Fixed Point Theory, 12(2011), No. 1, 217-232
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ALGORITHMS FOR APPROXIMATING MINIMIZATION PROBLEMS

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Abstract. In this paper, we study the following minimization problem

$$\min_{x\in F(S)\cap\Omega} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x\|^2,$$

where B is a bounded linear operator, $\mu \geq 0$ is some constant, $F(T)$ is the set of fixed points of nonexpansive mapping S and Ω is the solution set of an equilibrium problem. This paper introduces two new algorithms (one implicit and one explicit) that can be used to find the solution of the above minimization problem.

Key Words and Phrases: Nonexpansive mapping, monotone mapping, fixed point, equilibrium problem, minimization problem.
2010 Mathematics Subject Classification: 49J40, 47H10, 47H17, 49M05, 90C25, 90C99.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called α-inverse-strongly monotone if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \forall x, y \in C.$$

A mapping $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C.$$

Denote the set of fixed points of S by $F(S)$. Let B be a strongly positive bounded linear operator on H, that is, there exists a constant $\gamma > 0$ such that

$$\langle Bx, x\rangle \geq \gamma\|x\|^2, \forall x \in H.$$

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Let $A : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The solution set of (1.1) is denoted by Ω. If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \forall y \in C.$$

If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \forall y \in C.$$

Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. The equilibrium problems and the variational inequality problems have been investigated by many authors. Please see [6]-[35] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, e.g., [1], [3], [4], [5].

For solving equilibrium problem (1.1), Moudafi [5] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [3] introduced another iterative algorithm for finding an element of $F(S) \cap \Omega$ and they obtained a strong convergence result. Ceng and Yao [27] introduced an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Ceng, Schaible and Yao [29] introduced an implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings. Peng and Yao [32] introduced a new hybrid-extragradient method for generalized equilibrium problems and fixed point problems and variational inequality problems. In order to find a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping, very recently Yao et al [35] introduced the following two algorithms

$$x_t = SP_C[(1 - t)T_r(x_t - rAx_t)], \forall t \in (0, 1), \tag{1.2}$$

and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)SP_C[(1 - \alpha_n)T_r(x_n - rAx_n)], n \geq 0, \tag{1.3}$$

Furthermore, they proved that the proposed algorithms (1.2) and (1.3) converge strongly to a solution of the following minimization problem of finding $x^* \in F(S) \cap \Omega$ such that

$$\|x^*\| = \min_{x \in F(S) \cap \Omega} \|x\| \tag{1.4}$$

Motivated and inspired by the works in this direction in the literature, in this paper, we will study the following minimization problem

$$\min_{x\in F(S)\cap\Omega} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x\|^2, \tag{1.5}$$

where B is a bounded linear operator and $\mu \geq 0$ is some constant.

If we take $\mu = 0$ in (1.5), then the minimization problem (1.5) reduces to the minimization problem (1.4).

This paper introduces two new algorithms (one implicit and one explicit) that can be used to find the solution of the above minimization problem.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;
(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H. \tag{2.1}$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.2}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.3}$$

for all $x \in H$ and $y \in C$.

We need the following lemmas for proving our main results.

Lemma 2.1. *([2]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C.$$

Further, if $T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$, then the following hold:

(i) *T_r is single-valued and T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*
(ii) *Ω is closed and convex and $\Omega = F(T_r)$.*

Lemma 2.2. *([8]) Let C be a nonempty closed convex subset of a real Hilbert space H. Let the mapping $A : C \to H$ be α-inverse strongly monotone and $r > 0$ be a constant. Then, we have*

$$\|(I - rA)x - (I - rA)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \forall x, y \in H.$$

In particular, if $0 \leq r \leq 2\alpha$, then $I - rA$ is nonexpansive.

Lemma 2.3. *([19]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.4. *([10]) Let C be a closed convex subset of a real Hilbert space H and let $S : C \to C$ be a nonexpansive mapping. Then, the mapping $I - S$ is demiclosed. That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ weakly and $(I - S)x_n \to y$ strongly, then $(I - S)x^* = y$.*

Lemma 2.5. *Let C be a nonempty closed convex subset of a real Hilbert space H, and $g : C \to R \cup \{\infty\}$ be a proper lower-semicontinuous differentiable convex function. If z^* is a solution to the minimization problem*

$$g(z^*) = \inf_{x\in F(S)\cap\Omega} g(x),$$

then $\langle g'(x), z^* - x\rangle \leq 0, \forall x \in F(S) \cap \Omega$.

In particular, if z^* solves problem (1.5), then

$$\langle(I + \mu B))z^*, z^* - x\rangle \leq 0, \forall x \in F(S) \cap \Omega.$$

Proof. Since $F(S) \cap \Omega$ is convex , $z^* + t(x - z^*) \in F(S) \cap \Omega$ for all $x \in F(S) \cap \Omega$ and $0 < t < 1$. Hence

$$\lim_{t\to 0+} \frac{g(z^* + t(x - z^*)) - g(z^*)}{t} = \langle g'(z^*), x - z^*\rangle \geq 0.$$

In particular, if

$$\begin{aligned} g(x) &= \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x\|^2 \\ &= \frac{1}{2}\langle(I + \mu B)x, x\rangle, \end{aligned}$$

then

$$g'(x) = (I + \mu B)x.$$

Therefore, we obtain

$$\langle(I + \mu B))z^*, z^* - x\rangle \leq 0, \forall x \in F(S) \cap \Omega.$$

This completes the proof. □

Lemma 2.6. *([20]) Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping, $A : C \to H$ be an α-inverse strongly monotone mapping and $B : H \to H$ be a strongly positive bounded linear operator. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). In this section we will devote to find the solution of the minimization problem (1.5).

In order to find a solution of the minimization problem (1.5), we construct the following implicit algorithm

$$x_t = P_C\Big[\Big(I - t(I+\mu B)\Big)ST_r(x_t - rAx_t)\Big], \forall t \in (0,1), \tag{3.1}$$

where T_r is defined as Lemma 2.1. Now we show that $\{x_t\}$ is well-defined. As a matter of fact, we consider the mapping

$$W_t x := P_C\Big[\Big(I - t(I+\mu B)\Big)ST_r(x - rAx)\Big], \forall t \in (0,1), x \in C.$$

Since B is linear bounded self-adjoint operator on H, then

$$\|B\| = \sup\{|\langle Bu, u\rangle| : u \in H, \|u\| = 1\}.$$

For a small enough t, we have

$$\begin{aligned} \langle (I-(I+\mu B)t)u, u\rangle &= 1 - t - t\mu\langle Bu, u\rangle \\ &\geq 1 - t - t\mu\|B\| \\ &\geq 0, \end{aligned}$$

that is to say $I - t(I+\mu B)$ is positive for a small enough t.

Hence, we have

$$\begin{aligned} \|I - t(I+\mu B)\| &= \sup\{\langle (I - t(I+\mu B))u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - t - t\mu\langle Bu, u\rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - t - t\mu\gamma. \end{aligned}$$

Since S, T_r and $(I-rA)$ are nonexpansive, we get

$$\begin{aligned}\|W_tx-W_ty\| &= \|P_C[(I-(I+\mu B)t)ST_r(x-rAx)]\\ &\quad -P_C[(I-(I+\mu B)t)ST_r(y-rAy)]\|\\ &\leq \|[(I-(I+\mu B)t)ST_r(x-rAx)]\\ &\quad -[(I-(I+\mu B)t)ST_r(y-rAy)]\|\\ &\leq \|I-(I+\mu B)t\|\|ST_r(x-rAx)-ST_r(y-rAy)\|\\ &\leq (1-t-t\mu\gamma)\|x-y\|.\end{aligned}$$

This implies that W_t is a contraction. Using the Banach contraction principle, there exists a unique fixed point x_t of W_t in C, i.e.,

$$x_t = P_C\Big[\Big(I-t(I+\mu B)\Big)ST_r(x_t-rAx_t)\Big], \forall t\in(0,1).$$

If we take $\mu=0$ in (3.1), then we have

$$x_t = P_C\Big[(1-t)ST_r(x_t-rAx_t)\Big], \forall t\in(0,1), \tag{3.2}$$

Below is the first result of this paper which displays the behavior of the net $\{x_t\}$ as $t\to 0$.

Theorem 3.1. *Suppose $F(S)\cap\Omega\neq\emptyset$. Then the net $\{x_t\}$ defined by the implicit method (3.1) converges in norm, as $t\to 0$, to z^* which solves the minimization problem (1.5).*

Proof. First, we prove that $\{x_t\}$ is bounded. Set $u_t = T_r(x_t-rAx_t)$ for all $t\in(0,1)$. Take $z\in F(S)\cap\Omega$. It is clear that $z=T_r(z-rAz)$. Since T_r is nonexpansive and A is α-inverse-strongly monotone, we have from Lemma 2.2 that

$$\begin{aligned}\|u_t-z\|^2 &= \|T_r(x_t-rAx_t)-T_r(z-rAz)\|^2\\ &\leq \|x_t-rAx_t-(z-rAz)\|^2\\ &\leq \|x_t-z\|^2+r(r-2\alpha)\|Ax_t-Az\|^2\\ &\leq \|x_t-z\|^2.\end{aligned}\tag{3.3}$$

So, we have that

$$\|u_t-z\|\leq\|x_t-z\|.$$

It follows from (3.1) that

$$\begin{aligned}\|x_t-z\| &= \|P_C[(I-(I+\mu B)t)Su_t]-P_C[(I-(I+\mu B)t)Sz]\\ &\quad +P_C[(I-(I+\mu B)t)Sz]-P_C[Sz]\|\\ &\leq \|P_C[(I-(I+\mu B)t)Su_t]-P_C[(I-(I+\mu B)t)Sz]\|\\ &\quad +\|P_C[(I-(I+\mu B)t)Sz]-P_C[Sz]\|\\ &\leq \|I-(I+\mu B)t\|\|Su_t-Sz\|+\|[(I-(I+\mu B)t)Sz]-Sz\|\\ &\leq (1-(1+\mu\gamma)t)\|u_t-z\|+t\|(I+\mu B)z\|\\ &\leq (1-(1+\mu\gamma)t)\|x_t-z\|+t\|(I+\mu B)z\|\end{aligned}\tag{3.4}$$

that is,

$$\|x_t - z\| \le \frac{\|(I+\mu B)z\|}{1+\mu\gamma}.$$

So, $\{x_t\}$ is bounded. Hence $\{u_t\}$ is bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

From (3.3) and (3.4), we have

$$\begin{aligned} \|x_t - z\|^2 &\le [(1-(1+\mu\gamma)t)\|u_t - z\| + t\|(I+\mu B)z\|]^2 \\ &= (1-(1+\mu\gamma)t)^2\|u_t - z\|^2 + t^2\|(I+\mu B)z\|^2 \\ &\quad +2(1-(1+\mu\gamma)t)t\|(I+\mu B)z\|\|u_t - z\| \\ &\le \|u_t - z\|^2 + tM \qquad (3.5) \\ &\le \|x_t - z\|^2 + r(r-2\alpha)\|Ax_t - Az\|^2 + tM. \end{aligned}$$

This means that

$$r(2\alpha - r)\|Ax_t - Az\|^2 \le tM \to 0.$$

Since $r(2\alpha - r) > 0$, we deduce

$$\lim_{t\to 0}\|Ax_t - Az\| = 0. \qquad (3.6)$$

From Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned} \|u_t - z\|^2 &= \|T_r(x_t - rAx_t) - T_r(z - rAz)\|^2 \\ &\le \langle (x_t - rAx_t) - (z - rAz), u_t - z\rangle \\ &= \frac{1}{2}\Big(\|(x_t - rAx_t) - (z - rAz)\|^2 + \|u_t - z\|^2 \\ &\quad -\|(x_t - z) - r(Ax_t - Az) - (u_t - z)\|^2\Big) \\ &\le \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \|(x_t - u_t) - r(Ax_t - Az)\|^2\Big) \\ &= \frac{1}{2}\Big(\|x_t - z\|^2 + \|u_t - z\|^2 - \|x_t - u_t\|^2 \\ &\quad +2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2\Big), \end{aligned}$$

which implies that

$$\begin{aligned} \|u_t - z\|^2 &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + 2r\langle x_t - u_t, Ax_t - Az\rangle - r^2\|Ax_t - Az\|^2 \\ &\le \|x_t - z\|^2 - \|x_t - u_t\|^2 + M\|Ax_t - Az\|. \qquad (3.7) \end{aligned}$$

By (3.5) and (3.7), we have

$$\|x_t - z\|^2 \le \|x_t - z\|^2 - \|x_t - u_t\|^2 + (\|Ax_t - Az\| + t)M.$$

It follows that

$$\|x_t - u_t\|^2 \le (\|Ax_t - Az\| + t)M.$$

This together with (3.6) imply that

$$\lim_{t\to 0}\|x_t - u_t\| = 0.$$

Hence,

$$\begin{aligned}\|x_t - Sx_t\| &= \|P_C[(I-(I+\mu B)t)Su_t] - P_C Sx_t\| \\ &\le \|Su_t - Sx_t\| + t\|(I+\mu B)Su_t\| \\ &\le \|u_t - x_t\| + t\|(I+\mu B)Su_t\| \to 0. \end{aligned} \tag{3.8}$$

Next we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $u_n := u_{t_n}$. From (3.8), we get

$$\|x_n - Sx_n\| \to 0. \tag{3.9}$$

By (3.1), we deduce

$$\begin{aligned}\|x_t - z\|^2 &= \|P_C[(I-(I+\mu B)t)Su_t] - P_C z\|^2 \\ &\le \|(I-(I+\mu B)t)Su_t - z\|^2 \\ &= \|(I-(I+\mu B)t)(Su_t - z) - t(I+\mu B)z\|^2 \\ &= \|(I-(I+\mu B)t)(Su_t - z)\|^2 - 2t\langle (I+\mu B)z, Su_t - z\rangle \\ &\quad +2t^2\langle (I+\mu B)z, (I+\mu B)(Su_t - z)\rangle + t^2\|(I+\mu B)z\|^2 \\ &\le (1-t-t\mu\gamma)^2\|u_t - z\|^2 - 2t\langle (I+\mu B)z, Su_t - z\rangle \\ &\quad +2t^2\|(I+\mu B)z\|\|(I+\mu B)(Su_t - z)\| + t^2\|(I+\mu B)z\|^2 \\ &\le (1-t-t\mu\gamma)\|x_t - z\|^2 + 2t\langle (I+\mu B)z, z - Su_t\rangle + t^2 M. \end{aligned}$$

It follows that

$$\|x_t - z\|^2 \le \frac{2}{1+\mu\gamma}\langle (I+\mu B)z, z - Su_t\rangle + \frac{tM}{1+\mu\gamma}.$$

In particular,

$$\|x_n - z\|^2 \le \frac{2}{1+\mu\gamma}\langle (I+\mu B)z, z - Su_n\rangle + \frac{t_n M}{1+\mu\gamma}, z \in F(S)\cap\Omega. \tag{3.10}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to a point $z^* \in C$. Noticing (3.9) we can use Lemma 2.4 to get $z^* \in F(S)$.

Now we show $z^* \in \Omega$. Since $u_n = T_r(x_n - rAx_n)$, for any $y \in C$ we have

$$F(u_n, y) + \frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \ge 0.$$

From the monotonicity of F, we have

$$\frac{1}{r}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \ge F(y, u_n), \forall y \in C.$$

Hence,

$$\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \ge F(y, u_{n_i}), \forall y \in C. \tag{3.11}$$

Put $z_t = ty + (1-t)z^*$ for all $t \in (0,1]$ and $y \in C$. Then, we have $z_t \in C$. So, from (3.11) we have

$$\begin{aligned}\langle z_t - u_{n_i}, Az_t\rangle &\geq \langle z_t - u_{n_i}, Az_t\rangle - \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r} + Ax_{n_i}\rangle \\ &\quad + F(z_t, u_{n_i}) \\ &= \langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle + \langle z_t - u_{n_i}, Au_{n_i} - Ax_{n_i}\rangle \\ &\quad - \langle z_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r}\rangle + F(z_t, u_{n_i}). \end{aligned} \tag{3.12}$$

Note that $\|Au_{n_i} - Ax_{n_i}\| \leq \frac{1}{\alpha}\|u_{n_i} - x_{n_i}\| \to 0$. Further, from monotonicity of A, we have $\langle z_t - u_{n_i}, Az_t - Au_{n_i}\rangle \geq 0$. Letting $i \to \infty$ in (3.12), we have

$$\langle z_t - z^*, Az_t\rangle \geq F(z_t, z^*). \tag{3.13}$$

From (H1), (H4) and (3.13), we also have

$$\begin{aligned} 0 &= F(z_t, z_t) \leq tF(z_t, y) + (1-t)F(z_t, z^*) \\ &\leq tF(z_t, y) + (1-t)\langle z_t - z^*, Az_t\rangle \\ &= tF(z_t, y) + (1-t)t\langle y - z^*, Az_t\rangle \end{aligned}$$

and hence

$$0 \leq F(z_t, y) + (1-t)\langle Az_t, y - z^*\rangle. \tag{3.14}$$

Letting $t \to 0$ in (3.14), we have, for each $y \in C$,

$$0 \leq F(z^*, y) + \langle y - z^*, Az^*\rangle.$$

This implies that $z^* \in \Omega$. Therefore, $z^* \in F(S) \cap \Omega$.

We substitute z^* for z in (3.10) to get

$$\|x_n - z^*\|^2 \leq \frac{2}{1+\mu\gamma}\langle (I+\mu B)z^*, z^* - Su_n\rangle + \frac{t_n M}{1+\mu\gamma}.$$

Note that $Su_n \rightharpoonup z^*$ weakly. This facts and the last inequality imply that $x_n \to z^*$ strongly. This has proved the relative norm compactness of the net $\{x_t\}$ as $t \to 0$.

Now we return to (3.10) and take the limit as $n \to \infty$ to get

$$\|z^* - z\|^2 \leq \frac{2}{1+\mu\gamma}\langle (I+\mu B)z, z - z^*\rangle, z \in F(S) \cap \Omega. \tag{3.15}$$

In particular, z^* solves the following variational inequality

$$z^* \in F(S) \cap \Omega, \quad \langle (I+\mu B)z, z - z^*\rangle \geq 0, \quad z \in F(S) \cap \Omega,$$

or the equivalent dual variational inequality

$$z^* \in F(S) \cap \Omega, \quad \langle (I+\mu B)z^*, z - z^*\rangle \geq 0, \quad z \in F(S) \cap \Omega, \tag{3.16}$$

To show that the entire net $\{x_t\}$ converges to z^*, assume $x_{s_n} \to \tilde{z} \in F(S) \cap \Omega$, where $s_n \to 0$. We substitute $\tilde{z}$ for z in (3.16) to get

$$\langle (I+\mu B)z^*, \tilde{z} - z^*\rangle \geq 0. \tag{3.17}$$

Interchange z^* and $\tilde{z}$ to obtain

$$\langle (I+\mu B)\tilde{z}, z^* - \tilde{z}\rangle \geq 0. \tag{3.18}$$

Adding up (3.17) and (3.18) yields

$$(1+\mu\gamma)\|z^*-\tilde{z}\|^2 \le \langle (I+\mu B)(z^*-\tilde{z}), z^*-\tilde{z}\rangle \le 0,$$

which implies that $\tilde{z}=z^*$. By (3.16) and Lemma 2.5, we deduce immediately the desired result. This completes the proof. □

Theorem 3.2. *Suppose $F(S)\cap\Omega\neq\emptyset$. Then the net $\{x_t\}$ defined by the implicit method (3.2) converges in norm, as $t\to 0$, to z^* which solves the minimization problem (1.4).*

Next we introduce an explicit algorithm for finding a solution of minimization problem (1.5). This scheme is obtained by discretizing the implicit scheme (3.1). We will show the strong convergence of this algorithm.

Theorem 3.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S: C\to C$ be a nonexpansive mapping, $A: C\to H$ be an α-inverse strongly monotone mapping and $B: H\to H$ be a strongly positive bounded linear operator. Let $F: C\times C\to R$ be a bifunction which satisfies conditions (H1)-(H4). Suppose $F(S)\cap\Omega\neq\emptyset$. For given $x_0\in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1}=\beta_n x_n+(1-\beta_n)P_C\Big[\Big(I-\alpha_n(I+\mu B)\Big)ST_r(x_n-rAx_n)\Big], n\ge 0, \tag{3.19}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n=0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$*;*
(C2) $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1$.

Then the sequence $\{x_n\}$ converges strongly to z^* which solves the minimization problem (1.5).

Proof. We divide our proof into the following steps

(1) $\|x_{n+1}-x_n\|\to 0$.
(2) $\|ST_r(x_n-rAx_n)-T_r(x_n-rAx_n)\|\to 0$.
(3) $\limsup_{n\to\infty}\langle (I+\mu B)z^*, z^*-T_r(x_n-rAx_n)\rangle\le 0$.
(4) $x_n\to z^*$.

Proof of (1). Let $z\in F(S)\cap\Omega$. Set $u_n=T_r(x_n-rAx_n)$ for all $n\ge 0$.
From Lemma 2.1, we get

$$\begin{aligned}\|u_n-z\| &= \|T_r(x_n-rAx_n)-T_r(z-rAz)\| \\ &\le \|x_n-z\|.\end{aligned} \tag{3.20}$$

We write (3.19) as $x_{n+1}=\beta_n x_n+(1-\beta_n)z_n$ where

$$z_n=P_C[(I-\alpha_n(I+\mu B))Su_n], n\ge 0.$$

It follows that

$$\begin{aligned}
\|z_{n+1}-z_n\| &= \|P_C[(I-\alpha_{n+1}(I+\mu B))Su_{n+1}]-P_C[(I-\alpha_n(I+\mu B))Su_n]\| \\
&\leq \|(I-\alpha_{n+1}(I+\mu B))Su_{n+1}-(I-\alpha_n(I+\mu B))Su_n\| \\
&\leq \|Su_{n+1}-Su_n\|+\alpha_{n+1}\|(I+\mu B)Su_{n+1}\|+\alpha_n\|(I+\mu B)Su_n\| \\
&\leq \|u_{n+1}-u_n\|+\alpha_{n+1}\|(I+\mu B)Su_{n+1}\| \\
&\quad +\alpha_n\|(I+\mu B)Su_n\|. \qquad (3.21)
\end{aligned}$$

Note that the control conditions (C1) and (C2), we may assume, without loss of generality, that $\alpha_n \leq \min\{(1+\mu\|B\|)^{-1}, \frac{1}{1+\mu\gamma}\}$. Thus, we have

$$\|I-\alpha_n(I+\mu B)\| \leq 1-\alpha_n-\alpha_n\mu\gamma. \qquad (3.22)$$

From (3.19) and (3.22), we obtain

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\beta_n(x_n-z)+(1-\beta_n)(z_n-z)\| \\
&\leq \beta_n\|x_n-z\|+(1-\beta_n)\|z_n-z\| \\
&\leq \beta_n\|x_n-z\|+(1-\beta_n)\|(I-\alpha_n(I+\mu B))Su_n-z\| \\
&\leq \beta_n\|x_n-z\|+(1-\beta_n)\|(I-\alpha_n(I+\mu B))(Su_n-z)\| \\
&\quad +(1-\beta_n)\alpha_n\|(I+\mu B)z\| \\
&\leq \beta_n\|x_n-z\|+(1-\beta_n)(1-\alpha_n-\alpha_n\mu\gamma)\|u_n-z\| \\
&\quad +(1-\beta_n)\alpha_n\|(I+\mu B)z\| \\
&\leq [1-(1+\mu\gamma)\alpha_n(1-\beta_n)]\|x_n-z\|+(1-\beta_n)\alpha_n\|(I+\mu B)z\| \\
&= [1-(1+\mu\gamma)\alpha_n(1-\beta_n)]\|x_n-z\| \\
&\quad +(1+\mu\gamma)\alpha_n(1-\beta_n)\frac{1}{1+\mu\gamma}\|(I+\mu B)z\|.
\end{aligned}$$

By induction, we have

$$\|x_n-z\| \leq \max\{\|x_0-z\|, \frac{\|(I+\mu B)z\|}{1+\mu\gamma}\}, \quad n \geq 0.$$

Hence, $\{x_n\}$ is bounded. Consequently, we deduce that $\{Ax_n\}$ and $\{u_n\}$ are bounded. We shall use M to denote the possible different constants appearing in the following reasoning.

From Lemma 2.2, we have

$$\begin{aligned}
\|u_{n+1}-u_n\| &= \|T_r(x_{n+1}-rAx_{n+1})-T_r(x_n-rAx_n)\| \\
&\leq \|(x_{n+1}-rAx_{n+1})-(x_n-rAx_n)\| \\
&\leq \|x_{n+1}-x_n\|. \qquad (3.23)
\end{aligned}$$

By (3.21) and (3.23), we derive

$$\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| \leq \Big(\alpha_{n+1}+\alpha_n\Big)M.$$

Therefore,

$$\limsup_{n\to\infty}\Big(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\Big) \leq 0.$$

Hence by Lemma 2.3, we get

$$\lim_{n\to\infty} \|z_n - x_n\| = 0.$$

Thus,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0.$$

Proof of (2). We note that

$$\begin{aligned} \|z_n - Su_n\| &= \|P_C[(I-\alpha_n(I+\mu B))Su_n] - P_C[Su_n]\| \\ &\leq \alpha_n\|(I+\mu B)Su_n\| \to 0. \end{aligned}$$

Then we have

$$\|x_n - Su_n\| \leq \|x_n - z_n\| + \|z_n - Su_n\| \to 0.$$

From (3.19), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\beta_n(x_n - z) + (1-\beta_n)(z_n - z)\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|z_n - z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|(I-\alpha_n(I+\mu B))Su_n - z\|^2 \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)\|(I-\alpha_n(I+\mu B))(Su_n - z) \\ &\quad -\alpha_n(I+\mu B)z\|^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[(1-(1+\mu\gamma)\alpha_n)\|u_n - z\| \\ &\quad +\alpha_n\|(I+\mu B)z\|]^2 \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|u_n - z\| + \alpha_n\|(I+\mu B)z\|]^2 \\ &= \beta_n\|x_n - z\|^2 + (1-\beta_n)[\|u_n - z\|^2 + \alpha_n^2\|(I+\mu B)z\|^2 \\ &\quad +2\alpha_n\|u_n - z\|\|(I+\mu B)z\|] \\ &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 + \alpha_n M. \end{aligned} \tag{3.24}$$

From Lemma 2.2, we get

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r(x_n - rAx_n) - T_r(z - rAz)\|^2 \\ &\leq \|(x_n - rAx_n) - (z - rAz)\|^2 \\ &\leq \|x_n - z\|^2 + r(r-2\alpha)\|Ax_n - Az\|^2. \end{aligned} \tag{3.25}$$

Substituting (3.25) into (3.24), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq \beta_n\|x_n - z\|^2 + (1-\beta_n)(\|x_n - z\|^2 + r(r-2\alpha)\|Ax_n - Az\|^2) \\ &\quad +\alpha_n M \\ &= \|x_n - z\|^2 + (1-\beta_n)r(r-2\alpha)\|Ax_n - Az\|^2 + \alpha_n M. \end{aligned}$$

It follows that

$$\begin{aligned} (1-\beta_n)r(r-2\alpha)\|Ax_n - Az\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n M \\ &\leq (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\| \\ &\quad +\alpha_n M. \end{aligned}$$

Since $\liminf_{n\to\infty}(1-\beta_n)r(2\alpha-r)>0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we derive

$$\lim_{n\to\infty}\|Ax_n-Az\|=0.$$

From Lemma 2.1, we obtain

$$\begin{aligned}
\|u_n-z\|^2 &= \|T_r(x_n-rAx_n)-T_r(z-rAz)\|^2\\
&\le \langle (x_n-rAx_n)-(z-rAz),u_n-z\rangle\\
&= \frac{1}{2}\Big(\|(x_n-rAx_n)-(z-rAz)\|^2+\|u_n-z\|^2\\
&\quad -\|(x_n-z)-r(Ax_n-Az)-(u_n-z)\|^2\Big)\\
&\le \frac{1}{2}\Big(\|x_n-z\|^2+\|u_n-z\|^2-\|(x_n-u_n)-r(Ax_n-Az)\|^2\Big)\\
&= \frac{1}{2}\Big(\|x_n-z\|^2+\|u_n-z\|^2-\|x_n-u_n\|^2\\
&\quad +2r\langle x_n-u_n,Ax_n-Az\rangle-r^2\|Ax_n-Az\|^2\Big).
\end{aligned}$$

Thus, we deduce

$$\begin{aligned}
\|u_n-z\|^2 &\le \|x_n-z\|^2-\|x_n-u_n\|^2+2r\|x_n-u_n\|\|Ax_n-Az\|\\
&\le \|x_n-z\|^2-\|x_n-u_n\|^2+M\|Ax_n-Az\|. \qquad (3.26)
\end{aligned}$$

From (3.24) and (3.26), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \beta_n\|x_n-z\|^2+(1-\beta_n)(\|x_n-z\|^2-\|x_n-u_n\|^2\\
&\quad +M\|Ax_n-Az\|)+\alpha_nM\\
&\le \|x_n-z\|^2-(1-\beta_n)\|x_n-u_n\|^2+M(\|Ax_n-Az\|+\alpha_n).
\end{aligned}$$

Then we have

$$\begin{aligned}
(1-\beta_n)\|x_n-u_n\|^2 &\le \|x_n-z\|^2-\|x_{n+1}-z\|^2+M(\|Ax_n-Az\|+\alpha_n)\\
&\le (\|x_n-z\|+\|x_{n+1}-z\|)\times\|x_{n+1}-x_n\|\\
&\quad +M(\|Ax_n-Az\|+\alpha_n).
\end{aligned}$$

Since $\|Ax_n-Az\|\to 0$, $\|x_n-x_{n+1}\|\to 0$ and $\alpha_n\to 0$, we deduce

$$\lim_{n\to\infty}\|x_n-u_n\|=0.$$

Note that

$$\|Su_n-u_n\|\le\|Su_n-x_n\|+\|x_n-u_n\|.$$

Therefore,

$$\|Su_n-u_n\|\to 0. \qquad (3.27)$$

Proof of (3). Now we show that $\limsup_{n\to\infty}\langle (I+\mu B)z^*,z^*-u_n\rangle\le 0$, where z^* is a solution of (OP). To show this, we can choose a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\lim_{i\to\infty}\langle (I+\mu B)z^*,z^*-u_{n_i}\rangle=\limsup_{n\to\infty}\langle (I+\mu B)z^*,z^*-u_n\rangle.$$

Since $\{u_{n_i}\}$ is bounded, there exists a subsequence $\{u_{n_{i_j}}\}$ of $\{u_{n_i}\}$ which converges weakly to u. Without loss of generality, we can assume that $u_{n_i} \rightharpoonup u$. By the same argument as that of Theorem 3.1, we can deduce that $u \in F(S) \cap \Omega$. Hence, from Lemma 2.5, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (I+\mu B)z^*, z^* - u_n\rangle &= \lim_{i\to\infty}\langle (I+\mu B)z^*, z^* - u_{n_i}\rangle \\ &= \langle (I+\mu B)z^*, z^* - u\rangle \le 0.\end{aligned}$$

This together with $\|Su_n - u_n\| \to 0$ implies that

$$\limsup_{n\to\infty}\langle (I+\mu B)z^*, z^* - Su_n\rangle = \limsup_{n\to\infty}\langle (I+\mu B)z^*, z^* - u_n\rangle \le 0.$$

Proof of (4). From (3.19), we have

$$\begin{aligned}\|x_{n+1} - z^*\|^2 &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)\|z_n - z^*\|^2 \\ &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)\|(I - \alpha_n(I+\mu B))(Su_n - z^*) \\ &\quad -\alpha_n(I+\mu B)z^*\|^2 \\ &= \beta_n\|x_n - z^*\|^2 + (1-\beta_n)[\|(I - \alpha_n(I+\mu B))(Su_n - z^*)\|^2 \\ &\quad +\alpha_n^2\|(I+\mu B)z^*\|^2 - 2\alpha_n\langle Su_n - z^*, (I+\mu B)z^*\rangle \\ &\quad +2\alpha_n^2\langle (I+\mu B)(Su_n - z^*), (I+\mu B)z^*\rangle \\ &\le \beta_n\|x_n - z^*\|^2 + (1-\beta_n)[(1-(1+\mu\gamma)\alpha_n)\|x_n - z^*\|^2 \\ &\quad +\alpha_n^2\|(I+\mu B)z^*\|^2 + 2\alpha_n\langle z^* - Su_n, (I+\mu B)z^*\rangle \\ &\quad +2\alpha_n^2\|(I+\mu B)\|\|Su_n - z^*\|\|(I+\mu B)z^*\| \\ &\le [1-(1-\beta_n)(1+\mu\gamma)\alpha_n]\|x_n - z^*\|^2 \\ &\quad +\alpha_n^2 M + 2\alpha_n\langle z^* - Su_n, (I+\mu B)z^*\rangle \\ &= (1-\gamma_n)\|x_n - z^*\|^2 + \delta_n\gamma_n,\end{aligned}$$

where $\gamma_n = (1-\beta_n)(1+\mu\gamma)\alpha_n$ and

$$\delta_n = \frac{2}{(1+\mu\gamma)(1-\beta_n)}\langle (I+\mu B)z^*, z^* - Su_n\rangle + \frac{\alpha_n M}{(1+\mu\gamma)(1-\beta_n)}.$$

It is easy to see that $\sum_{n=1}^{\infty}\gamma_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Hence, by Lemma 2.6, the sequence $\{x_n\}$ converges strongly to z^*. This completes the proof. $\square$

From Theorem 3.3, we deduce immediately the following result.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $S : C \to C$ be a nonexpansive mapping and $A : C \to H$ be an α-inverse strongly monotone mapping. Let $F : C \times C \to R$ be a bifunction which satisfies conditions (H1)-(H4). Suppose $F(S) \cap \Omega \ne \emptyset$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)P_C\Big[(1-\alpha_n)ST_r(x_n - rAx_n)\Big], n \ge 0, \tag{3.28}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$ satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$*;*

(C2) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

Then the sequence $\{x_n\}$ converges strongly to z^* which solves the minimization problem (1.4).

Remark 3.5. We would like to point out that our algorithms (3.1) and (3.19) are different from those in the literature. The algorithms (3.2) and (3.28) are also different from those in the literature including algorithms (1.2) and (1.3).

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Received: May 19, 2009; Accepted: December 29, 2009.