \tag{3.12}$$

Since $t_n=\alpha_nu+(1-\alpha_n)y_n$, we have $t_n-y_n=\alpha_n(u-y_n)$ and hence

$$\|t_n-y_n\|=\alpha_n\|u-y_n\|\to 0. \tag{3.13}$$

On the other hand, by putting $u^* = T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)$, we observe that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(St_n - z)\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|t_n - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\alpha_n\|u - z\|^2 + (1-\alpha_n)\|y_n - z\|^2] \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|T_{\mu_1}^{G_1}(u_n - \mu_1 B_1 u_n) - T_{\mu_1}^{G_1}(u^* - \mu_1 B_1 u^*)\|^2 \\
&\quad +\alpha_n\|u - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|(u_n - \mu_1 B_1 u_n) - (u^* - \mu_1 B_1 u^*)\|^2 \\
&\quad +\alpha_n\|u - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|u_n - u^*\|^2 + \mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] \\
&\quad +\alpha_n\|u - z\|^2 \\
&= \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\|^2 \\
&\quad +\mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] + \alpha_n\|u - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|(z_n - \mu_2 B_2 z_n) - (z - \mu_2 B_2 z)\|^2 \\
&\quad +\mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] + \alpha_n\|u - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|z_n - z\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 z_n - B_2 z\|^2 \\
&\quad +\mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] + \alpha_n\|u - z\|^2 \\
&\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|x_n - z\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 z_n - B_2 z\|^2 \\
&\quad +\mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] + \alpha_n\|u - z\|^2 \\
&= \|x_n - z\|^2 + (1-\beta_n)[\mu_2(\mu_2 - 2\beta_2)\|B_2 z_n - B_2 z\|^2 \\
&\quad +\mu_1(\mu_1 - 2\beta_1)\|B_1 u_n - B_1 u^*\|^2] + \alpha_n\|u - z\|^2,
\end{aligned} \tag{3.14}$$

and hence

$$\begin{aligned}
&(1-d)\left[\mu_1(2\beta_1 - \mu_1)\|B_1 u_n - B_1 u^*\|^2 + \mu_2(2\beta_2 - \mu_2)\|B_2 z_n - B_2 z\|^2\right] \\
&\leq (1-\beta_n)\left[\mu_1(2\beta_1 - \mu_1)\|B_1 u_n - B_1 u^*\|^2 + \mu_2(\mu_2 - 2\beta_2)\|B_2 z_n - B_2 z\|^2\right] \\
&\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|u - z\|^2 \\
&\leq \|x_n - x_{n+1}\|(\|x_n - z\| + \|x_{n+1} - z\|) + \alpha_n\|u - z\|^2.
\end{aligned} \tag{3.15}$$

Since $\|x_n - x_{n+1}\| \to 0$, $\alpha_n \to 0$, $\mu_1 \in (0, 2\beta_1)$ and $\mu_2 \in (0, 2\beta_2)$, we obtain from (3.15) that $\|B_1 u_n - B_1 u^*\| \to 0$ and $\|B_2 z_n - B_2 z\| \to 0$.
By using Lemma 2.2, we obtain

$$\begin{aligned}
\|u_n - u^*\|^2 &= \|T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - T_{\mu_2}^{G_2}(z - \mu_2 B_2 z)\|^2 \\
&\leq \langle (z_n - \mu_2 B_2 z_n) - (z - \mu_2 B_2 z), u_n - u^* \rangle \\
&= \tfrac{1}{2}[\|(z_n - \mu_2 B_2 z_n) - (z - \mu_2 B_2 z)\|^2 + \|u_n - u^*\|^2 \\
&\quad -\|(z_n - \mu_2 B_2 z_n) - (z - \mu_2 B_2 z) - (u_n - u^*)\|^2] \\
&\leq \tfrac{1}{2}[\|z_n - z\|^2 + \|u_n - u^*\|^2 \\
&\quad -\|(z_n - u_n) - \mu_2(B_2 z_n - B_2 z) - (z - u^*)\|^2] \\
&= \tfrac{1}{2}[\|z_n - z\|^2 + \|u_n - u^*\|^2 - \|(z_n - u_n) - (z - u^*)\|^2 \\
&\quad +2\mu_2\langle (z_n - u_n) - (z - u^*), B_2 z_n - B_2 z \rangle - \mu_2^2\|B_2 z_n - B_2 z\|^2].
\end{aligned}$$

So, we get that

$$\begin{aligned}
&\|u_n - u^*\|^2 \leq \\
&\|z_n - z\|^2 - \|(z_n - u_n) - (z - u^*)\|^2 + 2\mu_2\langle (z_n - u_n) - (z - u^*), B_2 z_n - B_2 z \rangle - \mu_2^2\|B_2 z_n - B_2 z\|^2 \\
&\leq \|z_n - z\|^2 - \|(z_n - u_n) - (z - u^*)\|^2 + 2\mu_2\langle (z_n - u_n) - (z - u^*), B_2 z_n - B_2 z \rangle.
\end{aligned} \tag{3.16}$$

Hence, from (3.14) and (3.16) it follows that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \beta_n\|x_n-z\|^2+(1-\beta_n)[\|u_n-u^*\|^2+\mu_1(\mu_1-2\beta_1)\|B_1u_n-B_1u^*\|^2]\\ &\quad +\alpha_n\|u-z\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)\|u_n-u^*\|^2+\alpha_n\|u-z\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)[\|z_n-z\|^2-\|(z_n-u_n)-(z-u^*)\|^2\\ &\quad +2\mu_2\langle(z_n-u_n)-(z-u^*),B_2z_n-B_2z\rangle]+\alpha_n\|u-z\|^2\\ &\le \beta_n\|x_n-z\|^2+(1-\beta_n)[\|x_n-z\|^2-\|(z_n-u_n)-(z-u^*)\|^2\\ &\quad +2\mu_2\|(z_n-u_n)-(z-u^*)\|\|B_2z_n-B_2z\|]+\alpha_n\|u-z\|^2\\ &= \|x_n-z\|^2-(1-\beta_n)\|(z_n-u_n)-(z-u^*)\|^2\\ &\quad +2(1-\beta_n)\mu_2\|(z_n-u_n)-(z-u^*)\|\|B_2z_n-B_2z\|+\alpha_n\|u-z\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}(1-d)\|(z_n-u_n)-(z-u^*)\|^2 &\le (1-\beta_n)\|(z_n-u_n)-(z-u^*)\|^2\\ &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+\alpha_n\|u-z\|^2\\ &\quad +2(1-\beta_n)\mu_2\|(z_n-u_n)-(z-u^*)\|\|B_2z_n-B_2z\|\\ &\le \|x_n-x_{n+1}\|(\|x_n-z\|+\|x_{n+1}-z\|)+\alpha_n\|u-z\|^2\\ &\quad +2\mu_2\|(z_n-u_n)-(z-u^*)\|\|B_2z_n-B_2z\|.\end{aligned}$$

Since $\|x_n-x_{n+1}\|\to 0$, $\alpha_n\to 0$ and $\|B_2z_n-B_2z\|\to 0$, we have

$$\|(z_n-u_n)-(z-u^*)\|\to 0. \tag{3.17}$$

Now, utilizing Lemma 2.6 and the firm nonexpansivity of $T^{G_1}_{\mu_1}$ we have

$$\begin{aligned}&\|(u_n-y_n)+(z-u^*)\|^2\\ &=\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)-[T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)\\ &\quad -T^{G_1}_{\mu_1}(u^*-\mu_1B_1u^*)]+\mu_1(B_1u_n-B_1u^*)\|^2\\ &\le \left\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)-\left[T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)-T^{G_1}_{\mu_1}(u^*-\mu_1B_1u^*)\right]\right\|^2\\ &\quad +2\mu_1\langle B_1u_n-B_1u^*,(u_n-y_n)+(z-u^*)\rangle\\ &\le \|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|^2-\left\|T^{G_1}_{\mu_1}(u_n-\mu_1B_1u_n)-T^{G_1}_{\mu_1}(u^*-\mu_1B_1u^*)\right\|^2\\ &\quad +2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|\\ &=\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|^2-\|y_n-z\|^2\\ &\quad +2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|\\ &\le \|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|^2-\|Sy_n-z\|^2\\ &\quad +2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|\\ &=\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|^2-\|St_n-z\|^2+\|St_n-z\|^2-\|Sy_n-z\|^2\\ &\quad +2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|\\ &\le \|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)-(St_n-z)\|\\ &\quad \times(\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|+\|St_n-z\|)\\ &\quad +\|St_n-Sy_n\|(\|St_n-z\|+\|Sy_n-z\|)+2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|\\ &=\|z_n-x_n+x_n-St_n+z-u^*-(z_n-u_n)-\mu_1(B_1u_n-B_1u^*)\|\\ &\quad \times(\|u_n-\mu_1B_1u_n-(u^*-\mu_1B_1u^*)\|+\|St_n-z\|)\\ &\quad +\|St_n-Sy_n\|(\|St_n-z\|+\|Sy_n-z\|)+2\mu_1\|B_1u_n-B_1u^*\|\|(u_n-y_n)+(z-u^*)\|.\end{aligned} \tag{3.18}$$

Since $\|z_n - x_n\| \to 0$, $\|x_n - St_n\| \to 0$, $\|(z_n - u_n) - (z - u^*)\| \to 0$, $\|B_1 u_n - B_1 u^*\| \to 0$ and $\|St_n - Sy_n\| \to 0$, it follows from (3.18) that $\|(u_n - y_n) + (z - u^*)\| \to 0$.
We also observe that

$$\|St_n - t_n\| \leq \|St_n - x_n\| + \|x_n - z_n\| + \|(z_n - u_n) - (z - u^*)\| + \|(u_n - y_n) + (z - u^*)\| + \|y_n - t_n\|.$$

Thus, we get

$$\|St_n - t_n\| \to 0. \tag{3.19}$$

Next, putting $\bar{x} = P_{F(S)\cap EP\cap\mho} u$, we claim that

$$\limsup_{n\to\infty} \langle u - \bar{x}, t_n - \bar{x}\rangle \leq 0. \tag{3.20}$$

To show inequality (3.20) holds, we consider a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\limsup_{n\to\infty} \langle u - \bar{x}, t_n - \bar{x}\rangle = \lim_{i\to\infty} \langle u - \bar{x}, t_{n_i} - \bar{x}\rangle. \tag{3.21}$$

Without loss of generality, we may assume that $t_{n_i} \rightharpoonup w$. Since C is closed and convex, C is weakly closed. So, we have $w \in C$. Let us show $w \in F(S) \cap EP \cap \mho$. We first show $w \in EP$. Note that $\|St_n - t_n\| \to 0$, $\|St_n - x_n\| \to 0$ and $\|x_n - z_n\| \to 0$. Hence it follows that $\|t_n - z_n\| \leq \|t_n - St_n\| + \|St_n - x_n\| + \|x_n - z_n\| \to 0$, and so $\|t_n - z_n\| \to 0$. Consequently, we have $z_{n_i} \rightharpoonup w$. Since $z_n = T^F_{\lambda_n}(x_n - \lambda_n A x_n)$, for any $y \in C$ we have

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0.$$

From (A2), we obtain $\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq F(y, z_n)$. Replacing n by n_i, we get

$$\langle Ax_{n_i}, y - z_{n_i}\rangle + \left\langle y - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \geq F(y, z_{n_i}). \tag{3.22}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0, 1]$ and $y \in C$, we have $z_t \in C$. So, from (3.22) we obtain

$$\begin{aligned}
&\langle z_t - z_{n_i}, Az_t\rangle \geq \\
&\langle z_t - z_{n_i}, Az_t\rangle - \langle z_t - z_{n_i}, Ax_{n_i}\rangle - \left\langle z_t - z_{n_i}, \tfrac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(z_t, z_{n_i}) = \\
&\langle z_t - z_{n_i}, Az_t - Az_{n_i}\rangle + \langle z_t - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \left\langle z_t - z_{n_i}, \tfrac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(z_t, z_{n_i}).
\end{aligned}$$

Since $\|z_{n_i} - x_{n_i}\| \to 0$, we have $\|Az_{n_i} - Ax_{n_i}\| \to 0$. Further, from the monotonicity of A, we have $\langle z_t - z_{n_i}, Az_t - Az_{n_i}\rangle \geq 0$. So, from (A4), as $i \to \infty$, we have

$$\langle z_t - w, Az_t\rangle \geq F(z_t, w), \tag{3.23}$$

From (A1), (A4) and (3.23), we have

$$\begin{aligned}
0 &= F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, w) \leq tF(z_t, y) + (1-t)\langle z_t - w, Az_t\rangle \\
&= tF(z_t, y) + (1-t)t\langle y - w, Az_t\rangle
\end{aligned}$$

and hence $0 \leq F(z_t, y) + (1-t)\langle y - w, Az_t\rangle$. Letting $t \to 0$, we have, for each $y \in C$, $0 \leq F(w, y) + \langle y - w, Aw\rangle$. This implies that $w \in EP$.

Now, we show that $w \in F(S)$. Indeed, since $t_{n_i} \rightharpoonup w$ and $\|St_n - t_n\| \to 0$ due to (3.19), utilizing Lemma 2.5 we have $(I - S)w = 0$ and hence $w \in F(S)$.

Next, we show that $w \in \mho$. Indeed, utilizing Lemma 2.2 we have for all $x, y \in C$

$$\begin{aligned}
\|\Gamma(x)-\Gamma(y)\|^2 &= \|T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(x-\mu_2B_2x)-\mu_1B_1T_{\mu_2}^{G_2}(x-\mu_2B_2x)] \\
&\quad -T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(y-\mu_2B_2y)-\mu_1B_1T_{\mu_2}^{G_2}(y-\mu_2B_2y)]\|^2 \\
&\le \|T_{\mu_2}^{G_2}(x-\mu_2B_2x)-\mu_1B_1T_{\mu_2}^{G_2}(x-\mu_2B_2x) \\
&\quad -[T_{\mu_2}^{G_2}(y-\mu_2B_2y)-\mu_1B_1T_{\mu_2}^{G_2}(y-\mu_2B_2y)]\|^2 \\
&= \|T_{\mu_2}^{G_2}(x-\mu_2B_2x)-T_{\mu_2}^{G_2}(y-\mu_2B_2y) \\
&\quad -\mu_1(B_1T_{\mu_2}^{G_2}(x-\mu_2B_2x)-B_1T_{\mu_2}^{G_2}(y-\mu_2B_2y))\|^2 \\
&\le \|T_{\mu_2}^{G_2}(x-\mu_2B_2x)-T_{\mu_2}^{G_2}(y-\mu_2B_2y)\|^2 \\
&\quad +\mu_1(\mu_1-2\beta_1)\|B_1T_{\mu_2}^{G_2}(x-\mu_2B_2x)-B_1T_{\mu_2}^{G_2}(y-\mu_2B_2y)\|^2 \\
&\le \|T_{\mu_2}^{G_2}(x-\mu_2B_2x)-T_{\mu_2}^{G_2}(y-\mu_2B_2y)\|^2 \\
&\le \|(x-\mu_2B_2x)-(y-\mu_2B_2y)\|^2 \\
&\le \|x-y\|^2+\mu_2(\mu_2-2\beta_2)\|B_2x-B_2y\|^2 \\
&\le \|x-y\|^2.
\end{aligned}$$

It shows that $\Gamma : C \to C$ is nonexpansive. Since $\|St_n - t_n\| \to 0$, $\|St_n - x_n\| \to 0$ and

$$\|t_n - x_n\| \le \|St_n - t_n\| + \|St_n - x_n\|,$$

we conclude that $\|t_n - x_n\| \to 0$ as $n \to \infty$. Furthermore,

$$\begin{aligned}
&\|t_n-\Gamma(t_n)\| = \|t_n-y_n\|+\|y_n-\Gamma(t_n)\| \\
&= \alpha_n\|u-y_n\|+\|T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n-\mu_2B_2z_n)-\mu_1B_1T_{\mu_2}^{G_2}(z_n-\mu_2B_2z_n)]-\Gamma(t_n)\| \\
&= \alpha_n\|u-y_n\|+\|\Gamma(z_n)-\Gamma(t_n)\| \le \alpha_n\|u-y_n\|+\|z_n-t_n\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|t_n - z_n\| \to 0$, we obtain $\|t_n - \Gamma(t_n)\| \to 0$. In terms of Lemma 2.5 we have $(I - \Gamma)w = 0$ and so $w \in \mho$. Therefore, $w \in F(S) \cap EP \cap \mho$.

This together with (3.21) and the property of metric projection, implies that

$$\limsup_{n\to\infty}\langle u-\bar{x}, t_n-\bar{x}\rangle = \lim_{i\to\infty}\langle u-\bar{x}, t_{n_i}-\bar{x}\rangle = \langle u-\bar{x}, w-\bar{x}\rangle \le 0.$$

Finally, we prove $x_n \to \bar{x}$. Indeed, since $t_n - \bar{x} = \alpha_n u + (1-\alpha_n)y_n - \bar{x} = \alpha_n(u-\bar{x}) + (1-\alpha_n)(y_n - \bar{x})$, by utilizing Lemma 2.6, we derive from (3.2) and (3.3) that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\|St_n-\bar{x}\|^2 \\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)\|t_n-\bar{x}\|^2 \\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[(1-\alpha_n)^2\|y_n-\bar{x}\|^2+2\alpha_n\langle u-\bar{x}, t_n-\bar{x}\rangle] \\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[(1-\alpha_n)\|z_n-\bar{x}\|^2+2\alpha_n\langle u-\bar{x}, t_n-\bar{x}\rangle] \\
&\le \beta_n\|x_n-\bar{x}\|^2+(1-\beta_n)[(1-\alpha_n)\|x_n-\bar{x}\|^2+2\alpha_n\langle u-\bar{x}, t_n-\bar{x}\rangle] \\
&= (1-(1-\beta_n)\alpha_n)\|x_n-\bar{x}\|^2+2(1-\beta_n)\alpha_n\langle u-\bar{x}, t_n-\bar{x}\rangle.
\end{aligned} \tag{3.24}$$

Now, put $\gamma_n = (1-\beta_n)\alpha_n$ and $\delta_n = 2(1-\beta_n)\alpha_n\langle u-\bar{x}, t_n-\bar{x}\rangle$ for all $n \in \mathbb{N}$. Then (3.24) can be rewritten as

$$\|x_{n+1}-\bar{x}\|^2 \le (1-\gamma_n)\|x_n-\bar{x}\|^2+\delta_n, \quad \forall n \in \mathbb{N}. \tag{3.25}$$

Since $0 < c \le \beta_n \le d < 1$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, we have $\sum_{n=1}^{\infty}(1-\beta_n)\alpha_n = \infty$ and so $\sum_{n=1}^{\infty}\gamma_n = \infty$. Note that $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} = \limsup_{n\to\infty} 2\langle u-\bar{x}, t_n-\bar{x}\rangle \le 0$, due to (3.20). Consequently, by applying Lemma 2.4 to (3.25), we deduce that $\{x_n\}$ converges strongly to $\bar{x}$. This completes the proof. □

By using Theorem 3.1, we obtain the following strong convergence results in a Hilbert space.

Corollary 3.1. *Let $F, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions such that Condition A holds. Let the mappings $B_1, B_2 : C \to H$ be inverse-strongly monotone mappings with constants β_1 and β_2, respectively, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap EP(F) \cap \mho \neq \emptyset$, where $\mho$ is the same as in Theorem 3.1. Let $u \in C$, $x_1 \in C$ and $\{x_n\} \subset C$ be a sequence generated by the following scheme*

$$\begin{cases} F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = T_{\mu_1}^{G_1}[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S[\alpha_n u + (1-\alpha_n) y_n], \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$, and $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\lambda_n\} \subset (0,\infty)$ satisfy the following conditions $0 < c \leq \beta_n \leq d < 1, \qquad 0 < a \leq \lambda_n \leq b < \infty,$

$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0, \ \lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP(F)\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.2), where $\bar{y} = T_{\mu_2}^{G_2}(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. In Theorem 3.1, for all $n \in \mathbb{N}$, $z_n = T_{\lambda_n}^F(x_n - \lambda_n A x_n)$ is equivalent to

$$F(z_n, y) + \langle A x_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

By putting $A \equiv 0$, we obtain $F(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C$. Observe that for all $\alpha \in (0,\infty)$ $\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C$. So, taking $a, b \in (0,\infty)$ with $0 < a \leq b < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $a \leq \lambda_n \leq b$, we obtain the desired result from Theorem 3.1. □

Corollary 3.2. *Let $G_1, G_2 : C \times C \to \mathbb{R}$ be two bifunctions such that Condition A holds. Let the mappings $A, B_1, B_2 : C \to H$ be inverse-strongly monotone mappings with constants α, β_1, β_2, respectively, and let $S : C \to C$ be a nonexpansive mapping such that $VI(C,A) \cap F(S) \cap \mho \neq \emptyset$, where $\mho$ is the same as in Theorem 3.1. Let $u \in C$, $x_1 \in C$ and $\{x_n\} \subset C$ be a sequence generated by the following iterative scheme*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A x_n), \\ y_n = T_{\mu_1}^{G_1}\left[T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T_{\mu_2}^{G_2}(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + (1-\beta_n) S\left[\alpha_n u + (1-\alpha_n) y_n\right], \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$, $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, and $\{\lambda_n\} \subset [0, 2\alpha]$ satisfy the following conditions $0 < c \leq \beta_n \leq d < 1, \qquad 0 < a \leq \lambda_n \leq b < 2\alpha,$

$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0, \ \lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{VI(C,A)\cap F(S)\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.2), where $\bar{y} = T_{\mu_2}^{G_2}(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. In Theorem 3.1, for all $n \in \mathbb{N}$, $z_n = T^F_{\lambda_n}(x_n - \lambda_n A x_n)$ is equivalent to

$$F(z_n, y) + \langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C.$$

By taking $F \equiv 0$, we obtain $\langle Ax_n, y - z_n\rangle + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \geq 0, \quad \forall y \in C, \forall n \in \mathbb{N}$, which implies that $\langle y - z_n, x_n - \lambda_n A x_n - z_n\rangle \leq 0, \quad \forall y \in C$. So, it follows that $P_C(x_n - \lambda_n A x_n) = z_n$ for all $n \in \mathbb{N}$ and we obtain the desired result from Theorem 3.1. □

A mapping $T : C \to C$ is called *strictly pseudocontractive* if there exists k with $0 \leq k < 1$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$, for all $x, y \in C$. Notice that, if $T : C \to C$ is a strictly pseudocontractive mapping with constant k, then the mapping $A \equiv I - T$ is inverse-strongly monotone with constant $(1 - k)/2$.

Theorem 3.2. *Let $F, G_1, G_2 : C \times C \to \mathbb{R}$ be three bifunctions such that Condition A holds. Let $T : C \to C$ be a strictly pseudocontractive mapping with constant k and let $B_1, B_2 : C \to H$ be inverse-strongly monotone mappings with constants β_1, and β_2, respectively, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap EP \cap \mho \neq \emptyset$, where $A \equiv I - T$ and $\mho$ is the same as in Theorem 3.1. Let $u \in C$, $x_1 \in C$ and $\{x_n\} \subset C$ be a sequence generated by the following iterative scheme*

$$\begin{cases} z_n = T^F_{\lambda_n}\left((1 - \lambda_n)x_n + \lambda_n T x_n\right), \\ y_n = T^{G_1}_{\mu_1}\left[T^{G_2}_{\mu_2}(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 T^{G_2}_{\mu_2}(z_n - \mu_2 B_2 z_n)\right], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n) S[\alpha_n u + (1 - \alpha_n) y_n], \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\mu_1 \in (0, 2\beta_1)$, $\mu_2 \in (0, 2\beta_2)$, and $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$, $\{\lambda_n\} \subset [0, 1 - k]$ satisfy

$$0 < c \leq \beta_n \leq d < 1, \qquad 0 < a \leq \lambda_n \leq b < 1 - k,$$
$$\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0, \ \lim_{n\to\infty}\alpha_n = 0 \text{ and } \sum_{n=1}^{\infty}\alpha_n = \infty.$$

Then, $\{x_n\}$ converges strongly to $\bar{x} = P_{F(S)\cap EP\cap\mho}u$ and $(\bar{x}, \bar{y})$ is a solution of problem (1.2), where $\bar{y} = T^{G_2}_{\mu_2}(\bar{x} - \mu_2 B_2 \bar{x})$.

Proof. Since T is a strictly pseudocontractive mapping with contant k, the mapping $A \equiv I - T$ is inverse-strongly monotone with constant $(1 - k)/2$. Consider $\alpha = (1 - k)/2$. Then $z_n = T^F_{\lambda_n}(x_n - \lambda_n A x_n) = T^F_{\lambda_n}(x_n - \lambda_n(I - T)x_n) = T^F_{\lambda_n}((1 - \lambda_n)x_n + \lambda_n T x_n)$. By Theorem 3.1, we get the conclusion. □

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