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APPLICATIONS OF THE S-ITERATION PROCESS TO CONSTRAINED MINIMIZATION PROBLEMS AND SPLIT FEASIBILITY PROBLEMS

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Abstract. In this paper the S-iteration process introduced by Agarwal, O'Regan and Sahu [Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal., 8 (2007), 61-79] is further analyzed for contraction and nonexpansive mappings. It is shown, theoretically as well as numerically, that the S-iteration process is faster than the Picard and KM-iteration processes for contraction operators. We also propose a new iterative algorithm and prove a strong convergence theorem for computing fixed points of nonexpansive operators in a Banach space. Our results are applied for finding solutions of constrained minimization problems and split feasibility problems. Our iteration methods are of independent interest.

Key Words and Phrases: Accretive operator, nonexpansive mapping, sunny nonexpansive retraction, fixed point iterative algorithm, normal S-iteration process, rate of convergence of iterative algorithm, constrained optimization problem, split feasibility problem.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and $A : C \to H$ a nonlinear operator. A is said to be

(1) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in C$,
(2) *η-strongly monotone* if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2 \text{ for all } x, y \in C,$$

(3) *ν-inverse strongly monotone* (ν-ism) if there exists a constant $\nu > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \nu \|Ax - Ay\|^2 \text{ for all } x, y \in C,$$

(4) *L-Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Ax - Ay\| \leq L\|x - y\|^2 \text{ for all } x, y \in C.$$

An L-Lipschitzian operator is called *contraction* (respectively, *nonexpansive*) if $L < 1$ (respectively, $L = 1$).

The variational inequality $VI(C, A)$ is formulated as finding a point $z \in C$ such that $\langle Az, z - v\rangle \geq 0$ for all $v \in C$. The set of solutions of the variational inequality $VI(C, A)$ is denoted by $\Omega(C, A)$, i.e.,

$$\Omega(C, A) = \{z \in C : \langle Az, z - v\rangle \geq 0 \text{ for all } v \in C\}.$$

We denote by $F(T)$ the set of fixed points of mapping $T : C \to C$.

The variational inequalities were initially studied by Stampachhia [13, 15] and ever since have been widely studied. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in H$ satisfying $0 = Au$ and so on. The existence and approximation of solutions are important aspects of study of variational inequalities. The variational inequality problem $VI(C, A)$ is equivalent to the fixed point problem:

$$\text{find } x^* \in C \text{ such that } x^* = P_C(I - \mu A)x^*,$$

where $\mu > 0$ is a constant and P_C is the metric projection from H onto C. It is well known that if A is κ-Lipschitzian and ν-strongly monotone, then the operator $F_\mu x := P_C(I - \mu A)x$ is a contraction on C provided that $0 < \mu < 2\eta/\kappa^2$. In this case, the Banach contraction principle guarantees that $VI(C, A)$ has a unique solution x^* and the sequence of the Picard iteration process, given by,

$$x_{n+1} = P_C(I - \mu A)x_n, \ n \in \mathbb{N} \tag{1.1}$$

converges strongly to x^*. This method is called the *projected gradient method* (PGM) ([33]). It has been used widely in many practical problems, due partially to its fast convergence.

Now our concern is the following:

Qusetion 1.1. *Is it possible to develop an iterative sequence whose rate of convergence is faster than the Picard iteration process (1.1)?*

Construction of fixed points of nonexpansive operators is an important subject in the theory of nonexpansive operators and its applications in a number of applied areas, in particular, in image recovery and signal processing (see, e.g.,[7, 12, 31, 32]). For instance, split feasibility problem (SFP) is

$$\text{to find a point } x \in C \text{ such that } Ax \in Q, \tag{1.2}$$

here C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of another Hilbert space H_2 and $A : H_1 \to H_2$ is a bounded linear operator. The SFP is said to be consistent if (1.2) has a solution. It is easy to see that SFP is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \text{ such that } x = P_C(I - \gamma A^*(I - P_Q)A)x, \tag{1.3}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and A^* is the adjoint of A. Note that for sufficient small $\gamma > 0$, the operator $P_C(I - \gamma A^*(I - P_Q)A)$ in operator equation (1.3) is nonexpansive.

It is well known that the sequence $\{T^n x\}$ of iterates of nonexpansive operator T at a point $x \in C$ may, in general, not behave well. This means that it may not converge (even in the weak topology). One way to overcome this difficulty is to use the Krasnoselskii-Mann (KM) iteration method [7, 10] that produces a sequence $\{x_n\}$ via the recursive manner:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n \text{ for all } n \in \mathbb{N},$$

where the initial guess $x_1 \in C$ is chosen arbitrarily and $\{\alpha_n\}$ is a real sequence in [0,1]. It is worth noting that the KM iteration process is well known for finding fixed points of nonexpansive operators (see, [7]) and it is further developed in a general context in [30].

Recently, Agarwal, O'Regan and Sahu [2] have introduced the S-iteration process as follows: Let X be a normed linear space, C a nonempty convex subset of X and $T : C \to C$ an operator. Then, for arbitrary $x_1 \in C$, the S-iteration process is defined by

$$(S) \qquad \begin{cases} x_{n+1} &= (1 - \alpha_n)Tx_n + \alpha_n T y_n, \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in (0,1) satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) = \infty. \tag{1.4}$$

We remark that the S-iteration process is faster than the Picard iteration process for contraction operators (see, Theorem 3.5). One of the purposes of this paper is to apply the S-iteration process for finding solutions of SFP(1.2).

On the other hand, the strong convergence of the path $\{x_t = tu + (1 - t)Tx_t : t \in (0, 1)\}$ as $t \to 0^+$ for nonexpansive operator T on a bounded C was proved in a Hilbert space independently by Browder [6] and Halpern [11] in 1967 and in a uniformly smooth Banach space by Reich [16] in 1980. For a sequence $\{\alpha_n\}$ of real numbers in [0, 1] and an arbitrary $u \in C$, let the sequence $\{x_n\}$ in C be iteratively defined by $x_1 \in C$,

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n \text{ for all } n \in \mathbb{N}. \tag{1.5}$$

The explicit formula (1.5) was first introduced and its strong convergence to fixed points of nonexpansive operator T was proved in 1967 by Halpern [11] in the framework of Hilbert space with the choice $\alpha_n = 1/n^\theta$, where $\theta \in (0, 1)$. The strong convergence of the explicit formula (1.5) to fixed point of nonexpansive operator T was further studied by Cho, Kang and Zhou [9]; Lions [14]; Sahu, Kang and Liu [19]; Shioji and Takahashi [24]; Wittmann [25]; Wong, Sahu and Yao [26]; Xu [28, 29] under certain assumptions on iteration parameter α_n. One can unify these convergence results by a general theorem as follows:

Theorem 1.2. *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. Suppose that C has the fixed-point property for nonexpansive mappings. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C defined by (1.5). Suppose the sequence $\{\alpha_n\} \subset (0, 1)$ satisfies the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$,
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In addition, suppose $\{\alpha_n\}$ satisfies one of the following conditions:

(**C3**) $\lim_{n\to\infty} |\alpha_n - \alpha_{n+1}|/\alpha_{n+1}^2 = 0$,
(**C4**) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,
(**C5**) $\{\alpha_n\}$ *is decreasing*
(**C6**) $\lim_{n\to\infty} |\alpha_n - \alpha_{n+1}|/\alpha_{n+1} = 0$.

Then $\{x_n\}$ converges strongly to $R_{F(T)}u$, where $R_{F(T)}$ is the sunny nonexpansive retraction from C onto $F(T)$.

Recently, Suzuki [20] (see also Chidume and Chidume [8]) gave some variants on (1.5) as below:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)T((1-\lambda)x_n + \lambda T x_n) \text{ for all } n \in \mathbb{N} \tag{1.6}$$

and they proved that the conditions (**C1**) and (**C2**) on iteration parameter α_n are sufficient for strong convergence of iteration process (1.6).

The purpose of this paper is to further analyze the S-iteration process for different classes of nonlinear operators and give applications of investigated results in constrained minimization problems and split feasibility problems. The paper is organized as follows: The next section includes useful mathematical formulations and facts. In section 3, we compare the rate of convergence of the S-iteration process with the Picard and KM iteration processes. It is shown, theoretically as well as numerically, that the S-iteration process is faster than the Picard and KM iteration processes for contraction operators. An affirmative answer of Question 1.1 is also given in this section. Some properties of the S-iteration process for nonexpansive operators are given in section 4. In section 5, a new iterative algorithm, different from Suzuki [20], is designed suxh that it converges strongly to an element of the solution set. The last section contains applications our results investigated in sections 4 and 5 for finding solutions of constrained minimization problems and split feasibility problems.

2. Preliminaries

We use S_X to denote the unit sphere $S_X = \{x \in X : \|x\| = 1\}$ on Banach space X. A Banach space X is said to be *strictly convex* if

$$x, y \in S_X \text{ with } x \neq y \Rightarrow \|(1-\lambda)x + \lambda y\| < 1 \text{ for all } \lambda \in (0,1).$$

In a strictly convex Banach space X, we have that if $\|x\| = \|y\| = \|\alpha x + (1-\alpha)y\|$ for $x, y \in X$ and $\alpha \in (0,1)$, then $x = y$ (see, e.g., [1, 21]).

Recall that a Banach space X is said to be *smooth* provided the limit

$$\lim_{t\to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each x and y in $S_X = \{x \in X : \|x\| = 1\}$. In this case, the norm of X is said to be *Gâteaux differentiable.* It is said to be *uniformly Gâteaux differentiable* if for each $y \in S_X$, this limit is attained uniformly for $x \in S_X$. It is well known that every uniformly smooth space (e.g., L_p space, $1 < p < \infty$) has uniformly Gâteaux differentiable norm (see e.g., [1]).

A Banach space X is said to satisfy *Opial condition* (see [1]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{for all } \ y \in X \text{ with } \ y \neq x.$$

Let X be an arbitrary real normed space with dual space X^*. We denote by J the normalized duality mapping from X into 2^{X^*} defined by

$$J(x) := \{f^* \in X^* : \langle x, f^* \rangle = ||x||^2 = ||f^*||^2\}, x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Then for each $x, y \in X$, there exists $j(x+y) \in J(x+y)$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

It is well known that J is single-valued if and only if X is smooth (see, [1]).

A subset C of a Banach space X is called a *retract* of X if there exists a continuous mapping P from X onto C such that $Px = x$ for all x in C. We call such P a *retraction* of X onto C. It follows that if a mapping P is a retraction, then $Py = y$ for all y in the range of P. A retraction P is said to be *sunny* if $P(Px + t(x - Px)) = Px$ for each x in X and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be a *sunny nonexpansive retract* of X.

The following lemmas will be needed in the sequel for the proof of our main results:

Lemma 2.1. *Let X be a smooth Banach space. Then*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle \text{ for all } x, y \in X.$$

Lemma 2.2. (Xu [28, Lemma 2.5]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying:*

$$a_{n+1} \leq (1 - t_n)a_n + t_n b_n \text{ for all } n \in \mathbb{N},$$

where $\{b_n\}$ and $\{t_n\}$ are sequences of real numbers which satisfy the conditions:

(i) $\{t_n\} \subset [0, 1]$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,

(ii) $\limsup_{n\to\infty} b_n = 0$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3. (Takahashi and Ueda [22] and Wong, Sahu and Yao [26, Theorem 3.6]). *Let X be a reflexive Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. Suppose that every closed convex bounded subset of C has fixed point property for nonexpansive self-operators. Then $F(T)$ is the sunny nonexpansive retract of C. Moreover, if $u \in C$ and z_t be the unique point in C defined by*

$$z_t = tu + (1 - t)Tz_t, t \in (0, 1), \tag{2.1}$$

then $\{z_t\}$ converges strongly to $R_{F(T)}(u)$ as $t \to 0^+$, where $R_{F(T)}$ is the sunny nonexpansive retraction from C onto $F(T)$.

Lemma 2.4. (Wong, Sahu and Yao [26, Lemma 2.12]). *Let X be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty closed convex subset of X, $f : C \to C$ a continuous operator, $T : C \to C$ a nonexpansive operator and $\{x_n\}$ a bounded sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Suppose $\{z_t\}$ is a path in C defined by $z_t = tfz_t + (1-t)Tz_t,\ t \in (0,1)$ such that $z_t \to z$ as $t \to 0^+$. Then*

$$\limsup_{n\to\infty} \langle fz - z, J(x_n - z)\rangle \leq 0.$$

Lemma 2.5. (Agarwal, O'Regan and Sahu [1, Lemma 6.7.2]). *Let X be a normed space, C a nonempty convex subset of X and $T : C \to C$ a nonexpansive operator. If $\{x_n\}$ is the iterative process defined by (S), then $\lim\limits_{n\to\infty} \|x_n - Tx_n\|$ exists.*

Let X be a normed linear space, C a nonempty convex subset of X, $T : C \to C$ an operator with $F(T) \neq \emptyset$ and $\{x_n\}$ a sequence in C. We say that $\{x_n\}$ has

(D1) *limit existence property* (in short, LE property) if $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$,

(D2) *approximate fixed point property* (in short, AF point property) if $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

(D3) *LEAF point property* if $\{x_n\}$ has both LE property and AF point property.

It is not difficult to see that the Krasnoselskii-Mann (KM) iteration method and S-iteration method enjoy LEAF point property under suitable conditions of iteration parameters in Banach spaces (cf. [1, 17, 23]).

The following proposition shows that the LEAF point property plays an important role in approximation of fixed points of nonlinear operators.

Lemma 2.6. (Agarwal, O'Regan and Sahu [2, Lemma 2.10]). *Let X be a reflexive Banach space satisfying the Opial condition, C a nonempty closed convex subset of X and $T : C \to X$ a operator such that*

(i) $F(T) \neq \emptyset$,

(ii) $I - T$ is demiclosed at zero.

Let $\{x_n\}$ be a sequence in C satisfying the following properties:

(D_1) limit existence property: $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$;

(D_2) approximate fixed point property: $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$.

Then $\{x_n\}$ converges weakly to a fixed point of T.

3. Comparison of three fixed point iteration processes

In this section, we introduce S-operator and discuss its properties and then compare the rate of convergence of the S-iteration process with the Picard and KM iteration processes for contraction operators.

First, we introduce notion of S-operator. Let C be a nonempty convex subset of a vector space X and $T : C \to C$ an operator. Then

(1) an operator $G_{\alpha,\beta,T} : C \to C$ is said to be an *S-operator* generated by $\alpha \in (0, 1]$, $\beta \in (0, 1)$ and T if

$$G_{\alpha,\beta,T} = (1-\alpha)T + \alpha T((1-\beta)I + \beta T),$$

(2) an operator $G_{\beta,T} : C \to C$ is said to be an *S-operator* generated by $\beta \in (0,1)$ and T if

$$G_{\beta,T} = T((1-\beta)I + \beta T).$$

It is not difficult to see that $G_{\alpha,\beta,T}$ is contraction with contractivity factor $k(1-\alpha\beta(1-k))$ if T is a contraction with contractivity factor k and $G_{\alpha,\beta,T}$ is nonexpansive if T is a nonexpansive.

Proposition 3.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator. Assume that $\alpha \in (0,1]$ and $\beta \in (0,1)$. If $G_{\alpha,\beta,T}$ is an S-operator generated by α, β and T, then $F(G_{\alpha,\beta,T}) = F(T)$.*

Corollary 3.2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator. Assume that $\lambda \in (0,1)$. If $G_{\lambda,T}$ is an S-operator generated by λ and T, then $F(G_{\lambda,T}) = F(T)$.*

Proposition 3.3. *Let C be a nonempty closed convex subset of a strictly convex Banach space X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. Assume that $\alpha \in (0,1]$ and $\beta \in (0,1)$. If $G_{\alpha,\beta,T}$ is an S-operator generated by α, β and T, then $F(T) = F(G_{\alpha,\beta,T})$.*

Proof. Set $G_{\alpha,\beta,T} := G$. Note $F(T) \subseteq F(G)$. Let $z \in F(G)$ and $v \in F(T)$. Observe that

$$\begin{aligned}
\|z-v\| &= \|(1-\alpha)Tz + \alpha T((1-\beta)I + \beta T)z - v\| \\
&\leq (1-\alpha)\|Tz-v\| + \alpha\|T((1-\beta)I + \beta T)z - v\| \\
&\leq (1-\alpha)\|z-v\| + \alpha\|(1-\beta)z + \beta Tz - v\| \\
&\leq (1-\alpha)\|z-v\| + \alpha((1-\beta)\|z-v\| + \beta\|Tz-v\|) \\
&= (1-\alpha\beta)\|z-v\| + \alpha\beta\|Tz-v\| \\
&\leq \|z-v\|,
\end{aligned}$$

which implies that

$$||z-v|| = ||Tz-v|| = ||T((1-\beta)z+\beta Tz)-v|| = \|(1-\alpha)Tz+\alpha T((1-\beta)z+\beta Tz)-v||.$$

Since X is a strictly convex, it follows that $Tz = z$. ☐

Motivated by the S-operators generated by a sequence $\{\lambda_n\}$ of real numbers in (0,1) and operator T, we introduce the normal S-iteration process as follows:

Let X be a normed linear space, C a nonempty convex subset of X and $T : C \to C$ an operator. Then, for arbitrary $x_1 \in C$, the *normal S-iteration process* is defined by

$$x_{n+1} = S_n x_n = T[(1-\lambda_n)x_n + \lambda_n T x_n], n \in \mathbb{N},$$

where $\{S_n\}$ is a sequence of S-operators generated by a sequence $\{\lambda_n\}$ of real numbers in (0,1) and operator T.

In order to compare two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$ that converge to a certain fixed point p of a given operator T, Rhoades [18] considered that $\{u_n\}$ is better than $\{v_n\}$ if

$$||u_n - p|| \leq ||v_n - p|| \text{ for all } n \in \mathbb{N}.$$

Berinde [5] introduced a different formulation from that of Rhoades as below:

Definition 3.4. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(a) If $l = 0$, then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.

(b) If $0 < l < \infty$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose that for two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$||u_n - p|| \leq a_n \text{ for all } n \in \mathbb{N}, \tag{3.1}$$

$$||v_n - p|| \leq b_n \text{ for all } n \in \mathbb{N}, \tag{3.2}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). Then, in view of Definition 3.4, we will adopt the following concept.

Definition 3.5. (Berinde [5]). Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (3.1) and (3.2), respectively. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Theorem 3.6. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a contraction operator with contractivity factor $k \in [0, 1)$ and fixed point x^*. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in $[0, 1]$ such that $\alpha \leq \alpha_n \leq 1$ and $\beta \leq \beta_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences $\{u_n\}$ $\{v_n\}$ and $\{w_n\}$ in C as follows:*

S-iteration process: $u_{n+1} = (1 - \alpha_n)Tu_n + \alpha_n Ty_n,$

$y_n = (1 - \beta_n)u_n + \beta_n Tu_n, n \in \mathbb{N}.$

Picard iteration : $v_{n+1} = Tv_n, n \in \mathbb{N}.$

Mann iteration process : $w_{n+1} = (1 - \beta_n)w_n + \beta_n Tw_n, n \in \mathbb{N}.$

Then we have the following:

(a) $\|u_{n+1} - x^*\| \leq k^n[1 - (1 - k)\alpha\beta]^n \|u_1 - x^*\|$ *for all* $n \in \mathbb{N}$.

(b) $\|v_{n+1} - x^*\| \leq k^n \|v_1 - x^*\|$ *for all* $n \in \mathbb{N}$.

(c) $\|w_{n+1} - x^*\| \leq [1 - (1 - k)\beta]^n \|w_1 - x^*\|$ *for all* $n \in \mathbb{N}$.

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes.

Proof. (a) By the definition of the S-iteration process, we have

$$\begin{aligned}
\|u_{n+1} - x^*\| &\leq k[(1 - \alpha_n)\|u_n - x^*\| + \alpha_n \|y_n - x^*\|] \\
&\leq k[(1 - \alpha_n)\|u_n - x^*\| + \alpha_n((1 - \beta_n)\|u_n - x^*\| + \beta_n \|Tu_n - x^*\|)] \\
&\leq k[1 - (1 - k)\alpha_n\beta_n]\|u_n - x^*\| \\
&\leq a_n,
\end{aligned}$$

where $a_n = k^n[1-(1-k)\alpha\beta]^n\|u_1 - x^*\|$.

(b) Note

$$\|v_{n+1} - x^*\| \leq k\|v_n - x^*\| \leq b_n,$$

where $b_n = k^n\|v_1 - x^*\|$.

(b) By the definition of the Mann iteration process, we have

$$\begin{aligned}\|w_{n+1} - x^*\| &\leq (1-\beta_n)\|w_n - x^*\| + \beta_n\|Tw_n - x^*\|] \\ &\leq [1-(1-k)\beta_n]\|w_n - x^*\| \\ &\leq c_n,\end{aligned}$$

where $c_n = [1-(1-k)\beta]^n\|w_1 - x^*\|$.

It is easy to see that $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$, it follows from Definition 3.5 that the S-iteration process is faster than the Picard iteration process. Similarly, one can show that the normal S-iteration process is faster than the Mann iteration process. ☐

Examples 3.7. Let $X = \mathbb{R}$ and $C = [0,\infty)$. Let $T : C \to C$ be an operator defined by $Tx = (3x+18)^{1/3}$ for all $x \in C$. It is easy to see that T is a contraction on C with contractivity factor $k = 18^{-1/3}$ and $x^* = 3$. Choose $u_1 = v_1 = w_1 \in C$ and $\alpha_n = \beta_n = 1/2$, the corresponding S-iteration process, Picard iteration process and KM iteration process are given by $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$, respectively.

Table 1. : Comparison of S-iteration, Picard iteration and KM iteration processes

no. of iteration	S-iteration	Picard iteration	Mann iteration
1	1000	1000	1000
2	12.99923955	14.4512832	507.2256416
3	3.679603367	3.94409414	259.3864187
4	3.057482809	3.101431265	134.3273583
5	3.004958405	3.011228065	70.91103158
6	3.000428435	3.001247044	38.52222997
7	3.000037025	3.000138554	21.81696909
8	3.0000032	3.000015395	13.09346233
9	3.000000277	3.000001711	8.474131743

The sequences $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ converge to $x^* = 3$. The comparison of the S-iteration process with the Picard and KM iteration processes is given for the first 9 iterates in Table 1 and for initial value $x_1 = 1000$.

In the light of Theorem 3.6, we have the following sharper results which contain iterative sequences faster than the sequence defined by (1.1).

Theorem 3.8. *Let C be a nonempty closed convex subset of a Hilbert space H and $A : C \to H$ a κ-Lipschitzian and η-strongly monotone operator. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in (0,1) such that $a \leq \alpha_n$ and $\beta \leq \beta_n$ for all $n \in \mathbb{N}$ and for some*

$\alpha, \beta > 0$. Then for $\mu \in (0, 2\eta/\kappa^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$, and defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)P_C(I-\mu A)x_n + \alpha_n P_C(I-\mu A)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\mu A)x_n, n \in \mathbb{N} \end{cases} \tag{3.3}$$

converges strongly to $x^* \in \Omega(C, A)$.

Remark 3.9. Theorem 3.6 showes that the rate of convergence of sequence $\{x_n\}$ defined by (3.3) is faster than that the rate of convergence of the Picard iterative sequence defined by (1.1). Therefore, Theorem 3.8 provides an affirmative answer to Question 1.1.

Theorem 3.10. *Let C be a nonempty closed convex subset of a Hilbert space H and $A : C \to H$ a κ-Lipschitzian and η-strongly monotone operator. Let $\{\lambda_n\}$ be a sequence in (0,1) such that $\lambda \leq \lambda_n$ for all $n \in \mathbb{N}$ and for some $\lambda > 0$. Then for $\mu \in (0, 2\eta/\kappa^2)$, the iterative sequence $\{x_n\}$ generated from $x_1 \in C$, and defined by*

$$x_{n+1} = P_C(I-\mu A)[(1-\lambda_n)x_n + \lambda_n P_C(I-\mu A)x_n], n \in \mathbb{N}$$

converges strongly to $x^* \in \Omega$.

4. Nonexpansive operators and S-iteration process

First, we give an important characterization of the S-iteration process for nonexpansive operators in a uniformly convex Banach space which is slightly different from Lemma 2.5.

Theorem 4.1. [1, Theorem 6.7.3]. *Let C be a nonempty closed convex (not necessary bounded) subset of a uniformly convex Banach space X and $T : C \to C$ a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (S) with the restriction:*

$$\lim_{n\to\infty} \alpha_n\beta_n(1-\alpha_n) \text{ exists and } \lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0. \tag{4.1}$$

Then, for arbitrary initial value $x_1 \in C, \{\|x_n - Tx_n\|\}$ converges to some constant $r_C(T)$, which is independent of the choice of the initial value $x_1 \in C$.

Remark 4.2. In Theorem 4.1, the condition (4.1) can be replaced by (1.4).

By using the LEAF point property of S-iteration process, one can establish a convergence theorem for finding fixed points of nonexpansive operators. Indeed, applying Lemma 2.6, we have

Theorem 4.3. [1, Theorem 6.7.4]. *Let X a real uniformly convex Banach space with a Fréchet differentiable norm or which satisfies the Opial condition. Let C be a nonempty closed convex (not necessary bounded) subset of and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by (S) with the restriction (4.1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Next, we show that $\{x_n\}$ generated by the S-iteration process is an unbounded sequence if T has no fixed points.

Theorem 4.4. *Let H be a Hilbert space and $T : H \to H$ a nonexpansive operator. Let $\{x_n\}$ be the sequence defined by (S) with the restriction (4.1). If $F(T) = \emptyset$, then $\{x_n\}$ is unbounded.*

Proof. Let $F(T) = \emptyset$. Suppose, for contradiction, that $\{x_n\}$ is bounded. Then there exists a constant $K > 0$ such that $\|x_n\| \leq K$ for all $n \in \mathbb{N}$. By the nonexpansiveness of T, we have

$$\begin{aligned} \|Tx_n\| &\leq \|Tx_n - Tx_1\| + \|Tx_1\| \\ &\leq \|x_n - x_1\| + \|Tx_1\| \\ &\leq \|Tx_1\| + 2K \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Let $C := \{x \in H : \|Tx\| \leq \|Tx_1\| + 2K\}$. Note that C is a closed convex bounded set of H, there exists a metric projection operator P_C from H onto C. Define $\widehat{T} = P_C T$. It is not hard to check that $\widehat{T}$ is nonexpansive and $\widehat{T}x_n = Tx_n$ for all $n \in \mathbb{N}$. Thus, iterative sequence $\{x_n\}$ of (S) can be generated by the nonexpansive operator $\widehat{T}$ as below:

$$\begin{cases} x_{n+1} &= (1-\alpha_n)\widehat{T}x_n + \alpha_n \widehat{T}y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n \widehat{T}x_n, n \in \mathbb{N}. \end{cases}$$

It follows from Theorem 4.3 that $\{x_n\}$ converges weakly to x^*, a fixed point of $\widehat{T}$. Note that $\|x_n\| \leq K$, we have $\|x^*\| \leq K$. Observe that

$$\begin{aligned} \|\widehat{T}x^*\| &\leq \|\widehat{T}x^* - \widehat{T}x_1\| + \|\widehat{T}x_1\| \\ &\leq \|x^* - x_1\| + \|\widehat{T}x_1\| \\ &\leq \|\widehat{T}x_1\| + 2K \text{ for all } n \in \mathbb{N}. \end{aligned}$$

It shows that $x^* \in C$. Therefore, $x^* = \widehat{T}x^* = P_C Tx^* = Tx^*$, i.e., $x^* \in F(T)$, a contradiction. □

5. Strong convergence of S-iteration process of Helpern type

Motivated by works of Chidume and Chidume [8] and Suzuki [20], we propose the following algorithms:

Algorithm 5.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ an operator. Given $u, x_1 \in C$, a sequence $\{x_n\}$ in C is constructed as follows:*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n u, n \in \mathbb{N}, \end{cases} \tag{5.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1]$ satisfying the following condition:

$(\overline{C1})$ $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = \lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$ *and* $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$.

The iterative sequence $\{x_n\}$ defined by (5.1) is called S-iteration process of Halpern type.

Algorithm 5.2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ an operator. Given $u, x_1 \in C$, a sequence $\{x_n\}$ in C is constructed as follows:*

$$\begin{cases} x_{n+1} &= (1-\lambda)Tx_n + \lambda T y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n u, n \in \mathbb{N}, \end{cases} \tag{5.2}$$

where $\lambda \in (0,1]$ and $\{\beta_n\}$ is a sequence in $(0,1]$ satisfying the following condition:

($\overline{C2}$) $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} \frac{\beta_n}{\beta_{n+1}} = 1$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$.

Our Algorithms 5.1 $\sim$ 5.2 are independent from (1.5) and (1.6). Some basic properties of Algorithm 5.1 are detailed below:

Proposition 5.3. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ a nonexpansive operator such that $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 5.1. Then we have the following:*
(a) $\{x_n\}$ and $\{y_n\}$ are bounded.
(b) $\lim\limits_{n\to\infty} ||x_n - y_n|| = \lim\limits_{n\to\infty} ||x_{n+1} - x_n|| = \lim\limits_{n\to\infty} ||x_n - Tx_n|| = \lim\limits_{n\to\infty} ||y_n - Ty_n|| = 0.$

Proof. (a) Suppose $p \in F(T)$. From (5.1), we have

$$\|y_n - p\| \quad \leq \quad (1-\beta_n)\|x_n - p\| + \beta_n \|u - p\|. \tag{5.3}$$

Invoking (5.3), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\ &\leq (1-\alpha_n)\|x_n - p\| + \alpha_n \|y_n - p\| \\ &\leq (1-\alpha_n\beta_n)\|x_n - p\| + \alpha_n\beta_n \|u - p\| \\ &\leq \max\{\|x_n - p\|, \|u - p\|\} \\ &\vdots \\ &\leq \max\{\|x_1 - p\|, \|u - p\|\}. \end{aligned}$$

Thus, $\{x_n\}$ is bounded and hence, from (5.3), $\{y_n\}$ is bounded.

(b) Note that the condition $\lim_{n\to\infty} \beta_n = 0$ implies that

$$\|y_n - x_n\| = \beta_n \|x_n - u\| \to 0 \ \text{ as } n \to \infty.$$

Further, we have

$$\|x_{n+1} - Tx_n\| = \|Ty_n - Tx_n\| \leq \|y_n - x_n\| \to 0 \ \text{ as } n \to \infty.$$

Observe that

$$\begin{aligned} ||y_n - y_{n-1}|| &= ||(1-\beta_n)x_n + \beta_n u - (1-\beta_{n-1})x_{n-1} - \beta_{n-1}u\| \\ &= ||(1-\beta_n)x_n - (1-\beta_n)x_{n-1} + (1-\beta_n)x_{n-1} \\ &\quad -(1-\beta_{n-1})x_{n-1} + (\beta_n - \beta_{n-1})u\| \\ &\leq (1-\beta_n)||x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|(\|u\| + \|x_{n-1}\|) \\ &\leq (1-\beta_n)||x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|K_1 \end{aligned}$$

for some constant $K_1 > 0$. From (5.1), we have

$$\begin{aligned}
||x_{n+1} - x_n|| &= ||(1-\alpha_n)Tx_n + \alpha_n Ty_n - (1-\alpha_{n-1})Tx_{n-1} - \alpha_{n-1}Ty_{n-1}|| \\
&= ||(1-\alpha_n)Tx_n - (1-\alpha_n)Tx_{n-1} + (1-\alpha_n)Tx_{n-1} + \alpha_n Ty_n \\
&- \alpha_n Ty_{n-1} + \alpha_n Ty_{n-1} - (1-\alpha_{n-1})Tx_{n-1} - \alpha_{n-1}Ty_{n-1}|| \\
&\le (1-\alpha_n)||x_n - x_{n-1}\| + \alpha_n\|y_n - y_{n-1}\| \\
&+ |\alpha_n - \alpha_{n-1}| \; \|Tx_{n-1} - Ty_{n-1}|| \\
&\le (1-\alpha_n)||x_n - x_{n-1}\| + \alpha_n\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}| \; \|x_{n-1} - y_{n-1}|| \\
&= (1-\alpha_n)||x_n - x_{n-1}\| + \alpha_n[(1-\beta_n)||x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|K_1] \\
&+ |\alpha_n - \alpha_{n-1}| \; ||x_{n-1} - y_{n-1}|| \\
&\le (1-\alpha_n\beta_n)||x_n - x_{n-1}\| \\
&+ \alpha_n\beta_n\left(\left|1 - \frac{\beta_{n-1}}{\beta_n}\right|K_1 + \left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right|\frac{\beta_{n-1}}{\beta_n}\; K_2\right)
\end{aligned}$$

for some constant $K_2 > 0$. From condition $(\overline{C1})$, we have $\lim_{n\to\infty}\left|1 - \frac{\alpha_{n-1}}{\alpha_n}\right| = 0$ and $\lim_{n\to\infty}\left|1 - \frac{\beta_{n-1}}{\beta_n}\right| = 0$. It follows from Lemma 2.2 that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. Hence

$$\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \to 0 \text{ as } n \to \infty.$$

Moreover,

$$\begin{aligned}
||y_n - Ty_n|| &\le ||y_n - x_n|| + ||x_n - Tx_n|| + ||Tx_n - Ty_n|| \\
&\le 2||y_n - x_n|| + ||x_n - Tx_n|| \to 0 \text{ as } n \to \infty.
\end{aligned}$$

□

Now we are in a position to establish the main strong convergence theorems of this section.

Theorem 5.4. *Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 5.1. Then $\{x_n\}$ converges strongly to $R_{F(T)}(u)$, where $R_{F(T)}$ is the sunny nonexpansive retraction from C onto $F(T)$.*

Proof. Invoking Lemma 2.3, we see that the path $\{z_t\}$ defined by (2.1) for $t \in (0,1)$ is strongly convergent to $R_{F(T)}(u)$ as $t \to 0^+$. Set $z := R_{F(T)}(u) = \lim_{t\to 0^+} z_t$. By Lemma 2.4, we have

$$\limsup_{n\to\infty}\langle u - z, J(y_n - z)\rangle \le 0.$$

Applying Lemma 2.1, we get

$$\begin{aligned}
\|y_n - z\|^2 &= \|(1-\beta_n)(x_n - z) + \beta_n(u - z)\|^2 \\
&\le (1-\beta_n)\|x_n - z\|^2 + 2\beta_n\langle u - z, J(y_n - z)\rangle.
\end{aligned}$$

Now since X is uniformly convex, by [27], there exists a continuous strictly convex function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 = \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\varphi(\|x-y\|) \tag{5.4}$$

for all $x, y \in X$ with $\|x\| \le r$, $\|y\| \le r$ and for all $\lambda \in [0,1]$ and for some $r > 0$. Choose $r > 0$ large enough so that $\|Tx_n - z\| \le r$ and $\|Ty_n - z\| \le r$ for all $n \in \mathbb{N}$. From (5.4), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|(1-\alpha_n)(Tx_n - z) + \alpha_n(Ty_n - z)\|^2 \\ &\le (1-\alpha_n)\|Tx_n - z\|^2 + \alpha_n\|Ty_n - z\|^2 \\ &\le (1-\alpha_n)\|x_n - z\|^2 + \alpha_n\|y_n - z\|^2 \\ &\le (1-\alpha_n\beta_n)\|x_n - z\|^2 + 2\alpha_n\beta_n\langle u - z, J(y_n - z)\rangle \\ &\le (1-\alpha_n\beta_n)\|x_n - z\|^2 + \lambda_n\sigma_n, \end{aligned}$$

where $\lambda_n := \alpha_n\beta_n$ and $\sigma_n := 2\langle u - z, J(y_n - z)\rangle$. Since $\sum_{n=1}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\sigma_n \le 0$, we conclude from Lemma 2.2 that $\{x_n\}$ converges strongly to z. □

Corollary 5.5. *Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by Algorithm 5.2. Then $\{x_n\}$ converges strongly to $R_{F(T)}(u)$.*

Proof. Suppose $\alpha_n = \lambda$ for all $n \in \mathbb{N}$. Then the condition $(\overline{C1})$ is satisfied and hence Corollary 5.5 follows from Theorem 5.4. □

We now close this section with the following basic convergence theorem.

Theorem 5.6. *Let X be a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive operator with $F(T) \neq \emptyset$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = T[(1-\beta_n)x_n + \beta_n u], n \in \mathbb{N}, \tag{5.5}$$

where $\{\beta_n\}$ is a sequence in $(0,1]$ satisfying the condition $(\overline{C2})$. Then $\{x_n\}$ converges strongly to $R_{F(T)}(u)$, where $R_{F(T)}$ is the sunny nonexpansive retraction from C onto $F(T)$.

6. Applications

6.1. **Application to constrained optimization problems.** Let C be a closed convex subset of a Hilbert space H, $\nu > 0$ a constant, P_C the metric projection mapping from H onto C and $A : C \to H$ a ν-ism. It is well known that $P_C(I - \gamma A)$ is nonexpansive operator provided that $\gamma \in (0, 2\nu)$.

In view of above fact, we derive the following results from Theorems 4.3 and 5.4, respectively.

Theorem 6.1. *Let C be a closed convex subset of a Hilbert space H, $\nu > 0$ a constant, P_C the metric projection mapping from H onto C and $A : C \to H$ a ν-ism. Assume that $\Omega(C, A) \neq \emptyset$ and $\gamma \in (0, 2\nu)$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n)P_C(I-\gamma A)x_n + \alpha_n P_C(I-\gamma A)y_n, \\ y_n & = (1-\beta_n)x_n + \beta_n P_C(I-\gamma A)x_n, n \in \mathbb{N}, \end{cases} \tag{6.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1) satisfy the condition (1.4). Then $\{x_n\}$ converges weakly to a solution of the variational inequality $VI(C, A)$.

Theorem 6.2. *Let C be a nonempty closed convex subset of a Hilbert space H, $\nu > 0$ a constant, P_C the metric projection mapping from H onto C and $A : C \to H$ a ν-ism. Assume that $\Omega(C, A) \neq \emptyset$ and $\gamma \in (0, 2\nu)$. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n)P_C(I-\gamma A)x_n + \alpha_n P_C(I-\gamma A)y_n, \\ y_n & = (1-\beta_n)x_n + \beta_n u, n \in \mathbb{N}, \end{cases} \tag{6.2}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are two sequences in (0,1] satisfy the condition ($\overline{C1}$). Then $\{x_n\}$ converges strongly to a solution of the variational inequality $VI(C, A)$.

Algorithms for signal and image processing are often iterative constrained optimization procedures designed to minimize a convex differentiable function $f(x)$ over a closed convex set C in H. It is well known that every L-Lipschitzian operator is $2/L$-ism. Therefore, the following algorithms (6.3) and (6.4) converge to minimizer of f.

Corollary 6.3. *Let C be a closed convex subset of a Hilbert space H and f be a convex and differentiable function on an open set D containing the set C. Assume that ∇f is a L-Lipschitz continuous operator on D, $\gamma \in (0, 2/L)$ and minimizers of f relative to the set C exists. For given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n)P_C(I-\gamma\nabla f)x_n + \alpha_n P_C(I-\gamma\nabla f)y_n, \\ y_n & = (1-\beta_n)x_n + \beta_n P_C(I-\gamma\nabla f)x_n, n \in \mathbb{N}, \end{cases} \tag{6.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1) satisfy the condition (1.4). Then $\{x_n\}$ converges weakly to a minimizer of f.

Corollary 6.4. *Let C be a closed convex subset of a Hilbert space H and f be a convex and differentiable function on an open set D containing the set C. Assume that ∇f is a L-Lipschitz continuous operator on D, $\gamma \in (0, 2/L)$ and minimizers of f relative to the set C exists. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} & = (1-\alpha_n)P_C(I-\gamma\nabla f)x_n + \alpha_n P_C(I-\gamma\nabla f)y_n, \\ y_n & = (1-\beta_n)x_n + \beta_n u, n \in \mathbb{N}, \end{cases} \tag{6.4}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are two sequences in (0,1] satisfy the condition ($\overline{C1}$). Then $\{x_n\}$ converges strongly to a minimizer of f.

6.2. **Application to split feasibility problems.** Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1-\alpha)I+\alpha S$, where $\alpha \in (0,1)$ and S is nonexpansive on H.

Let C be a closed convex subset of a Hilbert space H_1, Q a closed convex subset of another Hilbert space H_2 and $A : H_1 \to H_2$ a bounded linear operator. Let P_C and P_Q be the orthogonal projections onto C and Q, respectively; $\gamma > 0$ a constant, and A^* the adjoint of A. Set $q(x) := \frac{1}{2}\|Ax - P_Q Ax\|^2, x \in C$.

Consider the minimization problem:

$$\text{find} \min_{x \in C} q(x).$$

By [3], the gradient of q is

$$\nabla q = A^*(I - P_Q)A.$$

Since $I - P_Q$ is nonexpansive, it follows that ∇q is L-Lipschitzian with $L = \|A\|^2$. Therefore, ∇q is $1/L$-ism (cf. [4]) and for any $0 < \gamma < 2/L$, $I - \gamma\nabla q$ is averaged. Therefore, the composite $P_C(I - \gamma\nabla q)$ is also averaged. Set $T := P_C(I - \gamma\nabla q)$. Note that solution set of SFP(1.2) is $F(T)$.

We now present some iterative algorithms that can be used to find solutions of SFP(1.2).

Theorem 6.5. *Assume that SFP(1.2) is consistent. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in (0,1) such that $\alpha \leq \alpha_n$ and $\beta \leq \beta_n$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)P_C(I-\gamma\nabla q)x_n + \alpha_n P_C(I-\gamma\nabla q)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n P_C(I-\gamma\nabla q)x_n, n \in \mathbb{N}, \end{cases}$$

where $0 < \gamma < 2/\|A\|^2$. Then $\{x_n\}$ converges weakly to a solution of SFP(1.2).

Proof. Since $T := P_C(I - \lambda\nabla q)$ is nonexpansive, Theorem 6.5 follows from Theorem 4.3. □

Theorem 6.6. *Assume that SFP(1.2) is consistent. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} &= (1-\alpha_n)P_C(I-\gamma\nabla q)x_n + \alpha_n P_C(I-\gamma\nabla q)y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n u, n \in \mathbb{N}, \end{cases}$$

where $0 < \gamma < 2/\|A\|^2$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in (0,1] satisfy the condition $(\overline{C}1)$. Then $\{x_n\}$ converges strongly to a solution of SFP(1.2) nearest to u.

Proof. Since $T := P_C(I - \lambda\nabla q)$ is nonexpansive, Theorem 6.6 follows from Theorem 5.4. □

The special case of Theorem 6.6 is the following corollary:

Corollary 6.7. *Assume that SFP(1.2) is consistent. For given $u, x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$x_{n+1} = P_C(I - \gamma\nabla q)[(1-\beta_n)x_n + \beta_n u], n \in \mathbb{N},$$

where $0 < \gamma < 2/\|A\|^2$ *and* $\{\beta_n\}$ *is a sequence in (0,1] satisfies the condition* $(\overline{C2})$*. Then* $\{x_n\}$ *converges strongly to a solution of SFP(1.2) nearest to* u*.*

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