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FUNCTION SPACES WITH THE MATKOWSKI PROPERTY AND DEGENERACY PHENOMENA FOR COMPOSITION OPERATORS

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Abstract. We give a condition, both necessary and sufficient, on a function $f\colon \mathbb{R} \to \mathbb{R}$, under which the nonlinear composition operator F defined by $Fx(t) = f(x(t))$ satisfies a local Lipschitz condition in the norm of the function spaces $C^1([a,b])$, $C^{0,\alpha}([a,b])$, $Lip([a,b])$, and $BV([a,b])$. In contrast to global Lipschitz conditions, this does not lead to a strong degeneracy of the generating function f, which is important to apply fixed point theorems.
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1. Statement of the problem

Given a function $f\colon \mathbb{R} \to \mathbb{R}$ or, more generally, $f\colon [a,b] \times \mathbb{R} \to \mathbb{R}$, the nonlinear *composition operator* (also called *superposition operator*, *substitution operator*, or *Nemytskij operator* in the literature) F generated by f is defined by

$$Fx(t) = f(x(t)) \tag{1}$$

and

$$Fx(t) = f(t, x(t)), \tag{2}$$

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respectively. In spite of its simple form, the behaviour of this operator exhibits many surprising and even pathological features in various function spaces. Thus, if one is interested in applying the Banach contraction mapping theorem or some of its generalizations to some nonlinear problem, one usually considers the operator (1) or (2) in some suitable Banach space X of functions $x\colon [a,b] \to \mathbb{R}$ and requires a (global) Lipschitz condition of the form

$$\|Fx - Fy\| \leq K\|x - y\| \qquad (x, y \in X) \tag{3}$$

for this operator in the norm of X. It turns out, however, that sometimes this leads to a strong degeneracy: in some classical function spaces X, the operator (1) satisfies (3) if and only if the corresponding function f has the form

$$f(u) = \alpha + \beta u \qquad (\alpha, \beta \geq 0); \tag{4}$$

similarly, the operator (2) satisfies (3) if and only if the corresponding function f has the form

$$f(t,u) = \alpha(t) + \beta(t)u \qquad (\alpha, \beta \in X). \tag{5}$$

This means, roughly speaking, that one may apply classical fixed point principles for contraction type maps only if the underlying problem is actually *linear*, and so of very limited interest. Several examples of this kind of degeneracy will be recalled in the following section.

Closer scrutiny of many nonlinear problems reveals, however, that it often suffices to impose, instead of (3), a *local* Lipschitz condition of the form

$$\|Fx - Fy\| \leq K(r)\|x - y\| \qquad (x, y \in X,\ \|x\|, \|y\| \leq r) \tag{6}$$

and to expect that this milder condition does not lead to the same drastic degeneracy for the generating function f. This is in fact true for several important function spaces like $C^1([a,b])$, $C^{0,\alpha}([a,b])$, $Lip([a,b])$, and $BV([a,b])$; to prove and illustrate this is the aim of the present paper.

2. Spaces with the Matkowski property

To the best of our knowledge, the first who observed the kind of degeneracy phenomenon for composition operators described above was Janusz Matkowski. More specifically, Matkowski (in part with coauthors) proved that Lipschitz continuous operators (2) in X are only generated by affine functions (5), if X is the space $C^m([a,b])$ of m-times continuously differentiable functions [10, 12], the Sobolev space $W_p^1([a,b])$ of functions with distributional first derivative in $L_p([a,b])$ [14], or the space $BV_p^2([a,b])$ of functions of bounded $(p,2)$-variation [13]. Likewise, an analogous result was proved by Matkowska [9] for the space $C^{0,\alpha}([a,b])$ of Hölder continuous functions of order $\alpha < 1$, by Lupa [8] for the space $C^{n,\alpha}([a,b])$ of functions with Hölder continuous n-th derivative, by Sieczko [20] for the space $AC^n([a,b])$ of functions with absolutely continuous n-th derivative, by Knop [7] for the space $Lip^n([a,b])$ of functions with Lipschitz continuous n-th derivative, by Merentes and Rivas [18] for the space $RV_p([a,b])$ of functions of bounded generalized p-variation in Riesz' sense, and by Merentes [16, 17] for the space $RV_\varphi([a,b])$ of functions of bounded generalized φ-variation in Riesz' sense.

Definition 1. Following [19], we say that a Banach space $(X, \|\cdot\|)$ has the *Matkowski property* if, whenever the operator (2) maps the space X into itself and satisfies (3), the underlying function f necessarily has the special form (5). (In case of the autonomous operator (1), the condition (3) takes of course the simpler form (4).)

We point out that there are some important function spaces which do *not* have the Matkowski property. For example, in [1] it was shown that the condition (3) in the space $C([a,b])$ with norm

$$\|x\|_C := \max_{a\le t\le b} |x(t)|$$

is equivalent to the Lipschitz condition

$$|f(t,u) - f(t,v)| \le k|u-v| \qquad (a \le t \le b,\ u,v \in \mathbb{R})$$

for the function $f(t,\cdot)$ (with the same constant $k = K$ as in (3)); this is of course what one should expect in "reasonable" function spaces. A similar result holds for the Lebesgue space $L_p([a,b])$ $(1 \le p < \infty)$ with norm

$$\|x\|_{L_p} := \Big(\int_a^b |x(t)|^p\, dt\Big)^{1/p},$$

see [1] or [4, Theorem 3.10]. For further reference, we state this as a theorem.

Theorem 1. *Suppose that the composition operator* (2) *maps the space* $L_p([a,b])$ *into the space* $L_q([a,b])$ $(1 \le q \le p < \infty)$. *Then condition* (3) *holds if and only if*

$$|f(t,u) - f(t,v)| \le g(t,w)|u-v| \qquad (a \le t \le b,\ |u|, |v| \le w)$$

where the composition operator G generated by g maps $L_p([a,b])$ into $L_{pq/(p-q)}([a,b])$. In particular, in case $p=q$ condition (3) *is equivalent to* (5).

In the space $BV([a,b])$ of functions of bounded variation, the degeneracy one encounters when imposing a Lipschitz condition is somewhat different. Recall that, given a function $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$, the *left regularization* $f^\#$ of f is defined by

$$f^\#(t,u) := \begin{cases} f(a,u) & \text{for } t = a,\\ \lim\limits_{s\to t-} f(s,u) & \text{for } a < t \le b, \end{cases} \tag{7}$$

while the *right regularization* $f^\flat$ of f is defined by

$$f^\flat(t,u) := \begin{cases} \lim\limits_{s\to t+} f(s,u) & \text{for } a \le t < b, \\ f(b,u) & \text{for } t = b. \end{cases} \tag{8}$$

Of course, these regularizations are different from f only if $f(\,\cdot\,,u)$ is discontinuous from the left or right, respectively. Recall that the *total variation* of a function $x \in BV([a,b])$ on $[a,b]$ is given by

$$var(x;[a,b]) := \sup\{var(x,P;[a,b]) : P \in \mathcal{P}([a,b])\},$$

where

$$var(x,P;[a,b]) = \sum_{j=1}^{m} |x(t_j) - x(t_{j-1})| \qquad (P = \{t_0, t_1, \ldots, t_m\}),$$

and the supremum is taken over the set $\mathcal{P}([a,b])$ of all partitions P of the interval $[a,b]$. The following result was proved by Matkowski and Miś in [15], see also [11].

Theorem 2. *Suppose that the composition operator* (2) *generated by some function* $f\colon [a,b]\times\mathbb{R}\to\mathbb{R}$ *maps the space* $BV([a,b])$ *with norm*

$$\|x\|_{BV} = |x(a)| + var(x;[a,b]) \tag{9}$$

into itself and satisfies a Lipschitz condition of type (3) *with respect to this norm. Then the left regularization* (7) *of* f *has the form*

$$f^{\#}(t,u) = \alpha(t) + \beta(t)u \tag{10}$$

for some functions $\alpha, \beta \in BV([a,b])$.

Clearly, an analogous statement is true for the right regularization (8). In view of Theorem 2 it seems reasonable to introduce a weaker form of Definition 1.

Definition 2. We say that a Banach space $(X, \|\cdot\|)$ has the *weak Matkowski property* if, whenever the operator (2) maps the space X into itself and satisfies (3), the left regularization (7) of the underlying function f necessarily has the special form (10).

So Theorem 2 states that the Banach space $(BV([a,b]), \|\cdot\|_{BV})$ has the weak Matkowski property. The following example (which is a slight modification of an example in [15]) shows that $(BV([a,b]), \|\cdot\|_{BV})$ does not have, however, the Matkowski property in the sense of Definition 1.

Example 1. Let $\{r_0, r_1, r_2, \dots\}$ be an enumeration of all rational numbers in $[0,1]$ ($r_0 := 0$), and let $g\colon \mathbb{R} \to \mathbb{R}$ be any functions satisfying $g(0) = 0$ and $|g(u) - g(v)| \leq L|u - v|$. We define $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$f(t,u) := \begin{cases} \dfrac{g(u)}{2^k} & \text{if } t = r_k, \\ 0 & \text{otherwise.} \end{cases}$$

For any partition $P = \{t_0, t_1, \dots, t_m\} \in \mathcal{P}([0,1])$ and $x \in BV([0,1])$ we have then

$$\sum_{j=1}^{m} |Fx(t_j) - Fx(t_{j-1})| \leq 2 \sum_{k=0}^{\infty} |f(r_k, x(r_k))| = 2 \sum_{k=0}^{\infty} \frac{|g(x(r_k))|}{2^k} \leq 2L,$$

which shows that F maps the space $BV([0,1])$ into itself. Furthermore, for $x, y \in BV([0,1])$ and $P \in \mathcal{P}([0,1])$ as above we obtain the estimate

$$\begin{aligned} var(Fx - Fy, P; [0,1]) &= \sum_{j=1}^{m} |Fx(t_j) - Fy(t_j) - Fx(t_{j-1}) + Fy(t_{j-1})| \\ &\leq 2 \sum_{j=1}^{m} |f(t_j, x(t_j)) - f(t_j, y(t_j))| \leq 2 \sum_{k=0}^{\infty} |f(r_k, x(r_k)) - f(r_k, y(r_k))| \\ &\leq 2 \sum_{k=0}^{\infty} \frac{|g(x(r_k)) - g(y(r_k))|}{2^k} \leq 2L \sum_{k=0}^{\infty} \frac{|x(r_k) - y(r_k)|}{2^k} \leq 2L\|x - y\|_{BV}. \end{aligned}$$

This together with the trivial estimate $|Fx(0) - Fy(0)| \leq L|x(0) - y(0)|$ shows that F satisfies the global Lipschitz condition (3) with $K = 2L$, although f is not of the form (5).

It is not hard to see that $f^{\#}(t,u) = f^{\flat}(t,u) \equiv 0$ for the function f in Example 1, in accordance with Theorem 2.

The strong degeneracy described above which occurs in many familiar function spaces emphasizes the need of dropping the Lipschitz condition (3) and replacing it by some weaker condition like (6). In fact, we will show in the next sections that the local Lipschitz condition (6) for the operator (1) in the spaces $C^1([a,b])$, $BV([a,b])$, $AC([a,b])$, $C^{0,\alpha}([a,b])$, and $Lip([a,b])$ leads to (in fact, is equivalent to) the local Lipschitz condition

$$|f'(u) - f'(v)| \leq k(r)|u - v| \qquad (u, v \in \mathbb{R},\ |u|, |v| \leq r) \tag{11}$$

for the *derivative* of f. This condition which is of course considerably less restrictive than (4), is fulfilled for a large variety of nonlinearities occurring in applications. The surprising (and somewhat unpleasant) fact is that, as we will see in the following three sections, each of these spaces requires a different proof.

3. Continuously differentiable functions

We start with the space $C^1([a,b])$ of continuously differentiable functions $x\colon [a,b] \to \mathbb{R}$, equipped with the norm

$$\|x\|_{C^1} := \|x\|_C + \|x'\|_C. \tag{12}$$

Sufficient conditions under which the operator (1) maps this space into itself and is continuous or bounded may be found in the monograph [4]. We point out that, in contrast to the space $C([a,b])$, the operator (2) has a quite unexpected behaviour in the space $C^1([a,b])$. For example, it may happen that F maps $C^1([a,b])$ into itself although the generating function f is *discontinuous* (and so F *does not* map $C([a,b])$ into itself). Since such pathologies seem to be interesting, we briefly recall an example (see [4, Section 8.2]).

Example 2. Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t,u) := \begin{cases} 0 & \text{if } u \le 0, \\ 3\dfrac{u^2}{t} - 2\dfrac{u^3}{t\sqrt{t}} & \text{if } 0 < u < \sqrt{t}, \\ 1 & \text{if } u \ge \sqrt{t}. \end{cases}$$

A rather cumbersome but straightforward calculation shows then that the operator (2) generated by this function maps $C^1([0,1])$ into itself, but f is *discontinuous* at $(0,0)$, and so F does not map $C([0,1])$ into itself!

The reason for the pathological behaviour of the function f in Example 2 is that the corresponding composition operator F is not continuous in the norm (12). In fact, one may prove the following result (see [4, Theorem 8.1]) which we state for further reference.

Theorem 3. *The composition operator* (2) *maps the space* $C^1([a,b])$ *into itself and is continuous with respect to the norm* (12) *if and only if* f *is* C^1 *on* $[a,b] \times \mathbb{R}$.

Now we show that (11) is necessary and sufficient for the operator (1) to satisfy condition (6) in the space $C^1([a,b])$ with norm (12).

This equivalence holds then also, if we pass to an equivalent norm in C^1 like

$$\|x\|_{C^1} := |x(a)| + \|x'\|_C \tag{13}$$

which is sometimes easier to calculate in applications.

However, we also give estimates for the corresponding Lipschitz constants for which the choice of the norm plays a role. In order to formulate good estimates for the norm (12), it will be convenient to introduce the quantity

$$\widetilde{k}(r) := \sup_{|u| \le r} |f'(u)| \tag{14}$$

which is finite for $r > 0$ if (11) holds; more explicitly, (11) implies

$$\widetilde{k}(r) \leq |f'(t)| + k(r) \max\{t - a, b - t\} \qquad (a \leq t \leq b).$$

Theorem 4. *If $f : \mathbb{R} \to \mathbb{R}$ is differentiable and* (11) *holds, then the composition operator F generated by f maps the space $C^1([a,b])$ into itself and satisfies* (6) *with respect to the norm* (12) *with some K satisfying*

$$K(r) \leq \max\{rk(r), \widetilde{k}(r)\}. \tag{15}$$

Conversely, if the composition operator F generated by some $f : \mathbb{R} \to \mathbb{R}$ maps $C^1([a,b])$ into itself and satisfies (6) *with respect to the norm* (12) *or* (13)*, then f' exists on $\mathbb{R}$ and satisfies* (11) *with some k satisfying*

$$k(r) \leq \frac{2K(2r) + 1}{r}. \tag{16}$$

Proof. Suppose first that the derivative f' of f satisfies the local condition (11). Fix $x, y \in C^1([a,b])$ with $\|x\|_{C^1} \leq r$ and $\|y\|_{C^1} \leq r$, and let $u := Fx$ and $v := Fy$. By the classical mean value theorem, we have

$$|u(t) - v(t)| = |f(x(t)) - f(y(t))| \leq |f'(\tau)||x(t) - y(t)| \tag{17}$$

for some τ between $x(t)$ and $y(t)$ (which may depend on t). Estimating the right hand side of (17) by $\widetilde{k}(r)|x(t) - y(t)|$ and passing to the maximum with respect to t on both sides yields

$$\|u - v\|_C \leq \widetilde{k}(r)\|x - y\|_C. \tag{18}$$

Now we estimate the derivatives of $u = Fx$ and $v = Fy$. By the chain rule we get

$$\begin{aligned} |u'(t) - v'(t)| &= |f'(x(t))x'(t) - f'(y(t))y'(t)| \\ &= |\big(f'(x(t)) - f'(y(t))\big)x'(t) + f'(y(t))\big(x'(t) - y'(t)\big)| \\ &\leq k(r)|x(t) - y(t)||x'(t)| + \widetilde{k}(r)|x'(t) - y'(t)|, \end{aligned}$$

and so

$$\|u' - v'\|_C \leq k(r)r\|x - y\|_C + \widetilde{k}(r)\|x' - y'\|_C. \tag{19}$$

Combining (18) and (19), we conclude that (6) holds with some K satisfying (15).

Now we suppose that F satisfies a local Lipschitz condition (6) in the norm (12) of the space $C^1([a,b])$. Putting, in particular, $x(t) \equiv u$ and $y(t) \equiv v$ in (6) we see that f satisfies a local Lipschitz condition

$$|f(u) - f(v)| \leq k(r)|u - v| \qquad (u, v \in \mathbb{R},\ |u|, |v| \leq r) \tag{20}$$

with $k(r) = K(r)$. Moreover, f is continuously differentiable on the real line, by Theorem 3.

Given $u, v \in [-r, r]$, where without loss of generality $v \neq 0$, put $\eta := -r \operatorname{sgn} v$ and consider the functions

$$x(t) := \frac{t-a}{b-a}(u-\eta) + \eta, \quad y(t) := \frac{t-a}{b-a}(v-\eta) + \eta.$$

Then $\|x\|_{C^1}, \|y\|_{C^1} \leq 2r$, and the functions $u := Fx$ and $v := Fy$ satisfy

$$|u'(b) - v'(b)| \leq \|u - v\|_{C^1} \leq K(2r)\|x - y\|_{C^1} = 2K(2r)|u - v|$$

by (6). On the other hand, the left hand side of this is

$$\begin{aligned} |u'(b) - v'(b)| &= |f'(u)(u-\eta) - f'(v)(v-\eta)| \\ &= |f'(u)(u-v) + \big(f'(u) - f'(v)\big)(v-\eta)|. \end{aligned}$$

The triangle inequality thus implies

$$|f'(u) - f'(v)||v - \eta| \leq |f'(u)(u-v)| + 2K(2r)|u - v|.$$

But from our choice of η we have $|v - \eta| \geq r$, and so we conclude that (11) holds with some k as in (16). □

The estimate (16) has an interesting consequence: If (3) holds, i.e. if K can be chosen independent of r, then this estimate implies $k(r) \to 0$ as $r \to \infty$. Hence (11) implies that f' is actually constant which means that f has the form (4), and so we have recovered *en passant* again the result from [10, 12] that the space $C^1([a, b])$ has the Matkowski property.

The same argument shows even more precisely that for non-affine f any function K satisfying (6) must not only be non-constant but even grow at least linearly at ∞, i.e. satisfy $\liminf_{r\to\infty} \frac{K(r)}{r} > 0$.

4. Functions of bounded variation or absolutely continuous functions

Now we give a parallel result for the space $BV([a, b])$ of functions of bounded variation $x\colon [a, b] \to \mathbb{R}$, equipped with either the norm (9) or the equivalent norm

$$\|x\|_{BV} := \|x\|_\infty + var(x; [a, b]), \tag{21}$$

where $\|x\|_\infty := \sup_{a \leq t \leq b} |x(t)|$.

Simultaneously, we will consider the subspace $AC([a, b])$ of absolutely continuous functions and $Lip([a, b])$ of Lipschitz continuous function with the same norm. It is well-known that

$$Lip([a, b]) \subseteq AC([a, b]) \subseteq BV([a, b]).$$

Note that, while $AC([a, b])$ is a Banach space with the norm (21), the subspace $Lip([a, b])$ is incomplete with the norm (21). However, completeness plays no role in our subsequent considerations. We will consider $\mathrm{Lip}([a, b])$ with its natural norm in Section 5.

Again, sufficient conditions under which the operators (1) and (2) map the space $BV([a,b])$ into itself and are continuous or bounded may be found in the monograph [4]. For the operator (1) even necessary and sufficient conditions are known; for further reference, we recall the following result of Chaica and Waterman [5].

Theorem 5. *The composition operator* (1) *maps the space* $BV([a,b])$ *into itself if and only if the function* f *satisfies the local Lipschitz condition* (20).

We prepare our result with a somewhat surprising generalization of Theorem 5.

Theorem 6. *The following statements about the composition operator* (1) *and the generating function* $f\colon \mathbb{R} \to \mathbb{R}$ *are equivalent:*

(1) *The operator* F *maps* $Lip([a,b])$ *into* $BV([a,b])$.
(2) *The operator* F *maps* $Lip([a,b])$ *into* $Lip([a,b])$.
(3) *The operator* F *maps* $AC([a,b])$ *into* $AC([a,b])$.
(4) *The operator* F *maps* $BV([a,b])$ *into* $BV([a,b])$.
(5) *The map* f *satisfies the local Lipschitz condition* (20).

In particular, the last property of Theorem 6 is independent of the interval $[a,b]$, and so must also be the first properties. We prove this consequence independently, since we use it as a tool to simplify the notation in the proof of Theorem 6:

Lemma 1. *For any two nondegenerate intervals* $[a,b]$ *and* $[c,d]$, *the composition operator* (1) *maps the space* $Lip([a,b])$ *into* $BV([a,b])$ *if and only if it maps* $Lip([c,d])$ *into* $BV([c,d])$.

Proof. Suppose that the composition operator defined by $Fu = f \circ u$ maps the space $Lip([a,b])$ into $BV([a,b])$. The function $\ell\colon [c,d] \to [a,b]$ defined by

$$\ell(t) := \frac{b-a}{d-c}(t-c) + a \qquad (c \le t \le d)$$

is a strictly increasing homeomorphism between $[c,d]$ and $[a,b]$ with inverse

$$\ell^{-1}(s) = \frac{d-c}{b-a}(s-a) + c \qquad (a \le s \le b)$$

which satisfies $\ell(c) = a$ and $\ell(d) = b$. So $\ell\colon \mathcal{P}([c,d]) \to \mathcal{P}([a,b])$ with

$$\ell(\{t_0, t_1, \dots, t_{m-1}, t_m\}) = \{\ell(t_0), \ell(t_1), \dots, \ell(t_{m-1}), \ell(t_m)\}$$

defines a one-to-one correspondence between all partitions of $[c,d]$ and all partitions of $[a,b]$.

Given $v \in Lip([c,d])$, the function $u := v \circ \ell^{-1}$ belongs to $Lip([a,b])$, and so $Fu = f \circ v \circ \ell^{-1}$ belongs to $BV([a,b])$, by assumption. But for $P \in \mathcal{P}([c,d])$ and $\ell(P) \in \mathcal{P}([a,b])$ as above we have that

$var(f \circ u, \ell(P); [a,b]) = var(f \circ v \circ \ell^{-1}, P; [a,b]) = \sum_{j=1}^{m} |f(u(\ell(t_j))) - f(u(\ell(t_{j-1})))| = \sum_{j=1}^{m} |f(v(t_j)) - f(v(t_{j-1}))| = var(f \circ v, P; [c,d])$.

This shows that $var(f \circ v; [c,d]) = var(f \circ u; [a,b])$, and so also $\|Fu\|_{BV} = \|Fv\|_{\mathrm{BV}}$, since $f(u(a)) = f(v(\ell^{-1}(a)) = f(v(c))$. □

Observe that our proof of Lemma 1 gives even a more precise result: the map $u \mapsto u \circ \ell$ is an *isometry* between $BV([a,b])$ and $BV([c,d])$ resp. $Lip([a,b])$ and $Lip([c,d])$ with respect to the norms (9) and (21), and so also the boundedness and continuity of F is preserved under this map.

Similar results (with an analogous proof) also hold if one replaces "*Lip*" or "*BV*" by any of "*BV*", "*AC*", or "*Lip*".

Proof. [Proof of Theorem 6] Suppose that f does not satisfy a local Lipschitz condition (6), i.e. there is some $r > 0$ for which (20) does not hold for any constant $k(r) > 0$. In particular, for any natural number n, we cannot have an estimate of the form

$$|f(u) - f(v)| \leq c|u - v| \qquad (u, v \in [-r, r],\ |u - v| \leq r/n),$$

since otherwise (20) would hold with $k(r) = nc$. Hence, there are $u_k < v_k$ such that

$$\delta_k := v_k - u_k < \frac{1}{k^2}, \quad |f(v_k) - f(u_k)| > k^2 |v_k - u_k| \qquad (k = 1, 2, \ldots). \quad (22)$$

Passing, if necessary, to a subsequence, we can assume without loss of generality that there exists $u_\infty \in [-r, r]$ such that $|u_k - u_\infty| \leq \frac{1}{2k^2}$ for all k, and so $|u_k - u_{k+1}| \leq \frac{1}{k^2}$. Let n_k be the unique natural number satisfying

$$\frac{1}{k^2 \delta_k} \leq n_k < \frac{1}{k^2 \delta_k} + 1 \qquad (k = 1, 2, \ldots). \quad (23)$$

Then $\delta_k n_k \leq \frac{1}{k^2} + \delta_k < \frac{2}{k^2}$. Hence, the strictly increasing sequence $(t_k)_k$, defined recursively by $t_1 := 0, \quad t_{k+1} := t_k + |u_k - u_{k+1}| + 2n_k\delta_k$, is actually bounded by

$$t_\infty := \sum_{k=1}^{\infty} \big(|u_k - u_{k+1}| + 2n_k\delta_k\big) \leq \sum_{k=1}^{\infty} \frac{5}{k^2} < \infty.$$

Now we define a function $x\colon [0, t_\infty] \to \mathbb{R}$ by

$$x(t) := \begin{cases} u_k & \text{if } t = t_k + 2m\delta_k \text{ for } m \in \{0, \ldots, n_k\}, \\ v_k & \text{if } t = t_k + (2m-1)\delta_k \text{ for } m \in \{1, \ldots, n_k\}, \\ \text{affine} & \text{if } t_k + (m-1)\delta_k \leq t \leq t_k + m\delta_k \text{ for } m \in \{1, \ldots, 2n_k\}, \\ \text{affine} & \text{if } t_k + 2n_k\delta_k \leq t \leq t_{k+1}, \\ u_\infty & \text{if } t = t_\infty. \end{cases}$$

Note that $|u_k - v_k| = \delta_k$ and $|u_k - u_{k+1}| \leq |t_{k+1} - (t_k + 2n_k\delta_k)|$, and so x is actually Lipschitz on $[0, t_\infty)$ with Lipschitz constant $L \leq 1$. Hence, $x|_{[0,t_\infty)}$ has a unique continuous extension to a Lipschitz function with Lipschitz constant L on $[0, t_\infty]$, and since $u_k \to u_\infty$, actually x itself is this extension.

Assume now by contradiction that (1) maps $Lip([a,b])$ into $BV([a,b])$. Applying Lemma 1 with $[c,d] = [0, u_\infty]$, this would imply that the function $y = f \circ x$ belongs to $BV([0, u_\infty])$. On the other hand, by construction of x, (22), and (23), we have for any $k_0 \in \mathbb{N}$

$$var(y; [0, u_\infty]) \geq \sum_{k=1}^{k_0} 2n_k |f(v_k) - f(u_k)| > \sum_{k=1}^{k_0} 2n_k k^2 \delta_k \geq 2k_0,$$

which is a contradiction. Hence, we have shown that the last statement of Theorem 6 follows if any of the other statements holds.

Conversely, assume that f satisfies the local Lipschitz condition (20). Then F maps the space $BV([a,b])$ into itself by Theorem 5. Moreover, F maps each of the spaces $X = Lip([a,b])$ and $X = AC([a,b])$ into itself. Indeed for any function $x \in X$ there is some r with $|x(t)| \leq r$ for all $t \in [a,b]$, and thus $y = Fx$ satisfies

$$|y(t) - y(s)| = |f(u(t)) - f(u(s))| \leq k(r)|u(t) - u(s)| \qquad (t, s \in [a,b]),$$

and thus also

$$\sum_{k=1}^{m} |y(t_k) - y(s_k)| \leq k(r) \sum_{k=1}^{m} |x(t_k) - x(s_k)|$$

for every finite collection of non-overlapping intervals $[s_k, t_k] \subseteq [a,b]$. These estimates show that if x is Lipschitz or absolutely continuous, then also $y = Fx$ has the respective property. □

Building on Theorem 6, we give now a necessary and sufficient condition for the operator (1) to fulfill (6) in any of the spaces $BV([a,b])$, $AC([a,b])$, or $Lip([a,b])$ with the norm (9) or (21). In fact, the following result shows even much more for both implications of the equivalence.

Theorem 7. *If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and satisfies* (11)*, then the composition operator F generated by f maps each of the spaces $X = BV([a,b])$, $X = AC([a,b])$ and $X = Lip([a,b])$ into itself and satisfies* (6) *with respect to the norm* (21) *with some K satisfying*

$$K(r) \leq \max\{4rk(r), \widetilde{k}(r)\}, \tag{24}$$

where $\widetilde{k}(r)$ is defined by (14)*.*

Conversely, if the composition operator F generated by $f\colon \mathbb{R} \to \mathbb{R}$ maps any of the three spaces $X = BV([a,b])$, $X = AC([a,b])$, or $X = Lip([a,b])$ into any of these three spaces and satisfies (6) *with respect to the norm* (9) *or* (21)*, then f is differentiable on $\mathbb{R}$ and satisfies* (11) *with some k satisfying the relation $k(r) \leq K(3r+1)$.*

To prove Theorem 7 we state two technical lemmas which seems to be interesting on their own and which we will also use in the proof of Theorem 10 below.

Lemma 2. *Suppose that* (11) *holds for some* $r > 0$*, and let* $\widetilde{k}(r)$ *be defined by* (14)*. Then for all* $|x_i|, |y_i| \leq r$ $(i = 1, 2)$ *we have*

$$|f(x_1) - f(y_1) - f(x_2) + f(y_2)|$$
$$\leq k(r)\big(|x_1 - x_2| + |y_1 - y_2|\big)\big(|x_1 - y_1| + |x_2 - y_2|\big) + \widetilde{k}(r)|x_1 - y_1 - x_2 + y_2|.$$

Proof. We distinguish the cases

$$|x_1 - y_1| + |x_2 - y_2| \leq |x_1 - x_2| + |y_1 - y_2| \tag{25}$$

and

$$|x_1 - y_1| + |x_2 - y_2| > |x_1 - x_2| + |y_1 - y_2|. \tag{26}$$

In the first case we choose, by the mean value theorem, some ξ_i between x_i and y_i $(i = 1, 2)$ with

$$f(x_i) - f(y_i) = f'(\xi_i)(x_i - y_i) \qquad (i = 1, 2).$$

Using (25), a straightforward but cumbersome case distinction shows that

$$|\xi_1 - \xi_2| \leq |x_1 - x_2| + |y_1 - y_2|.$$

Consequently,

$$\begin{aligned} |f(x_1) - f(y_1) - f(x_2) + f(y_2)| &= |f'(\xi_1)(x_1 - y_1) - f'(\xi_2)(x_2 - y_2)| \\ &= |\big(f'(\xi_1) - f'(\xi_2)\big)\big(x_1 - y_1\big) + f'(\xi_2)\big(x_1 - y_1 - x_2 + y_2\big)| \\ &\leq k(r)\big(|x_1 - x_2| + |y_1 - y_2|\big)|x_1 - y_1| + \widetilde{k}(r)|x_1 - y_2 - x_2 + y_2|. \end{aligned}$$

In the second case we choose, by the mean value theorem, some η_x between x_1 and x_2 and some η_y between y_1 and y_2 satisfying

$$f(x_1) - f(x_2) = f'(\eta_x)(x_1 - x_2), \qquad f(y_1) - f(y_2) = f'(\eta_y)(y_1 - y_2).$$

As before, a straightforward but cumbersome case distinction, now building on (26), shows that

$$|\eta_x - \eta_y| \leq |x_1 - y_1| + |x_2 - y_2|.$$

So in this case we obtain

$$\begin{aligned} |f(x_1) - f(y_1) - f(x_2) + f(y_2)| &= |f'(\eta_x)(x_1 - x_2) - f'(\eta_y)(y_1 - y_2)| \\ &= |\big(f'(\eta_x) - f'(\eta_y)\big)\big(x_1 - x_2\big) + f'(\eta_y)\big(x_1 - y_1 - x_2 + y_2\big)| \\ &\leq k(r)\big(|x_1 - y_1| + |x_2 - y_2|\big)|x_1 - x_2| + \widetilde{k}(r)|x_1 - y_1 - x_2 + y_2|, \end{aligned}$$

as claimed. □

Lemma 3. *Let I be an interval and $f\colon I \to \mathbb{R}$ be absolutely continuous. Let N be a null set such that f' exists on $I \setminus N$. If $f'|_{I\setminus N}$ is uniformly continuous or satisfies a Lipschitz condition then f' exists on I (one-sided in the boundary points of I) and is uniformly continuous or satisfies a Lipschitz condition with the same constant, respectively.*

Proof. The function $f'|_{I\setminus N}$ has an extension to a uniformly continuous or Lipschitz (with the same Lipschitz constant) function g on I. Since f and the primitive G of g are absolutely continuous with the same derivative a.e., they differ only by some constant. Since g is continuous, $G'(x) = g(x)$ for all $x \in I$, and so $f'(x) = g(x)$ for all $x \in I$. □

Proof. [Proof of Theorem 7] Suppose first that the derivative f' of f satisfies the local Lipschitz condition (11), and define $\widetilde{k}(r)$ as in (14). Fix $x, y \in BV([0,1])$ with $\|x\|_{BV} \le r$ and $\|y\|_{BV} \le r$. Given a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}([0,1])$, we apply Lemma 2 to the choice $x_1 := x(t_j)$, $y_1 := y(t_j)$, $x_2 := x(t_{j-1})$, and $y_2 := y(t_{j-1})$ $(j = 1, 2, \ldots, m)$. As a result, we get the estimate

$$\begin{aligned}
&|f(x(t_j)) - f(y(t_j)) - f(x(t_{j-1})) + f(y(t_{j-1}))|\\
&\le k(r)\big(|x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})|\big)\times\\
&\quad\big(|x(t_j) - y(t_j)| + |x(t_{j-1}) - y(t_{j-1})|\big)\\
&\quad+\widetilde{k}(r)|x(t_j) - y(t_j) - x(t_{j-1}) + y(t_{j-1})|.
\end{aligned}$$

Taking the sum over $j = 1, 2, \ldots, m$ we obtain for $u := Fx$ and $v := Fy$

$$\begin{aligned}
&\sum_{j=1}^{m}|u(t_j) - v(t_j) - u(t_{j-1}) + v(t_{j-1})|\\
&\le 2k(r)\|x-y\|_\infty \sum_{j=1}^{m}\big(|x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})|\big)\\
&\quad+\widetilde{k}(r)\sum_{j=1}^{m}|x(t_j) - y(t_j) - x(t_{j-1}) + y(t_{j-1})|\\
&\le 2k(r)\|x-y\|_\infty\big(var(x;[a,b]) + var(y;[a,b])\big) + \widetilde{k}(r)var(x-y;[a,b])\\
&\le \widehat{k}(r)\|x-y\|_{BV}
\end{aligned}$$

with $\widehat{k}(r) := \max\{4rk(r), \widetilde{k}(r)\}$. This together with Theorem 6 proves the first part of the theorem.

Now we suppose that F acts at least from $Lip([a,b])$ into $BV([a,b]$ and satisfies a local Lipschitz condition (6) in the norm (9) or (21). By Theorem 6, f satisfies a local Lipschitz condition (20). We will only use that f is absolutely continuous on $[-r, r]$. There is a null set $N \subseteq \mathbb{R}$ such that f' exists on $[-r, r] \setminus N$. By Lemma 3, we are to show that the function $f'|_{[-r,r]\setminus N}$ satisfies a Lipschitz condition with constant at most $L := K(3r + 1)$.

Thus, assume by contradiction that there are $u_0, v_0 \in [-r, r] \setminus N$ with

$$|f'(u_0) - f'(v_0)| > L|u_0 - v_0|.$$

Let m be the unique integer number satisfying

$$\frac{1}{|u_0 - v_0|} \leq m < \frac{1}{|u_0 - v_0|} + 1.$$

Let $P = \{t_0, \ldots, t_m\}$ be a partition of $[a, b]$, and let $x \in Lip([a, b])$ be defined by

$$x(t) = \begin{cases} u_0 & \text{if } t \in \{t_j : j \text{ even}\}, \\ v_0 & \text{if } t \in \{t_j : j \text{ odd}\}, \\ \text{affine} & \text{if } t \in [t_{k-1}, t_k] \text{ for some } k \in \{1, \ldots, m\}. \end{cases}$$

Then $var(x; [a, b]) = m|u_0 - v_0| < 1 + 2r$, and so $\|x\|_{BV} < 1 + 3r$. Hence, if n is sufficiently large, the function $x_n \in Lip([a, b])$, defined by $x_n(t) := x(t) + 1/n$ also satisfies $\|x_n\|_{BV} < 1 + 3r$. By hypothesis, the function

$$w_n := \frac{Fx_n - Fx}{n^{-1}}$$

thus satisfies

$$\|w_n\|_{BV} \leq \frac{L\|x_n - x\|_{BV}}{n^{-1}} = L$$

for all large n. In particular,

$$L \geq var(w_n, P; [a, b])) = \sum_{j=1}^{m} |w_n(t_j) - w_n(t_{j-1})|$$

for all large n. Since the definition of the derivative implies

$$\lim_{n \to \infty} w_n(t_j) = \begin{cases} f'(u_0) & \text{if } j \text{ is even}, \\ f'(v_0) & \text{if } j \text{ is odd}, \end{cases}$$

we obtain

$$L \geq \sum_{j=1}^{m} |f'(u_0) - f'(v_0)| > Lm|u_0 - v_0| \geq L,$$

which is not possible. □

We remark that an analogous problem was studied in [2] for the space $BV_\varphi([a, b])$ (in particular, $BV_p([a, b])$ with $p > 1$) of functions of generalized bounded variation. However, the proof there is different, since all functions in those spaces are continuous, while the space $BV([a, b])$ contains many discontinuous functions.

5. Hölder continuous functions

For $0 < \alpha \leq 1$, we consider now the Banach space $C^{0,\alpha}([a,b])$ of all Hölder continuous (in particular, Lipschitz continuous for $\alpha = 1$) functions on $[a,b]$, equipped with the usual norm

$$\|x\|_{C^{0,\alpha}} := \|x\|_C + h_\alpha(x), \tag{27}$$

where

$$h_\alpha(x) := \sup_{s \neq t} \frac{|x(s) - x(t)|}{|s-t|^\alpha} \qquad (s, t \in [a,b])$$

denotes the minimal Hölder constant of x on $[a,b]$. As before, we also consider the equivalent norm

$$\|x\|_{C^{0,\alpha}} := |x(a)| + h_\alpha(x) \tag{28}$$

which is sometimes easier to calculate in applications.

Interestingly, the composition operator (2) exhibits similar pathologies in the space $C^{0,\alpha}([a,b])$ as in the space $C^1([a,b])$. The following example (see [4, Section 7.3]) is in a certain sense parallel to Example 2.

Example 3. Let $f\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t,u) := \begin{cases} 0 & \text{if } u \leq t^{\alpha/2}, \\ \dfrac{1}{u^{2/\alpha}} - \dfrac{t}{u^{4/\alpha}} & \text{if } u > t^{\alpha/2}. \end{cases}$$

Again, a somewhat cumbersome calculation shows then that the operator (2) generated by this function maps $C^{0,\alpha}([0,1])$ into itself, but f is discontinuous at $(0,0)$, and so F does not map $C([0,1])$ into itself.

In contrast to Example 2, the reason for the pathological behaviour of the function f in Example 3 is the lack of boundedness of the corresponding composition operator F in the norm (27). In fact, one may prove (see [4, Theorem 7.3]) the following result.

Theorem 8. *The composition operator* (2) *maps the space* $C^{0,\alpha}([a,b])$ *into itself and is bounded with respect to the norm* (27) *if and only if* f *satisfies the mixed local Hölder-Lipschitz condition*

$$|f(s,u) - f(t,v)| \leq k(r)\big(|s-t|^\alpha + |u-v|\big) \qquad (|u|, |v| \leq r). \tag{29}$$

In particular, f *is then necessarily continuous on* $[a,b] \times \mathbb{R}$.

Of course, in the special case of the autonomous operator (1), condition (29) takes the simpler form (20). The following result generalizes this special case in the sense that it is also applicable if f is known only on a certain subset M.

Theorem 9. *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *generate the composition operator* (1). *Then for any bounded set* $M \subseteq \mathbb{R}$ *and real numbers* $0 < \alpha \leq 1$, $K \geq 0$, *and* $r \geq \operatorname{diam} M/(b-a)^\alpha$ *the following statements are equivalent:*

(1) $f|_M$ *satisfies a Lipschitz condition with constant* $L \leq K$.
(2) *For any* $x \in C^{0,\alpha}([a,b])$ *with* $x([a,b]) \subseteq \operatorname{conv} M$ *and* $h_\alpha(x) \leq r$ *the estimate*

$$|Fx(t) - Fx(s)| \leq Kr|t-s|^\alpha \tag{30}$$

holds for all $s, t \in [a,b]$ *with* $x(s), x(t) \in M$, *where* $\operatorname{conv} M$ *denotes the convex hull of* M.

Proof. If $f|_M$ is Lipschitz with constant K, then we have for any $x \in C^{0,\alpha}([a,b])$ (even without the assumption $x([a,b]) \subseteq \operatorname{conv} M$) and any $s, t \in [a,b]$ with $x(s), x(t) \in M$ that

$$|f(x(t)) - f(x(s))| \leq K|x(t) - x(s)| = Kh_\alpha(x)|t-s|^\alpha,$$

which implies (30) even for any $r \geq h_\alpha(x)$.

Conversely, given $u, v \in M$, $u \neq v$, we find by the hypothesis $r \geq \operatorname{diam} M/(b-a)^\alpha$ numbers $s, t \in [a,b]$, $s < t$, with $|u - v| = r|t-s|^\alpha$. Let $x\colon [a,b] \to \mathbb{R}$ be defined by

$$x(\tau) := \begin{cases} u & \text{if } \tau \leq s, \\ v & \text{if } \tau \geq t, \\ u + \Big(\dfrac{\tau - s}{t - s}\Big)^\alpha (v - u) & \text{if } \tau \in [s,t]. \end{cases}$$

The concavity of $\rho \mapsto \rho^\alpha$ implies

$$|x(\tau) - x(\sigma)| \leq |v - u|\Big|\frac{\tau - \sigma}{t - s}\Big|^\alpha \qquad (\tau, \sigma \in [a,b]),$$

and so $h_\alpha(x) \leq r$. Hence, (30) implies

$$|f(u) - f(v)| = |Fx(s) - Fx(t)| \leq Kr|t-s|^\alpha = K|u-v|,$$

i.e. $f|_M$ is Lipschitz with constant K. □

We return to the problem of verifying a local Lipschitz condition for the operator (1). Suppose that f' exists and satisfies the local condition (11), and consider the constant $\widetilde{k}(r)$ from (14). In [21], the author claims that the operator F generated by f satisfies then the local condition (6) in the norm (27); however, the proof given there is false. In fact, for $x, y \in C^{0,\alpha}([0,1])$

with $\|x\|_{C^{0,\alpha}} \leq r$ and $\|y\|_{C^{0,\alpha}} \leq r$ the author uses the estimate

$$|f(x(s)) - f(y(s)) - f(x(t)) + f(y(t))|$$

$$= \Big|\big(x(s) - x(t)\big) \int_0^1 \big(f'(\theta x(s) + (1-\theta)x(t)) - f'(\theta y(s) + (1-\theta)y(t))\big)\, d\theta$$

$$+ \big(x(s) - x(t) - y(s) + y(t)\big) \int_0^1 f'\big(\theta y(s) + (1-\theta)y(t)\big)\, d\theta\Big|$$

$$\leq 2rk(r) \int_0^1 |\theta\big(x(s) - y(s)\big) + (1-\theta)\big(x(t) - y(t)\big)|\, d\theta +$$

$$\widetilde{k}(r)|x(s) - x(t) - y(s) + y(t)| \leq \big(rk(r) + \widetilde{k}(r)\big)|x(s) - x(t) - y(s) + y(t)|.$$

Unfortunately, the estimate

$$\int_0^1 |\theta\big(x(s) - y(s)\big) + (1-\theta)\big(x(t) - y(t)\big)|\, d\theta \leq \frac{1}{2}|x(s) - x(t) - y(s) + y(t)|$$

used in the last step is obviously false, and so the proof does not work. In the following theorem we give an alternative proof which builds again on Lemma 2.

Theorem 10. *If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and satisfies* (11)*, then the composition operator F generated by f maps $X = C^{0,\alpha}([a,b])$ into itself and satisfies* (6) *with respect to the norm* (27) *with some K satisfying* (24)*.*

Conversely, if the composition operator F generated by a function $f\colon \mathbb{R} \to \mathbb{R}$ maps the space $X = C^{0,\alpha}([a,b])$ into itself and satisfies (6) *with respect to the norm* (27) *or* (28)*, then f' exists on $\mathbb{R}$ and satisfies* (11) *with some k such that*

$$k(r) \leq \frac{K(r+R)}{R} \text{ for any } R \geq \frac{2r}{(b-a)^\alpha}. \tag{31}$$

Again, the estimate (31) contains as a special case the Matkowski property for the spaces $X = C^{0,\alpha}$ which was proved for $\alpha < 1$ in [9] and for $\alpha = 1$ in [7]. Indeed, if $K(r) \equiv K$ is independent of r, then letting $R \to \infty$ in (31) leads to $k(r) = 0$ which by (11) means that f has the form (4).

More generally, in the same manner, one can show that for non-affine f any function K satisfying (6) must grow at least linearly at ∞.

Proof. Suppose first that the derivative f' of f exists and satisfies the local Lipschitz condition (11). Define $\widetilde{k}(r)$ as in (14). Fix $s,t \in [a,b]$ and $x,y \in C^{0,\alpha}([a,b])$ with $\|x\|_{C^{0,\alpha}} \leq r$ and $\|y\|_{C^{0,\alpha}} \leq r$, and let $u := Fx$ and $v := Fy$. Applying now Lemma 2 to the choice $x_1 := x(s)$, $y_1 := y(s)$, $x_2 := x(t)$, and $y_2 := y(t)$ (with $s \neq t$) yields

$$|u(s) - v(s) - u(t) + v(t)| \leq$$

$$2k(r)\big(|x(s) - x(t)| + |y(s) - y(t)|\big)\|x - y\|_C + \widetilde{k}(r)|x(s) - y(s) - x(t) + y(t)|.$$

Dividing by $|s-t|^\alpha$ and passing to the supremum over $s \neq t$, we arrive at

$$h_\alpha(u-v) \le 2k(r)\big(h_\alpha(x)+h_\alpha(y)\big)\|x-y\|_C + \widetilde{k}(r)h_\alpha(x-y) \le \widehat{k}(r)\|x-y\|_{C^{0,\alpha}},$$

with the same $\widehat{k}(r)$ as in the proof of Theorem 7, which proves the "if" part of Theorem 10.

To prove the "only if" part, let us now suppose that F satisfies a local Lipschitz condition (6) in the norm (27) of the space $C^{0,\alpha}([a,b])$. Putting again $x(t) \equiv u$ and $y(t) \equiv v$ in (6) (or applying Theorem 8 or 9) we see that f satisfies the local Lipschitz condition (20). We will only use that f is absolutely continuous on $I = [-r,r]$. There is a null set $N \subseteq \mathbb{R}$, without loss of generality $\pm r \in N$, such that f' exists on $M := (-r,r) \setminus N = I \setminus N$. By Lemma 3, we are to show that the function $f'|_M$ satisfies a Lipschitz condition with constant $L \le K(r+R)/R$ where $R \ge 2r/(b-a)^\alpha$.

Fix $x \in C^{0,\alpha}([a,b])$ with $x([a,b]) \subseteq (-r,r)$ and $h_\alpha(x) \le R$. For sufficiently small $\lambda > 0$ we have $\|x_\lambda\|_{C^{0,\alpha}} < r+R$, where $x_\lambda(t) := x(t)+\lambda$. By assumption, we have, for all $s,t \in [a,b]$, $s \neq t$, that

$$\frac{|f(x(t)+\lambda) - f(x(t)) - f(x(s)+\lambda) + f(x(s))|}{\lambda|t-s|^\alpha} =$$

$$\frac{Fx_\lambda(t) - Fx(t) - Fx_\lambda(s) + Fx(s)}{\lambda|t-s|^\alpha} \le \frac{h_\alpha(Fx_\lambda - Fx)}{\lambda} \le K(r+R).$$

Letting $\lambda \to 0$, we conclude that

$$\frac{|f'(x(s)) - f'(x(t))|}{|s-t|^\alpha} \le K(r+R)$$

for all $s,t \in [a,b]$, $s \neq t$, with $x(s), x(t) \in M$. Since this holds for all $x \in C^{0,\alpha}([a,b])$ satisfying $x([a,b]) \subseteq \operatorname{conv} M$ and $h_\alpha(x) \le R$, and since $R \ge \operatorname{diam} M/(b-a)$ we can apply Theorem 9 with an arbitrary extension of $f'|_M$ and obtain that $f'|_M$ is Lipschitz on M with constant $L \le K(r+R)/R$. □

We point out that the abstract Theorems 4, 7 and 10 given above apply to any problem involving a nonlinearity f whose derivative satisfies (11). For example, we may apply Theorem 10 to weakly singular nonlinear integral equations of Hammerstein-Volterra type

$$x(s) - \lambda \int_0^s \frac{k(s,t)f(x(t))}{|s-t|^\nu}\, dt = y(s) \qquad (0 \le s \le 1), \tag{32}$$

where $k\colon [0,1]\times[0,1] \to \mathbb{R}$ is a continuous kernel function and λ, ν are real parameters satisfying $\lambda > 0$ and $0 < \nu < 1$. To this end, we may write (32) as operator equation

$$x - \lambda K_\nu F x = y \qquad (x \in X), \tag{33}$$

where K_ν is the linear weakly singular integral operator defined by

$$K_\nu x(s) = \int_0^s \frac{k(s,t)x(t)}{|s-t|^\nu}\, dt \qquad (0 \le s \le 1), \tag{34}$$

and F is the nonlinear composition operator (1) generated by the function f. A suitable space for studying equation (33) is the space $C_0^{0,\alpha}([0,1])$ of all functions $x \in C^{0,\alpha}([0,1])$ satisfying $x(0) = 0$. From the second Hardy-Littlewood theorem (see [3] or [6]), the linear weakly singular integral operator (34) maps $C_0^{0,\alpha}([0,1]) \cap L_p([0,1])$ into $C_0^{0,\alpha}([0,1])$ and is bounded if

$$\frac{1}{1-\nu} < p \le \infty, \qquad 0 < \alpha \le 1 - \nu - \frac{1}{p}.$$

For simplicity, let us take $p = \infty$, hence $0 < \alpha \le 1 - \nu$ and $C_0^{0,\alpha}([0,1]) \cap L_\infty([0,1]) = C_0^{0,\alpha}([0,1])$. By Theorem 10, the local Lipschitz condition

$$|f'(u) - f'(v)| \le k(r)|u - v| \qquad (|u|, |v| \le r) \tag{35}$$

is necessary and sufficient for the corresponding composition operator F to satisfy (6). Clearly, a sufficient condition for (35) in turn is the boundedness of the second derivative of f on the interval $[-r, r]$. From Banach's fixed point theorem we get then existence and uniqueness of a solution x of the operator equation (33), and so also of the integral equation (32), in a ball of suitable radius r. On the other hand, we usually have $k(r) \to \infty$ as $r \to \infty$ in (35), and so we cannot expect global existence and uniqueness in the whole space $C^{0,\alpha}([0,1])$. This again justifies replacing the global condition (3) by the local condition (6).

In view of similar applications to partial differential equations, we formulate the sufficient part of Theorem 10 in a more general setting. For a metric space Ω (not necessarily compact or of finite diameter), we define $C^{0,\alpha}(\Omega)$ as the set of all *bounded* and Hölder/Lipschitz continuous functions $x\colon \Omega \to \mathbb{R}$ with exponent $0 < \alpha \le 1$. We equip $C^{0,\alpha}(\Omega)$ with the norm

$$\|x\| := \|x\|_\infty + h_\alpha(x) \tag{36}$$

(with the obvious modifications in the definitions of $\|\cdot\|_\infty$ and h_α). Then the following result holds.

Theorem 11. *If $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R}$ and satisfies* (11) *(in particular if $f \in C^2(\mathbb{R})$), then the composition operator F generated by f maps $X = C^{0,\alpha}(\Omega)$ into itself and satisfies* (6) *with the norm* (36)*. More precisely, one can choose K satisfying* (24)*.*

The proof consists in an obvious modification of the arguments used in the first parts of the proofs of Theorems 9 and 10.

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