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# EXISTENCE OF THREE POSITIVE SOLUTIONS FOR MULTIPLE-POINT BOUNDARY VALUE PROBLEM OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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**Abstract.** By using a fixed point theorem, some new results for multiplicity of positive solutions for a class of multiple-point boundary value problem of second-order functional differential equations are obtained. The associated Green's functions of the problem are also given.
**Key Words and Phrases**: Boundary value problem, functional differential equation, positive solution, cone, fixed point.
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## 1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have received a great deal of attentions. To identify a few, we refer the reader to [1, 2, 3, 4, 5] and references therein. But most work were done under the assumption that the first order derivative $x'$ is not involved explicitly in the nonlinear term. In [6], Bai et al.studied the two-point boundary value problem

$$x''(t) + a(t)f(t, x, x') = 0 \tag{1.1}$$

subject to one of the following two pairs of boundary conditions

$$x(0) = x(1) = 0, \ \ or \ x(0) = x'(1) = 0. \tag{1.2}$$

By using a new fixed-point theorem introduced by Avery and Peterson [7], they obtained sufficient conditions for the existence of at least three positive solutions for this system.

In recent years,accompanied by the development of the theory of functional differential equations, many authors have paid attention to boundary value problem of functional differential equations (for example, see [8]-[10]). In [10], Jiang and Zhang used fixed-point index theorem in cones to study the existence of at least one positive

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solution for the boundary value problem of second-order delay differential equations of the form

$$x''(t) + f(t, x(t - \tau)) = 0, 0 < t < 1, \tau > 0, \tag{1.3}$$

$$x(t) = 0, -\tau \leq t \leq 0, \quad x(1) = 0. \tag{1.4}$$

In [11], by using the fixed-point theorem by Avery and Peterson, the authors studied the existence of three positive solutions for boundary value problem of delay differential equation

$$x''(t) + f(t, x(t), x'(t - 1)) = 0, 0 < t < 1, \tag{1.5}$$

$$x(t) = F(t), -1 \leq t \leq 0, x(1) = 0. \tag{1.6}$$

In this paper we consider existence of positive solutions for delay differential equations

$$x''(t) + f(t, x(t), x'(t - 1)) = 0, 0 < t < 1, \tag{1.7}$$

$$x(t) = F(t), -1 \leq x \leq 0, x(1) = \beta x(\xi), \tag{1.8}$$

where

$$F(-1) = 0, F(0) = \mu x(\eta), 0 < \eta < \xi < 1, \tag{1.9}$$

and problem

$$x''(t) + f_1(t, x(t), x'(t - 1)) = 0, 0 < t < 1, \tag{1.10}$$

$$x(t) = F_1(t), -1 \leq x \leq 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \tag{1.11}$$

where

$$F_1(-1) = 0, F_1(0) = \beta_1 x'_+(0), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1. \tag{1.12}$$

In this article it is assumed that:

$C_1$) $f, f_1 \in C([0, 1] \times [0, +\infty) \times R, [0, +\infty))$;

$C_2$) $0 \leq \mu < \dfrac{1}{1 - \eta}, 0 \leq \beta < \dfrac{1}{\xi}, \alpha\eta(1 - \beta) + (1 - \alpha)(1 - \beta\xi) > 0$;

$C_3$) $\alpha_i \geq 0, \beta_1 \geq 0, i = 1, 2, \cdots, m - 2$ satisfying $0 < \sum\limits_{i=1}^{m-2} \alpha_i \xi_i < 1$.

$C_4$) $F(t), F_1(t) \in C^1[-1, 0]$ is a nonnegative concave functional on [-1,0].

## 2. Background definitions and preliminaries

For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces. This definitions can be found in recent literature.

**Definition 2.1.** Let E be a real Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that

(*i*) $au \in P$, for all $u \in P, a \geq 0$;

(*ii*) $u, -u \in P$ implies $u = 0$.

Note that every one cone $P \subset E$ induces an ordering in E given by $x \leq y$ if $y - x \in P$.

**Definition 2.2.** An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

**Definition 2.3.** The map $\alpha$ is said to be a nonnegative continuous convex functional on cone P of a real Banach space E provided that $\alpha : P \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ *and* $t \in [0, 1]$.

**Definition 2.4.** The map $\beta$ is said to be a nonnegative continuous concave functional on cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in P$ and $t \in [0, 1]$.

Our main results will depend on an application of a fixed-point theorem due to Avery and Peterson .

Let $\gamma, \theta$ be nonnegative continuous convex functionals on P, $\alpha$ be a nonnegative continuous concave functional on P and $\psi$ be a nonnegative continuous functional on P. Then for positive numbers $a,\ b,\ c$ and $d$, we define the following convex sets:

$$P(\gamma, d) = \{x \in P | \gamma(x) < d\},$$
$$P(\gamma, \alpha, b, d) = \{x \in P | b \le \alpha(x), \gamma(x) \le d\},$$
$$P(\gamma, \theta, \alpha, b, c, d) = \{x \in P | b \le \alpha(x), \theta(x) \le c, \gamma(x) \le d\}$$

and a closed set

$$R(\gamma, \psi, a, d) = \{x \in P | a \le \psi(x), \gamma(x) \le d\}.$$

**Lemma 2.5.** *Let $P$ be a cone in a real Banach space $E$. Let $\gamma,\ \theta$ be nonnegative continuous convex functionals on $P$, $\alpha$ be a nonnegative continuous concave functional on $P$ and $\psi$ be a nonnegative continuous functional on $P$ satisfying:*

$$\psi(\lambda x) \le \lambda\psi(x), for\ 0 \le \lambda \le 1, \tag{2.1}$$

*such that for some positive numbers $l$ and $d$,*

$$\alpha(x) \le \psi(x), \|x\| \le l\gamma(x) \tag{2.2}$$

*for all $x \in \overline{P(\gamma, d)}$. Suppose $T : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive numbers $a,\ b,\ c$ with $a < b$ such that*

$(S_1)$ $\{x \in P(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \ne \emptyset$, *and* $\alpha(Tx) > b$ *for* $x \in P(\gamma, \theta, \alpha, b, c, d)$;

$(S_2)$ $\alpha(Tx) > b$ *for* $x \in P(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;

$(S_3)$ $0\overline{\in} R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

*Then $T$ has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma, d)}$ such that:*

$$\gamma(x_i) \le d, i = 1, 2, 3;\ b < \alpha(x_1); a < \psi(x_2),\ \alpha(x_2) < b; \psi(x_3) < a. \tag{2.3}$$

## 3. Solutions of (1.7)-(1.9)

In this section we impose growth conditions on $f$, $F$ and apply fixed-point theorem we mentioned above to establish the existence of three positive solutions of (1.7)-(1.9). Firstly we give some lemmas which are useful in the proof of our main results.

**Lemma 3.1.** *Denote $\rho = (1-\beta\xi)(1-\mu)+\mu\eta(1-\beta)$, the Green's function of boundary value problem*

$$-x'' = 0, \tag{3.1}$$

$$x(0) = \mu x(\eta), x(1) = \beta x(\xi), \tag{3.2}$$

*is*

$$G(t,s)=\frac{1}{\rho}\begin{cases} s[(1-\beta\xi)+(\beta-1)t] & s\le t, 0\le s\le\eta \\ t[(1-\beta\xi)+(\beta-1)s]+\mu(1-\eta+\beta\eta-\beta\xi)(s-t) & t\le s, 0\le s\le\eta \\ (\mu\eta+s-\mu s)[(1-\beta\xi)+(\beta-1)t] & s\le t,\eta\le s\le\xi \\ (\mu\eta+t-\mu t)[(1-\beta\xi)+(\beta-1)s] & t\le s,\eta\le s\le\xi \\ (1-s)(t-\mu t+\mu\eta)+\rho(s-t) & s\le t,\xi\le s\le 1 \\ (1-s)(\mu\eta-\mu t+t) & t\le s,\xi\le s\le 1 \end{cases}$$

*Further if condition $C_2$) holds, then $G(t,s)>0$ for $0\le t,\ s\le 1$.*

Let $X=C^1([-1,1]\setminus\{0\})\cup C^1[-1,1]$ be endowed with the ordering $x\le y$ if $x(t)\le y(t)$ for all $t\in[-1,1]$ and the norm

$$\|x\|=\max\{\max_{-1\le t\le 1}|x(t)|,\max\{\max_{t\in[-1,1]\setminus\{0\}}|x'(t)|,\max\{x'_-(0),x'_+(0)\}\}\}.$$

It is easy to see $X$ is a Banach space relative to the norm defined above.

From $x''(t)=-f(t,x(t),x'(t-1))\le 0$, we know that $x$ is concave on [0,1]. We define the cone $P\subset X$ by $P=\{x\in X: x(t)$ is nonnegative on [-1,1] and concave on $[-1,0]$ and $[0,1]$ respectively$\}$.

**Lemma 3.2.** *If $x(t)\in P$ is a solution of problem (1.7)-(1.9), then*

$$\max_{-1\le t\le 1}|x(t)|\le l\ \max\{\max_{t\in[-1,1]\setminus\{0\}}|x'(t)|,\max\{x'_+(0),x'_-(0)\}\},$$

*where*

$$l=\begin{cases} \min\{1+|\frac{\mu\eta}{1-\mu}|,1+|\frac{\beta(1-\xi)}{1-\beta}|\} & \mu\ne 1,\beta\ne 1 \\ 1+|\frac{\mu\eta}{1-\mu}| & \beta=1 \\ 1+|\frac{\beta(1-\xi)}{1-\beta}| & \mu=1 \end{cases}$$

*is a constant.*

*Proof.* It's easy to see that

$$\max\{\max_{t\in[-1,1]\setminus\{0\}}|x'(t)|,\max\{x'_+(0),x'_-(0)\}\}=\max\{\max_{-1\le t\le 0}|x'(t)|,\max_{0\le t\le 1}|x'(t)|\}.$$

For $F(t)=F(-1)+\int_{-1}^t F'(s)ds=\int_{-1}^t F'(s)ds, -1\le t\le 0$, we see

$$|F(t)|\le\int_{-1}^t|F'(s)|ds\le\max_{-1\le t\le 0}|F'(t)|. \tag{3.3}$$

For $t\in[0,1]$, if $\mu\ne 1$,as $x(0)=\mu x(\eta)$ and mean value theorem, there exists $t_0\in(0,\eta)$ such that $x(0)=\frac{\mu\eta}{1-\mu}x'(t_0)$.

Considering $x(t)=x(0)+\int_0^t x'(s)ds$ we have

$$|x(t)|\le|\frac{\mu\eta}{1-\mu}x'(t_0)|+\int_0^t|x'(s)|ds\le(|\frac{\mu\eta}{1-\mu}|+1)\max_{0\le t\le 1}|x'(t)|. \tag{3.4}$$

If $\beta \neq 1$,considering $x(1) = \beta x(\xi)$, similarly we get

$$|x(t)| \leq (|\frac{\beta(1-\xi)}{1-\beta}| + 1) \max_{0\leq t\leq 1} |x'(t)|. \tag{3.5}$$

Let

$$l = \begin{cases} \min\{1 + |\frac{\mu\eta}{1-\mu}|, 1 + |\frac{\beta(1-\xi)}{1-\beta}|\} & \mu \neq 1, \beta \neq 1 \\ 1 + |\frac{\mu\eta}{1-\mu}| & \beta = 1 \\ 1 + |\frac{\beta(1-\xi)}{1-\beta}| & \mu = 1 \end{cases}$$

From (3.3),(3.4), (3.5) we can obtain that

$$\max_{-1\leq t\leq 1} |x(t)| \leq l \ \max\{\max_{t\in[-1,1]\setminus\{0\}} |x'(t)|, \max\{x'_+(0), x'_-(0)\}\}.$$

**Lemma 3.3.** *If $x \in P$ is a solution of problem (1.7)-(1.9), we have*

$$\min_{t\in[\eta,\xi]} \geq \delta \max_{0\leq t\leq 1} |x(t)|, \tag{3.6}$$

*where $\delta = \min\{\eta, 1-\xi\} < 1$ is a constant.*
*Proof.* Let $x(t_1) = \max_{0\leq t\leq 1} |x(t)|,\ t_1 \in [0,1]$. From the concavity of $x(t)$,

$$\min_{t\in[\eta,\xi]} x(t) = min\{x(\xi_{j-1}),\ x(\xi_j)\}.$$

Here we distinguish two cases. $(1):\ \min_{t\in[\eta,\xi]} x(t) = x(\eta)$. Here $\eta < t_1$. We get

$$\frac{x(\eta)-x(0)}{\eta} \geq \frac{x(t_1)-x(0)}{t_1} \tag{3.7}$$

Arranging (3.7) and considering $x(0) \geq 0$, we have $x(\eta) \geq \eta x(t_1)$.
$(2):\ \min_{t\in[\eta,\xi]} x(t) = x(\xi)$. Here $\xi > t_1$. From

$$\frac{x(1)-x(t_1)}{1-t_1} \geq \frac{x(1)-x(\xi)}{1-\xi} \tag{3.8}$$

we see $x(\xi) \geq (1-\xi)x(t_1)$.

Considering (3.7), (3.8) we get that

$$\min_{t\in[\eta,\xi]} \geq \min\{\eta, 1-\xi\} \max_{0\leq t\leq 1} |x(t)| = \delta \max_{0\leq t\leq 1} |x(t)|. \tag{3.9}$$

Let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functionals $\gamma,\ \theta$ and the nonnegative continuous functional $\psi$ be defined on the cone by

$$\gamma(x) = \max\{\max_{t\in[-1,1]\setminus\{0\}} |x'(t)|, \max\{x'_+(0), x'_-(0)\}\},$$

$$\psi(x) = \min\{\min_{t\in[-\xi,-\eta]} |x(t)|, \max_{0\leq t\leq 1} |x(t)|\},$$

$$\theta(x) = \max_{0\leq t\leq 1} |x(t)|, \alpha(x) = \min_{t\in[-\xi,\eta]\cap[\eta,\xi]} |x(t)|.$$

By Lemma 3.2, 3.3 the functionals defined above satisfy:

$$\delta\theta(x) \leq \alpha(x) \leq \theta(x), \|x\| = \max\{\theta(x), \gamma(x)\} \leq l\gamma(x), \alpha(x) \leq \psi(x). \tag{3.10}$$

Therefore conditions (2.1) and (2.2) are satisfied.

Let

$$M = \max\{\int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=0} ds, \int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=1} ds\},$$

$$m = \min\{\int_\eta^\xi G(\eta,s)ds, \int_\eta^\xi G(\xi,s)ds\},$$

$$N = \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds, \lambda = \min\{\frac{m}{M}, \delta l\}.$$

We assume that there exist positive constants $a, b, c, d$ with $a < b < \lambda d$ such that:

$A_1$) $f(t,u,v) \leq d/M, (t,u,v) \in [0,1] \times [0,\ ld] \times [-d,\ d]$;

$A_2$) $f(t,u,v) > b/m, (t,u,v) \in [\eta,\xi] \times [b,\ b/\delta] \times [-d,\ d]$;

$A_3$) $f(t,u,v) < a/N, (t,u,v) \in [0,1] \times [0,\ a] \times [-d,\ d]$.

**Theorem 3.4.** *Under assumption $A_1) - A_3)$ and $C_1),\ C_2),\ C_4)$, problem (1.7)-(1.9) has at least three positive solutions $x_1,\ x_2,\ x_3$ satisfying*

$$\max\{\max_{t\in[-1,1]\setminus\{0\}} |x_i'(t)|,\ \max\{x'_{(i)+}(0),\ x'_{(i)-}(0)\}\} \leq d \text{ for } i = 1,2,3;$$

$$b < \min_{t\in[\eta,\xi]} |x_1(t)|, \quad a < \max_{0\leq t\leq 1} |x_2(t)| \text{ with } \min_{t\in[\eta,\xi]} |x_2(t)| < b;$$

$$\max_{0\leq t\leq 1} |x_3(t)| < a. \tag{3.11}$$

*Proof.* Suppose $x(t)$ is a solution of boundary value problem (1.7)-(1.9). Then $x(t)$ can be expressed as

$$x(t) = \begin{cases} F(t) & -1 \leq t \leq 0 \\ \int_0^1 G(t,s) f(s, x(s), x'(s-1)) ds & 0 < t < 1 \end{cases}$$

Define an operator $T : P \to P$ by

$$(Tx)(t) = \begin{cases} F(t) & -1 \leq t \leq 0 \\ \int_0^1 G(t,s) f(s, x(s), x'(s-1)) ds & 0 < t < 1 \end{cases}$$

It is well known that the operator $T$ is completely continuous. Now we show all conditions of Lemma 2.5 are satisfied.

If $x \in \overline{P(\gamma,d)}$, then $\gamma(x) = \max\{\max_{t\in[-1,1]\setminus\{0\}} |x'(t)|, \max\{x'_+(0), x'_-(0)\}\} \leq d$.

It is easy to see that $\max_{-1\leq t\leq 0} |x'(t)| \leq d$ and $\max_{0\leq t\leq 1} |x'(t)| \leq d$. So, when $t \in [-1,0]$, we have $\gamma(Tx) = \max_{-1\leq t\leq 0} |F'(t)| = \gamma(x(t)) \leq d$.

When $t \in [0,1]$, for $x \in \overline{P(\gamma,d)}$, we have $\max_{0\leq t\leq 1} |x'(t)| \leq d$. With Lemma 3.3, assumption $(A_1)$ implies $f(t, x(t), x'(t-1)) \leq d/M$.

Thus

$$\gamma(Tx) = \max_{0<t<1} |Tx'(t)| = \max\{|Tx'(0)|, |Tx'(1)|\}$$

$$\leq \max\{\int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=0} f(s,x(s),x'(s-1))ds, \int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=1} f(s,x(s),x'(s-1))ds\}$$

$$\leq \max\{\int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=0} ds, \int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=1} ds\}\frac{d}{M} = \frac{d}{M}M = d.$$

Hence $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$.

To check condition $(S_1)$ of Lemma 2.5, we choose $x(t) = \dfrac{b}{\delta} = c$. We can see that $x(t) = \dfrac{b}{\delta} \in P(\gamma,\theta,\alpha,b,c,d)$ and $\alpha(\dfrac{b}{\delta}) > b$. So $\{x \in P(\gamma,\theta,\alpha,b,c,d|\alpha(x) > b)\} \neq \emptyset$. Next we show that $\alpha(Tx) > b$,for $x \in P(\gamma,\theta,\alpha,b,c,d)$.

As $x \in P(\gamma,\theta,\alpha,b,c,d)$, we have $b \leq x(t) \leq b/\delta, |x'(t)| \leq d$ for $t \in [\eta,\xi]$. From assumption $(A_2)$ we have $f(t,x(t),x'(t-1)) \geq b/m$.

Then

$$\alpha(Tx) = \min\{(Tx)(\eta),(Tx)(\xi)\}$$

$$= \min\{\int_0^1 G(\eta,s)f(s,x(s),x'(s-1))ds, \int_0^1 G(\xi,s)f(s,x(s),x'(s-1))ds\}$$

$$\geq \frac{b}{m}\ \min\{\int_\eta^\xi G(\eta,s)ds, \int_\eta^\xi G(\xi,s)ds\} = \frac{b}{m}m = b.$$

Thus, condition $(S_2)$ of Lemma 2.5 is satisfied. Finally we show that $(S_3)$ also holds.

Clearly, $\psi(0) = 0 < a$ shows $0 \overline{\in} R(\gamma,\psi,a,d)$. Suppose $x \in R(\gamma,\psi,a,d)$ with $\psi(x) = a$. For assumption $(A_3)$we have

$$\max_{0\leq t\leq 1} |(Tx)(t)| = \max_{0\leq t\leq 1} \int_0^1 G(t,s)f(s,x(s),x'(s-1))ds < \frac{a}{N} \max_{0\leq t\leq 1} \int_0^1 G(t,s)ds = a.$$

Thus $\psi(Tx) = \max\limits_{0\leq t\leq 1} |(Tx)(t)| < a$. So condition $(S_3)$ of Lemma 2.5 is satisfied.

Therefore, an application of Lemma 2.5 implies problem (1.7)-(1.9) has at least three positive solutions $x_1, x_2, x_3$ and (3.11) is satisfied. The proof is complete.

**Remark 3.5.** In [11], the author defined the functionals $\theta, \alpha$ with

$$\theta(x) = \max_{-1\leq t\leq 1} |x(t)|, \alpha(x) = \min_{t\in[-\frac{3}{4},-\frac{1}{4}]\cup[\frac{1}{4},\frac{3}{4}]} |x(t)|.$$

By the definitions and the concavity of $x(t)$, they claimed

$$\alpha(x) \geq \frac{1}{4}\theta(x). \tag{3.12}$$

In fact, considering the concavity of $x(t)$, we can get that

$$\min_{t\in[-\frac{3}{4},-\frac{1}{4}]} x(t) \geq \frac{1}{4} \max_{t\in[-1,0]} x(t)$$

and

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} x(t) \geq \frac{1}{4} \max_{t\in[0,1]} x(t). \tag{3.13}$$

But it's easy to see that we can't get (3.12) by (3.13). Thus we define the functionals different with [11].

**Remark 3.6.** To apply Lemma 2.5, we only need $T : \overline{P(\gamma,d)} \to \overline{P(\gamma,d)}$, therefore condition $C_1$) can be substituted with a weaker condition

$$H_1) : f \in C([0,1]\times[0,ld]\times[-d,d],[0,+\infty)).$$

## 4. Solutions of (1.10)-(1.12)

In this section we deal with problem (1.10)-(1.12). The method and existence results are remarkable analogous to those in section 3. Also we give some Lemmas firstly.

**Lemma 4.1.** *Denote* $\rho_1 = \beta_1(1-\sum_{i=0}^{m-1}\alpha_i)+1-\sum_{i=0}^{m-1}\alpha_i\xi_i$, *then boundary value problem*

$$x'' + y(t) = 0, \tag{4.1}$$

$$x(0) = \beta_1 x'(0),\ x(1) = \sum_{i=1}^{m-2}\alpha_i x(\xi_i), \tag{4.2}$$

*has solution* $x(t) = \int_0^1 G_1(t,s)y(s)ds$, *where*

$$G_1(t,s)=\frac{1}{\rho}\begin{cases}[(1-s)+\sum_{k=i}^{m-1}\alpha_k(s-\xi_k)](\beta_1+t) & t\le s\\ (s+\beta_1)(1-t)+\sum_{k=0}^{i-1}\alpha_k(t-s)(\beta_1+\xi_k)+\sum_{k=i}^{m-1}\alpha_k(t-\xi_k)(s+\beta_1) & t\ge s\end{cases}$$

*for* $\xi_{i-1}\le s\le \xi_i$.

*Proof.* Suppose $\overline{G}(t,s)$ is the Green's function of problem (4.1), (4.2).

For $\xi_{i-1}<s<\xi_i$, $i=1,2\cdots m-1$, we let

$$\overline{G}(t,s)=\begin{cases}A+Bt & t\le s\\ C+Dt & t\ge s\end{cases}$$

From the definition and properties of Green's function and (4.2) we have

$$\begin{cases}A+Bs=C+Ds\\ B-D=-1\\ A=\beta_1 B\\ C+D=\sum_{k=0}^{i-1}\alpha_k(A+B\xi_k)+\sum_{k=i}^{m-1}\alpha_k(C+D\xi_k)\end{cases}$$

We get

$$A=\frac{\beta_1}{\rho}[(s-1)+\sum_{k=i}^{m-1}\alpha_k(\xi_k-s)],$$

$$B=\frac{1}{\rho}[(s-1)+\sum_{k=i}^{m-1}\alpha_k(\xi_k-s)],$$

$$C=\frac{1}{\rho}[\sum_{k=0}^{i-1}\alpha_k\beta_1 s+s(\sum_{k=0}^{m-1}\alpha_k\xi_k-1)+\beta_1(\sum_{k=i}^{m-1}\alpha_k\xi_k-1)],$$

$$D = \frac{1}{\rho}[(\beta_1 + s)(1 - \sum_{k=i}^{m-1} \alpha_k) - \sum_{k=0}^{i-1} \alpha_k(\beta_1 + \xi_k)].$$

Thus

$$\overline{G}(t,s) = \frac{1}{\rho} \begin{cases} [(s-1) + \sum\limits_{k=i}^{m-1} \alpha_k(\xi_k - s)](\beta_1 + t) & t \le s \\ (s+\beta_1)(t-1) + \sum\limits_{k=0}^{i-1} \alpha_k(s-t)(\beta_1 + \xi_k) + \sum\limits_{k=i}^{m-1} \alpha_k(\xi_k - t)(s+\beta_1) & t \ge s \end{cases}$$

We show the expression of the Green's function for problem (4.1), (4.2).
Let $G_1(t,s) = -\overline{G}(t,s)$, then the solution of boundary value problem $(4.1) - (4.2)$ is

$$x(t) = \int_0^1 \overline{G}(t,s)(-y(s))ds = \int_0^1 G_1(t,s)y(s)ds. \tag{4.3}$$

**Lemma 4.2.** *If $C_3)$ holds, we claim $G_1(t,s) \ge 0, t, s \in [0,1]$.*
*Proof.* For $\xi_{i-1} \le s \le \xi_i$, if $t \le s$,

$$(1-s) + \sum_{k=i}^{m-1} \alpha_k(s - \xi_k) \ge \sum_{k=i}^{m-1} \alpha_k \xi_k (1-s) + \sum_{k=i}^{m-1} \alpha_k(s-\xi_k)$$

$$\ge \sum_{k=i}^{m-1} \alpha_k(1-\xi_k)s \ge 0.$$

If $t \ge s$,

$$(s+\beta_1)(1-t) + \sum_{k=0}^{i-1} \alpha_k(t-s)(\beta_1+\xi_k) + \sum_{k=i}^{m-1} \alpha_k(t-\xi_k)(s+\beta_1)$$

$$\ge \sum_{k=0}^{i-1} \alpha_k(t-s)(\beta_1+\xi_k) + \sum_{k=i}^{m-1} \alpha_k(1-\xi_k)(s+\beta_1)t \ge 0.$$

Then $G(t,s) \ge 0, t, s \in [0,1]$.

**Lemma 4.3.** *If $x \in P$ and is a solution of problem (1.10)-(1.12), then*

$$\max_{-1 \le t \le 1} |x(t)| \le (\beta_1 + 1) \max\{\max_{t \in [-1,1]\setminus\{0\}} |x'(t)|, \max\{x'_-(0), x'_+(0)\}\}.$$

*Proof.* When $t \in [-1,0]$, $\max\limits_{t\in[-1,0]} |x(t)| \le \max\limits_{t \in [-1,0]} |x'(t)|$ obviously.

When $t \in [0,1]$, for $x(t) = x(0) + \int_0^t x'(t)dt = \beta_1 x'(0) + \int_0^1 x'(t)dt$, we have

$$|x(t)| \le |\beta_1 x'(0)| + \int_0^1 |x'(t)|dt \le (\beta_1 + 1) \max_{t\in[0,1]} |x'(t)|.$$

We sum up the conclusions above to obtain that

$$\max_{-1 \le t \le 1} |x(t)| \le (\beta_1 + 1) \max\{\max_{t \in [-1,1]\setminus\{0\}} |x'(t)|, \max\{x'_-(0), x'_+(0)\}\}.$$

Similar to Lemma 3.3, we have

$$\min_{t\in[\xi_{j-1},\xi_j]} \ge \delta_1 \max_{-1\le t\le 1} |x(t)|,$$

where $\xi_j \in \{\xi_1, \xi_2, \cdots, \xi_{m-1}\}, \delta_1 = \min\{\xi_{j-1}, 1-\xi_j\}$.

Let the nonnegative continuous concave functional $\alpha_1$, the nonnegative continuous convex functionals $\gamma_1, \theta_1$ and the nonnegative continuous functional $\psi_1$ be defined on the cone similarly to Theorem 3.4.

By Lemma 4.3 the functionals satisfy:

$$\delta_1\theta_1(x) \le \alpha_1(x) \le \theta_1(x), \|x\| = \max\{\theta_1(x), \gamma_1(x)\} \le (\beta_1+1)\gamma_1(x), \alpha_1(x) \le \psi_1(x). \tag{4.4}$$

Therefore conditions (2.1) and (2.2) are satisfied.

Let

$$M_1 = \max\{\int_0^1 |\frac{\partial G_1(t,s)}{\partial t}|_{t=0} ds, \int_0^1 |\frac{\partial G_1(t,s)}{\partial t}|_{t=1} ds\},$$

$$m_1 = \min\{\int_{\xi_{j-1}}^{\xi_j} G_1(\xi_{j-1}, s) ds, \int_{\xi_{j-1}}^{\xi_j} G_1(\xi_j, s) ds\},$$

$$N_1 = \max_{0\le t\le 1} \int_0^1 G_1(t,s) ds, \lambda_1 = \min\{\frac{m_1}{M_1}, \delta_1(\beta_1+1)\}.$$

To present our main results, we assume there exist constants $0 < a_1, b_1, c_1, d_1$, $a_1 < b_1 < \lambda_1 d_1$ such that

$A_4)$ $f(t,u,v) \le d_1/M_1, (t,u,v) \in [0,1] \times [0, (\beta_1+1)d_1] \times [-d_1, d_1]$;

$A_5)$ $f(t,u,v) > b_1/m_1, (t,u,v) \in [\xi_{j-1}, \xi_j] \times [b_1, b_1/\delta_1] \times [-d_1, d_1]$;

$A_6)$ $f(t,u,v) < a_1/N_1, (t,u,v) \in [0,1] \times [0, a_1] \times [-d_1, d_1]$.

**Theorem 4.4.** *Under assumption $A_4) - A_6)$ and $C_1), C_3), (C_4)$, the boundary value problem (1.10)-(1.12) has at least three positive solutions $x_1, x_2, x_3$ satisfying*

$$\max\{\max_{t\in[-1,1]\setminus\{0\}} |x_i'(t)|, \max\{x'_{(i)+}(0), x'_{(i)-}(0)\}\} \le d_1 \text{ for } i = 1,2,3;$$

$$b_1 < \min_{t\in[\xi_{j-1},\xi_j]} |x_1(t)|; \quad a_1 < \max_{0\le t\le 1} |x_2(t)|$$

*with*

$$\min_{t\in[\xi_{j-1},\xi_j]} |x_2(t)| < b_1; \quad \max_{0\le t\le 1} |x_3(t)| < a_1. \tag{4.5}$$

*Proof.* Define an operator $T_1 : P_1 \to P_1$ by

$$(T_1x)(t) = \begin{cases} F_1(t) & -1 \le t \le 0 \\ \int_0^1 G_1(t,s) f(s, x(s), x'(s-1)) ds & 0 < t < 1 \end{cases}$$

It is well known that the operator $T_1$ is completely continuous and $x(t)$ is a solution of problem (1.10)-(1.12) if and only if it solves operator equation

$$x(t) = (T_1)x(t). \tag{4.6}$$

If $x \in \overline{P_1(\gamma_1, d_1)}$, when $t \in [-1, 0], \gamma_1(T_1x) \le d_1$ obviously. When $t \in [0,1]$, it is easy to see that $f(t, x(t), x'(t-1)) \le d_1/M_1$. Then

$$\gamma(T_1x) = \max_{0<t<1} |(T_1x)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$$

$$\le \max\{\int_0^1 |\frac{\partial G_1(t,s)}{\partial t}|_{t=0} ds, \int_0^1 |\frac{\partial G_1(t,s)}{\partial t}|_{t=1} ds\} \frac{d_1}{M_1} \le \frac{d_1}{M_1} M_1 = d_1.$$

Hence $T_1 : \overline{P_1(\gamma_1, d_1)} \to \overline{P_1(\gamma_1, d_1)}$.

From the proof of Theorem 3.4 and the definitions of $M_1, m_1, N_1$, all the conditions of lemma 1 are satisfied obviously. Therefore, problem (1.10)-(1.12) has at least three positive solutions $x_1, x_2, x_3$ and (4.5) is satisfied.

**Remark 4.5.** To apply Lemma 2.5,we only need that $T_1 : \overline{P_1(\gamma_1, d_1)} \to \overline{P_1(\gamma_1, d_1)}$, therefore condition $C_1)$ can be substituted with a weaker condition, namely

$H_2)$ $f_1 \in C([0,1] \times [0,(\beta_1+1)d_1] \times [-d_1, d_1], [0,+\infty))$.

## 5. Example

Finally we present an example to check our main results. Consider the boundary value problem

$$x''(t) + f(t, x(t), x'(t-1)) = 0, \quad 0 < t < 1, \tag{5.1}$$

$$x(t) = F(t) = 7 + 6t - t^2, \quad -1 \le t \le 0, \tag{5.2}$$

$$x(0) = x(\frac{1}{3}), \quad x(1) = \frac{1}{2}x(\frac{2}{3}), \tag{5.3}$$

where

$$f(t,u,v) = \begin{cases} \frac{1}{5}e^t + u^3 + \frac{1}{5}(\frac{v}{3001})^3 & 0 \le u \le 5 \\ \frac{1}{5}e^t + 125 + \frac{1}{5}(\frac{v}{3001})^3 & u > 5 \end{cases}$$

Choose $a = 1, b = 4, d = 3000, \eta = \frac{1}{3}, \xi = \frac{2}{3}$, we note that $\delta = \frac{1}{3}, l = \frac{4}{3}$ and the Green's function

$$G(t,s) = \begin{cases} s(4-3t) & s \le t, 0 \le s \le \frac{1}{3} \\ 5s - t - 3st & t \le s, 0 \le s \le \frac{1}{3} \\ \frac{4}{3} - t & s \le t, \frac{1}{3} \le s \le \frac{2}{3} \\ \frac{4}{3} - s & t \le s, \frac{1}{3} \le s \le \frac{2}{3} \\ 2 - s - t & s \le t, \frac{2}{3} \le s \le 1 \\ 2 - 2s & t \le s, \frac{2}{3} \le s \le 1 \end{cases}$$

Conditions $H_1), C_2), C_4)$ hold and $F(-1) = 0$ obviously. By the definitions above, we get

$$M = \max\{\int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=0} ds, \int_0^1 |\frac{\partial G(t,s)}{\partial t}|_{t=1} ds\} = \frac{5}{6},$$

$$m = \min\{\int_{\frac{1}{3}}^{\frac{2}{3}} G(\frac{1}{3}, s) ds, \int_{\frac{1}{3}}^{\frac{2}{3}} G(\frac{2}{3}, s) ds\} = \frac{2}{9},$$

$$N = \max_{0\le t\le 1}\int_0^1 G(t,s)ds = \frac{37}{72}, \lambda = \min\{\frac{m}{M}, \delta l\} = \frac{4}{15}.$$

Consequently $f(t,u,v)$ satisfy

$$f(t,u,v) \le 3600, (t,u,v) \in [0,1]\times[0,4000]\times[-3000,3000];$$

$$f(t,u,v) \ge 18, (t,u,v) \in [\frac{1}{3},\frac{2}{3}]\times[4,12]\times[-3000,3000];$$

$$f(t,u,v) \le \frac{72}{37}, (t,u,v) \in [0,1]\times[0,1]\times[-3000,3000].$$

Then all assumptions of Theorem 3.4 are satisfied. Thus problem (5.1) − (5.2) has three positive solutions $x_1, x_2, x_3$ satisfying

$$\max\{\max_{t\in[-1,1]\setminus\{0\}} |x_i'(t)|, \max\{x'_{(i)+}(0), x'_{(i)-}(0)\}\} \le 3000 \text{ for } i = 1,2,3;$$

$$4 < \min_{t\in[\frac{1}{3},\frac{2}{3}]} |x_1(t)|; \quad 1 < \max_{0\le t\le 1} |x_2(t)|$$

with

$$\min_{t\in[\frac{1}{3},\frac{2}{3}]} |x_2(t)| < 4; \max_{0\le t\le 1} |x_3(t)| < 1.$$

**Remark 5.1.** The early results about positive solutions of boundary value problems, to author's best knowledge, are not applicable to this four-point boundary value problem of functional differential equation.

**Remark 5.2.** If $\mu = 0$, $\beta = 0$, or $\beta_1 = 0$, $\alpha_i = 0, i = 1,2,\cdots,m-2$, Theorem 3.4 or 4.4 gives Theorem 3.2 of [11]. So our main results extend the results of [6], [11].

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