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# A VISCOSITY APPROXIMATION METHOD OF COMMON SOLUTIONS FOR QUASI VARIATIONAL INCLUSION AND FIXED POINT PROBLEMS

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**Abstract.** In this paper, we introduce a new iterative scheme for finding solutions the common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inclusion problem with a multivalued maximal monotone mapping and an $\alpha$-inverse-strongly monotone mapping. We show that the sequence converges strongly to a common solutions for quasi variational inclusion and fixed point problems under some parameters controlling conditions. This main theorem extends a recent result of Zhang et al. [Algorithms of common solutions to quasi variational inclusion and fixed point problems. *Appl. Math. Mech.* Engl. Ed., 2008, 29(5)(2006), 571-581.] and some other authors.

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## 1. Introduction

Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$. Let $A : H \to H$ be a single-valued nonlinear mapping and $M : H \to 2^H$ be a multi-valued mapping. We are interested to the problem so-called *quasi-variational inclusion problem*, that is, determine an element $u \in H$ such that

$$0 \in A(u) + M(u). \tag{1.1}$$

The set of solutions of the problem (1.1) is denoted by $VI(H, A, M)$.

Examples of the variational inclusion (1.1):

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(I) If $M = \partial\varphi$, where $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semi-continuous functional $\varphi : H \to (-\infty, +\infty]$, then problem (1.1) reduces to the following problem: find $u \in H$ such that

$$\langle A(u), v - u\rangle + \varphi(v) - \varphi(u) \geq 0, \ \forall v \in H. \tag{1.2}$$

which is called a nonlinear variational inequality and has been studied by many authors.

(II) If $M = \partial\delta_C$, where $C$ is a nonempty closed convex subset of $H$, and $\delta_C : H \to [0, \infty]$ is the indicator function of C, i.e.,

$$\delta_C(x) = \begin{cases} 0, \ x \in C, \\ +\infty, \ x \notin C. \end{cases} \tag{1.3}$$

Then the variational inclusion problem (1.1) is equivalent (see [32]) to find $u \in C$ such that

$$\langle A(u), v - u\rangle \geq 0, \ \forall v \in C. \tag{1.4}$$

This problem is called Hartman-Stampacchia variational inequality problem (or the *classical variational inequality*) denoted by $VI(C, A)$.

Inspired by their wide applications, many researchers have studied the classical variational inequality and generalized it in various directions. Many computational methods for solving variational inequalities have been proposed in Hilbert spaces, see [1, 4, 8] and [18] and the references therein. In fact, it is worth noting that, the variational inequalities problems are among the most interesting and important mathematical problems and have been studied intensively in the past years since they have wide applications in mechanics, physics, optimization and control, nonlinear programming, economics and transportation equilibrium, and engineering sciences, etc. Meanwhile, the variational inclusions problems have been generalized and extended in different directions using the novel and innovative techniques. Various kinds of iterative algorithms to solve the variational inequalities and variational inclusions have been developed by many authors, see [8, 5, 10, 11, 12, 13, 15, 16, 17, 27, 28, 30] and [32] and the references therein.

In 1976, Korpelevich [14] introduced the following so-called extragradient method: Let $C$ be a closed convex subset of $\mathbb{R}^N$,

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n) \end{cases} \tag{1.5}$$

for all $n \geq 0$, where $A$ is a monotone and $k$-Lipschitz continuous mapping of $C$ in to $\mathbb{R}^N$ . He showed that if $VI(C, A)$ is nonempty then, under some suitable conditions, the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.5), converge to the same point $z \in VI(C, A)$.

Related to the variational inequalities problems, we also have the problems of finding the fixed points of the nonlinear mappings, which is the subject of current interest in functional analysis. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of the set of solutions of the equilibrium problems and the set of fixed points of nonlinear mappings. In 2003, Takahashi and Toyoda [26] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) SP_C(x_n - \lambda_n A x_n) \tag{1.6}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C, \alpha_n$ is a sequence in (0, 1), and $\lambda_n$ is a sequence in $(0, 2\alpha)$. They proved that if $F(S) \cap VI(C, A) \neq \emptyset$, where $F(S)$ is denoted for the set of fixed points of $S$, then the sequence $\{x_n\}$ generated by (1.6) converges weakly to some $z \in F(S) \cap VI(C, A)$. Moreover, in 2006, Yao and Yao [31] introduced the following iterative scheme:

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n SP_C(y_n - \lambda_n A y_n), \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that, under some suitable control conditions, the sequence $\{x_n\}$ defined by (1.7) converges strongly to common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for $\alpha$-inverse-strongly monotone mappings under some parameters controlling conditions. Recently, in 2008, Zhang Shi-sheng et al. [32], introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x + (1-\alpha_n) SJ_M^{\lambda}(x_n - \lambda A x_n) \tag{1.8}$$

for every $n = 0, 1, 2, ...$, where $J_M^{\lambda}$ the resolvent operator associated with $M$ and $M$ is maximal monotone and $x_0 = x \in H, \{\alpha_n\}$ is a sequence in (0, 1), and $\lambda \in (0, 2\alpha]$ satisfy some parameters controlling conditions. They proved that if $F(S) \cap VI(H, A, M) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(H,A,M)} x_0$.

On the other hand, in 2000, Moudafi [21] introduced the so-called *viscosity approximation method* for nonexpansive mappings: Let $f$ be a contraction on $H$. Starting with an arbitrary $x_0 \in H$, defined a sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) S x_n, \ \ n \geq 0, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. Moreover, Takahashi and Takahashi [25] introduced an iterative scheme by using the viscosity approximation method to proved a strong convergence theorem, which is connected with Combettes and Hirstoaga's result [7] and Wittmann's result [29].

In this paper, motivated by the iterative schemes considered in [21, 25, 31, 32], we will introduce the new following iterative process: given $x_0 \in H$ arbitrarily and

$$\begin{cases} x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n SJ_M^{\lambda}(y_n - \lambda A y_n), \\ y_n = J_M^{\lambda}(x_n - \lambda A x_n), \ \ n \geq 0. \end{cases}$$

We will use the sequence $\{x_n\}$ for finding a common element of the set of fixed points of a nonexpansive mapping and the solutions set of the quasi variational inequality inclusion problem for an $\alpha$-inverse-strongly monotone mapping in a real Hilbert space. Furthermore, we also provide some strong convergence theorems which are connected with Yao and Yao's result [31] and Takahashi and Takahashi's result [25]. Our results extend and improve the results of Sheng et al. [32].

## 2. Preliminaries

Let $H$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let $C$ be a closed convex subset of $H$. In the following, we denote by $\to$ strong convergence and by $\rightharpoonup$ weak convergence. It is well known that for any $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

For every point $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x-y\| \quad \text{for all } y \in C.$$

$P_C$ is called the metric projection of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2 \tag{2.2}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{2.3}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.4}$$

for all $x \in H, y \in C$. It is easy to see that the following is true :

$$u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda Au), \lambda > 0. \tag{2.5}$$

We now recall some basic definitions and well-known results.

A mapping $S : H \to H$ is called *L-Lipschitz* continuous, if

$$\|Sx - Sy\| \le L\|x-y\| \quad \forall x, y \in H.$$

In particular, if $L = 1$ then $S$ is called a *nonexpansive* mapping. Moreover, if $L \in [0,1)$ then $S$ is called a *contraction* mapping.

Recall that a mapping $A : H \to H$ is said to be:

(i) *monotone* if $\langle Au - Av, u - v\rangle \ge 0,\ \forall u, v \in H$;

(ii) *$\alpha$-inverse-strongly monotone* [2, 18] if there exists a positive real number $\alpha$ such that

$$\langle Au - Av, u - v\rangle \ge \alpha\|Au - Av\|^2, \quad \forall u, v \in C.$$

Notice that any $\alpha$-inverse-strongly monotone mapping $A$ is monotone and $\frac{1}{\alpha}$-Lipschitz continuous. Moreover, if $A$ is $\alpha$-inverse-strongly monotone, then $I - \lambda A$ is a nonexpansive mapping from $C$ to $H$, provided $\lambda \le 2\alpha$.

A set-valued mapping $M : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \ge 0$. A monotone mapping $M : H \to 2^H$ is *maximal* if the graph of $G(M)$ of $M$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $M$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \ge 0$ for every $(y, g) \in G(M)$ implies $f \in Mx$. Let

$M$ be a monotone map of $H$ into $H$, $L$-Lipschitz continuous mapping and let $N_C v$ be the normal cone to $C$ at $v \in C$, i.e., $N_C v = \{w \in H : \langle u - v, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Mv + N_C v, & v \in C; \\ \emptyset, & v \notin C. \end{cases}$$

Then $T$ is the maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, B)$; see [23].

**Proposition 2.1.** [32, Proposition 1.2] *Let $M : H \to H$ be a maximal monotone mapping. For any $\lambda > 0$, define the operator $J_M^\lambda : H \to H$ by $J_M^\lambda = (I + \lambda M)^{-1}$, where $I$ is the identity operator on $H$. Then $J_M^\lambda$ possesses the following properties:*

(i) *$J_M^\lambda$ is single-valued and nonexpansive mapping;*
(ii) *$J_M^\lambda$ is 1-inverse-strongly monotone, i.e.,*

$$\|J_M^\lambda x - J_M^\lambda y\|^2 \leq \langle x - y, J_M^\lambda x - J_M^\lambda y\rangle, \ \forall x, y \in H. \tag{2.6}$$

**Definition 2.2.** Let $M : H \to H$ be a maximal monotone mapping. Then for any $\lambda > 0$, the mapping $J_M^\lambda : H \to H$ associated with $M$ defined by

$$J_M^\lambda(x) = (I + \lambda M)^{-1}(x), \quad \forall x \in H,$$

is called the resolvent operator.

**Lemma 2.3.** [32]. *Let $M : H \to H$ be a maximal monotone mapping.*

(1) *For any $\lambda > 0$, we have*

$$u \in VI(H, A, M) \Leftrightarrow u = J_M^\lambda(u - \lambda Au), \forall \lambda > 0. \tag{2.7}$$

(2) *If $\lambda \in (0, 2\alpha]$, then $VI(H, A, M)$ is a closed convex subset in $H$.*

In order to prove our main results, we need the following lemmas.

**Lemma 2.4.** [22] *Let $(E, \langle ., .\rangle)$ be an inner product space. Then for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2.$$

**Lemma 2.5.** ([24]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space $E$ and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 1$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.*

**Lemma 2.6.** ([29]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \quad \forall n \geq n_0,$$

*where $n_0$ is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $[0, 1]$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = \circ(\lambda_n)$, then $\lim_{n\to\infty} a_n = o$.*

## 3. Main results

In this section, we prove strong convergence theorems.

**Theorem 3.1.** *Let $H$ be a real Hilbert space and let $A : H \to H$ be an $\alpha-$inverse-strongly monotone mapping $M : H \to 2^H$ be a maximal monotone mapping and $S$ be a nonexpansive mapping of $H$ into itself. Let $f$ be a contraction of $H$ into itself. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in H, \text{ chosen arbitrary },\\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n\\ z_n = J_M^\lambda(y_n - \lambda A y_n),\\ y_n = J_M^\lambda(x_n - \lambda A x_n) \end{cases} \tag{3.1}$$

*for all $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\lambda$ are chosen so that $\lambda \in (a, b]$ for some $a, b$ with $0 < a < \lambda \le b < 2\alpha$ and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

*If $F(S) \cap VI(H, A, M) \neq \emptyset$, then $\{x_n\}$ converges strongly to $z_0$ which is the unique solution in $F(S) \cap VI(H, A, M)$ to the following variational inequality:*

$$\langle (f - I)z_0, z_0 - z\rangle \le 0 \text{ for all } z \in F(S) \cap VI(H, A, M). \tag{3.2}$$

*Equivalently, we have $z_0 = P_{F(S)\cap VI(H,A,M)} f(z_0)$.*

*Proof.* Let $v \in F(S) \cap VI(H, A, M)$, then $v = J_M^\lambda(v - \lambda A v)$. By the nonexpansiveness of $J_M^\lambda$ and $I - \lambda A$, we note that

$$\begin{aligned} \|z_n - v\| &= \|J_M^\lambda(y_n - \lambda A y_n) - J_M^\lambda(v - \lambda A v)\|\\ &\le \|(y_n - \lambda A y_n) - (v - \lambda A v)\|\\ &= \|(I - \lambda A)y_n - (I - \lambda A)v\|\\ &\le \|y_n - v\| && (3.3)\\ &= \|J_M^\lambda(x_n - \lambda A x_n) - J_M^\lambda(v - \lambda A v)\|\\ &\le \|(x_n - \lambda A x_n) - (v - \lambda A v)\|\\ &\le \|x_n - v\|. && (3.4) \end{aligned}$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - v\| &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S z_n - v\| \\
&\leq \alpha_n \|f(x_n) - v\| + \beta_n \|x_n - v\| + \gamma_n \|z_n - v\| \\
&\leq \alpha_n \|f(x_n) - v\| + \beta_n \|x_n - v\| + \gamma_n \|x_n - v\| \\
&\leq \alpha_n \|f(x_n) - f(v)\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n)\|x_n - v\| \\
&\leq \alpha_n \alpha \|x_n - v\| + \alpha_n \|f(v) - v\| + (1 - \alpha_n)\|x_n - v\| \\
&= (1 - \alpha_n(1 - \alpha))\|x_n - v\| + \alpha_n (1 - \alpha) \frac{\|f(v) - v\|}{(1 - \alpha)} \\
&\leq \max\{\|x_0 - v\|, \frac{\|f(v) - v\|}{(1 - \alpha)}\}.
\end{aligned}$$

Therefore $\{x_n\}$ is bounded. Consequently, the sets $\{x_n\}, \{z_n\}, \{Sz_n\}, \{Ax_n\}$ and $\{Ay_n\}$ are also bounded. Moreover, we observe that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|J_M^\lambda(y_{n+1} - \lambda A y_{n+1}) - J_M^\lambda(y_n - \lambda A y_n)\| \\
&\leq \|(y_{n+1} - \lambda A y_{n+1}) - (y_n - \lambda A y_n)\| \\
&= \|(I - \lambda A) y_{n+1} - (I - \lambda A) y_n\| \\
&\leq \|y_{n+1} - y_n\| \\
&= \|J_M^\lambda(x_{n+1} - \lambda A x_{n+1}) - J_M^\lambda(x_n - \lambda_n A x_n)\| \\
&\leq \|(x_{n+1} - \lambda A x_{n+1}) - (x_n - \lambda_n A x_n)\| \\
&\leq \|x_{n+1} - x_n\|. \qquad (3.5)
\end{aligned}$$

Let $x_{n+1} = (1 - \beta_n) w_n + \beta_n x_n$. Thus, we note that

$$w_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} = \frac{\alpha_n f(x_n) + \gamma_n S z_n}{1 - \beta_n} = \frac{\alpha_n f(x_n) + \gamma_n S J_M^\lambda (y_n - \lambda A y_n)}{1 - \beta_n}$$

and hence we have

$$\begin{aligned}
w_{n+1} - w_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S z_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S z_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} S z_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n+1} f(x_n) + \gamma_{n+1} S z_n}{1 - \beta_{n+1}} \\
&\quad + \frac{\alpha_{n+1} f(x_n) + \gamma_{n+1} S z_n}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n S z_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(f(x_{n+1}) - f(x_n)) + (\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}) f(x_n) \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(S z_{n+1} - S z_n) + (\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}) S z_n. \qquad (3.6)
\end{aligned}$$

Combining (3.5) and (3.6), we obtain

$$\begin{aligned}
\|w_{n+1} - w_n\| - \|x_{n+1} - x_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - f(x_n)\| + |\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}| \|f(x_n)\| \\
&\quad + \frac{\gamma_{n+1}}{1 - \beta_{n+1}} \|z_{n+1} - z_n\| + |\frac{\gamma_{n+1}}{1 - \beta_{n+1}} - \frac{\gamma_n}{1 - \beta_n}| \|S z_n\| - \|x_{n+1} - x_n\|
\end{aligned}$$

$$\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\alpha\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|f(x_n)\|$$
$$+\frac{\gamma_{n+1}}{1-\beta_{n+1}}\|x_{n+1}-x_n\| + \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sz_n\| - \|x_{n+1}-x_n\|$$
$$\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\alpha\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|\|f(x_n)\|$$
$$+|\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}|\|Sz_n\| + (\frac{\gamma_{n+1}}{1-\beta_{n+1}} - 1)\|x_{n+1}-x_n\|$$
$$\le |\frac{\alpha_{n+1}(\alpha-1)}{1-\beta_{n+1}}|\|x_{n+1}-x_n\| + |\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}|(\|f(x_n)\| + \|Sz_n\|)$$

This together with (ii) and (iii) imply that

$$\limsup_{n\to\infty}(\|w_{n+1}-w_n\| - \|x_{n+1}-x_n\|) \le 0.$$

Hence, by Lemma 2.5, we obtain $\|w_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|w_n - x_n\| = 0. \tag{3.7}$$

From (iii), (3.5) and (3.7), we also have $\|z_{n+1}-z_n\| \to 0, \|x_{n+1}-x_n\| \to 0$ and $\|y_{n+1}-y_n\| \to 0$ as $n \to \infty$. Since

$$x_{n+1} - x_n = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sz_n - x_n = \alpha_n(f(x_n) - x_n) + \gamma_n(Sz_n - x_n),$$

it follows by (ii) and (3.7) that

$$\|x_n - Sz_n\| \to 0 \text{ as } n \to \infty. \tag{3.8}$$

From Lemma 2.4, equations (3.1) and (3.3), we get

$$\begin{aligned}
\|x_{n+1}-v\|^2 &\le \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|z_n - v\|^2 \\
&\le \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|y_n - v\|^2 \\
&\le \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 \\
&+ \gamma_n\{\|J_M^\lambda(x_n - \lambda Ax_n) - J_M^\lambda(v - \lambda Av)\|^2\} \\
&\le \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 \\
&+ \gamma_n\{\|(x_n - \lambda Ax_n) - (v - \lambda Av)\|^2\} \\
&\le \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 \\
&+ \gamma_n\{\|x_n - v\|^2 + \lambda(\lambda - 2\alpha)\|Ax_n - Av\|^2\} \\
&= \alpha_n\|f(x_n)-v\|^2 + \beta_n\|x_n - v\|^2 \\
&+ \gamma_n\|x_n - v\|^2 + \gamma_n\lambda(\lambda - 2\alpha)\|Ax_n - Av\|^2 \\
&\le \alpha_n\|f(x_n)-v\|^2 + (1-\alpha_n)\|x_n - v\|^2 + \gamma_n a(b - 2\alpha)\|Ax_n - Av\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&-\gamma_n a(b-2\alpha)\|Ax_n - Av\|^2 \le \alpha_n\|f(x_n)-v\|^2 + \|x_n - v\|^2 - \|x_{n+1}-v\|^2 \\
&= \alpha_n\|f(x_n)-v\|^2 + (\|x_n - v\| + \|x_{n+1}-v\|) \times (\|x_n - v\| - \|x_{n+1}-v\|) \\
&\le \alpha_n\|f(x_n)-v\|^2 + (\|x_n - v\| + \|x_{n+1}-v\|) \times \|x_n - x_{n+1}\|.
\end{aligned} \tag{3.9}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, from (3.9), we obtain $\|Ax_n - Av\| \to 0$ as $n \to \infty$. Since $J_M^\lambda$ is 1-inverse-strongly monotone mapping, we have

$$\begin{aligned}
\|y_n - v\|^2 &= \|J_M^\lambda(x_n - \lambda Ax_n) - J_M^\lambda(v - \lambda Av)\|^2 \\
&\leq \langle (x_n - \lambda Ax_n) - (v - \lambda Av), y_n - v\rangle \\
&= \frac{1}{2}\{\|(x_n - \lambda Ax_n) - (v - \lambda Av)\|^2 + \|y_n - v\|^2 \\
&- \|(x_n - \lambda Ax_n) - (v - \lambda Av) - (y_n - v)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - v\|^2 + \|y_n - v\|^2 - \|(x_n - y_n) - \lambda(Ax_n - Av)\|^2\} \\
&= \frac{1}{2}\{\|x_n - v\|^2 + \|y_n - v\|^2 - \|x_n - y_n\|^2 \\
&+ 2\lambda\langle x_n - y_n, Ax_n - Av\rangle - \lambda^2\|Ax_n - Av\|^2\}
\end{aligned}$$

so, we obtain

$$\|y_n - v\|^2 \leq \|x_n - v\|^2 - \|x_n - y_n\|^2 + 2\lambda\langle x_n - y_n, Ax_n - Av\rangle - \lambda^2\|Ax_n - Av\|^2.$$

Hence

$$\begin{aligned}
\|x_{n+1} - v\|^2 &\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|z_n - v\|^2 \\
&\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\|y_n - v\|^2 \\
&\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\{\|x_n - v\|^2 - \|x_n - y_n\|^2 \\
&+ 2\lambda\langle x_n - y_n, Ax_n - Av\rangle - \lambda^2\|Ax_n - Av\|^2\} \\
&\leq \alpha_n\|f(x_n) - v\|^2 + \beta_n\|x_n - v\|^2 + \gamma_n\{\|x_n - v\|^2 - \|x_n - y_n\|^2 \\
&+ 2\lambda\langle x_n - y_n, Ax_n - Av\rangle - \lambda^2\|Ax_n - Av\|^2\} \\
&\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \gamma_n\|x_n - y_n\|^2 \\
&+ 2\gamma_n\lambda\|x_n - y_n\|\|Ax_n - Av\|
\end{aligned}$$

which implies that

$$\begin{aligned}
\gamma_n\|x_n - y_n\|^2 &\leq \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&+ 2\gamma_n\lambda\|x_n - y_n\|\|Ax_n - Av\| \\
&\leq \alpha_n\|f(x_n) - v\|^2 + 2\gamma_n\lambda\|x_n - y_n\|\|Ax_n - Av\| \\
&+ \|x_n - x_{n+1}\| \times (\|x_n - v\| + \|x_{n+1} - v\|). \qquad (3.10)
\end{aligned}$$

From $\alpha_n \to 0$, $\|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Av\| \to 0$ as $n \to \infty$, we have $\|x_n - y_n\| \to 0$ as $n \to \infty$. Since

$$\begin{aligned}
\|Sz_n - z_n\| &\leq \|Sz_n - x_n\| + \|x_n - y_n\| + \|y_n - z_n\| \\
&= \|Sz_n - x_n\| + \|x_n - y_n\| + \|J_M^\lambda(x_n - \lambda Ax_n) - J_M^\lambda(y_n - \lambda Ay_n)\| \\
&\leq \|Sz_n - x_n\| + \|x_n - y_n\| + \|(x_n - \lambda Ax_n) - (y_n - \lambda Ay_n)\| \\
&\leq \|Sz_n - x_n\| + 2\|x_n - y_n\|,
\end{aligned}$$

we get $\|Sz_n - z_n\| \to 0$ as $n \to \infty$. Hence

$$\|z_n - y_n\| \le \|z_n - Sz_n\| + \|Sz_n - x_n\| + \|x_n - y_n\| \to 0, n \to \infty.$$

It is clear that $P_{F(S)\cap VI(H,A,M)}f$ is contractive, then $P_{F(S)\cap VI(H,A,M)}f$ has a unique fixed point, say $z_0 \in H$. That is $z_0 = P_{F(S)\cap VI(H,A,M)}fz_0$.
Next, we show that

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle \le 0.$$

To this end, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle f(z_0) - z_0, Sz_n - z_0\rangle = \lim_{i\to\infty}\langle f(z_0) - z_0, Sz_{n_i} - z_0\rangle.$$

Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to $z$. Without loss of generality, we can assume that $z_{n_i} \rightharpoonup z$. From $\|Sz_n - z_n\| \to 0$, we obtain $Sz_{n_i} \rightharpoonup z$. By the Opial's condition, we obtain $z \in F(S)$.

Finally, by the same argument as that in the proof of [32, Theorem 2.1, p. 578-579], we can show that $z \in VI(H, A, M)$. Hence $z \in F(S) \cap VI(H, A, M)$.

Now from (3.8), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle f(z_0) - z_0, x_n - z_0\rangle &= \limsup_{n\to\infty}\langle f(z_0) - z_0, Sz_n - z_0\rangle \\ &= \lim_{i\to\infty}\langle f(z_0) - z_0, Sz_{n_i} - z_0\rangle \\ &= \langle f(z_0) - z_0, z - z_0\rangle \le 0. \end{aligned} \tag{3.11}$$

Therefore,

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \langle \alpha_n f(x_n) + \beta_n x_n + \gamma_n Sz_n - z_0, x_{n+1} - z_0\rangle \\ &= \alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle + \beta_n\langle x_n - z_0, x_{n+1} - z_0\rangle \\ &+ \gamma_n\langle Sz_n - z_0, x_{n+1} - z_0\rangle \\ &\le \alpha_n\langle f(x_n) - z_0, x_{n+1} - z_0\rangle + \frac{1}{2}\beta_n(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &+ \frac{1}{2}\gamma_n(\|z_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &= \frac{1}{2}(\beta_n + \gamma_n)(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) \\ &+ \alpha_n\langle f(x_n) - f(z_0), x_{n+1} - z_0\rangle + \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &= \frac{1}{2}(1 - \alpha_n)(\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2) + \alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\ &+ \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle \\ &\le \frac{1}{2}\{(1 - \alpha_n)\|x_n - z_0\|^2 + \|x_{n+1} - z_0\|^2\} + \alpha_n\alpha\|x_n - z_0\|\|x_{n+1} - z_0\| \\ &+ \alpha_n\langle f(z_0) - z_0, x_{n+1} - z_0\rangle\end{aligned}$$

which implies that

$$\|x_{n+1}-z_0\|^2\le(1-\alpha_n)\|x_n-z_0\|^2+2\alpha_n\alpha\|x_n-z_0\|\|x_{n+1}-z_0\|+2\alpha_n\langle f(z_0)-z_0, x_{n+1}-z_0\rangle.$$

Finally by (3.11) and Lemma 2.6, we get that $\{x_n\}$ converges to $z_0$. This completes the proof. □

Using the same argument as in the proof in Theorem 3.1, we can obtain the following theorems in Hilbert Spaces.

**Theorem 3.2.** *Let $H$ be a real Hilbert space and let $A$ be an $\alpha-$inverse-strongly monotone mapping of $C$ into $H$, $M : H \to 2^H$ be a maximal monotone mapping and $S$ be a nonexpansive mapping of $H$ into itself. Suppose that $F(S) \cap VI(H, A, M) \neq \emptyset$. Let $f$ be a contraction of $H$ into itself and given $x_0 \in H$ arbitrarily and $\{x_n\}$ is the sequences defined by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S J_M^{\lambda}(x_n - \lambda A x_n), \tag{3.12}$$

*for all $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$. If $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

*then $\{x_n\}$ converges strongly to $z_0$ which is the unique solution in $F(S) \cap VI(H, A, M)$ to the following variational inequality:*

$$\langle (f - I)z_0, z_0 - z\rangle \leq 0 \text{ for all } z \in F(S) \cap VI(H, A, M). \tag{3.13}$$

**Corollary 3.3.** *Let $H$ be a real Hilbert space and let $A$ be an $\alpha-$inverse-strongly monotone mapping of $C$ into $H$, $M : H \to 2^H$ be a maximal monotone mapping and $S$ be a nonexpansive mapping of $H$ into itself. Suppose that $F(S) \cap VI(H, A, M) \neq \emptyset$. Given $x_0 = u \in H$ arbitrarily and $\{x_n\}$ is the sequences defined by*

$$x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n S J_M^{\lambda}(x_n - \lambda A x_n) \tag{3.14}$$

*for all $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0, 1]$ and $\lambda \in (0, 2\alpha]$ . If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\lambda\}$ are chosen so that $\lambda \in (a, b]$ for some $a, b$ with $0 < a < \lambda \leq b < 2\alpha$ and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

*then $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(H,A,M)} x_0$.*

*Proof.* Put $f(x_n) = x_0$, for all $n \in \mathbf{N}$ in Theorem 3.2, we obtain the desired easily. □

## 4. Applications

Using Theorem 3.1, we obtain the following results.

**Theorem 4.1.** *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $A$ be an $\alpha-$inverse-strongly monotone mapping of $C$ into $H$ and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = u \in C$ and $\{x_n\}, \{y_n\}$ are given by*

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S P_C(y_n - \lambda A y_n), \end{cases}$$

*where* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are three sequences in* $[0,1]$ *and* $\lambda \in (0,2\alpha]$ *. If* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\lambda_n\}$ *are chosen so that* $\lambda_n \in [a,b]$ *for some* $a,b$ *with* $0 < a < b < 2\alpha$ *and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

*then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)} f(z_0)$.

*Proof.* Take $M = \partial\delta_C : H \to 2^H$, where $\delta_C : H \to [0,\infty]$ is the indicator function of $C$, i.e.,

$$\delta_C(x) = \begin{cases} 0, & x \in C; \\ +\infty, & x \notin C. \end{cases}$$

Then the variational inclusion problem (1.1) is equivalent to variational inequality problem (1.4) (see [32]). Putting $J^{\lambda}_{M_{|C}} = I$, and we get

$$y_n = P_C(x_n - \lambda A x_n) = J^{\lambda}_M(P_C(x_n - \lambda A x_n))$$

and

$$z_n = P_C(y_n - \lambda A y_n) = J^{\lambda}_M(P_C(y_n - \lambda A y_n)).$$

The conclusion of Theorem 4.1 can be obtained from Theorem 3.1 immediately. $\square$

Next, we will apply the main results to the problem for finding a common element of the set of fixed points of a nonexpansive mappings and the set of fixed points of a $k$-strictly pseudocontractive mapping.

**Definition 4.2.** A mapping $T : C \to H$ is said to be a $k$-strictly pseudocontractive mapping, if there exists $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \quad \leq \quad \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \ \forall x,y \in C. \tag{4.1}$$

**Remark 4.3.** If $T : C \to H$ is a $k$-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-k}{2}$-inverse-strongly monotone. Indeed, Let $T : C \to H$ be a $k$-strictly pseudocontractive mapping for some $0 \leq k < 1$. Set the mapping $A = I - T : C \to H$. From

$$\|Tx - Ty\|^2 \quad \leq \quad \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \ \forall x,y \in C,$$

we have

$$\|(I-A)x - (I-A)y\|^2 \leq \|x-y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand, we observe that

$$\|(I-A)x - (I-A)y\|^2 = \|x-y\|^2 - 2\langle x-y, Ax-Ay\rangle + \|Ax-Ay\|^2.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \quad \geq \quad \frac{1-k}{2}\|Ax-Ay\|^2.$$

This shows that $A$ is $\frac{1-k}{2}$-inverse-strongly monotone.

**Theorem 4.4.** *Let $C$ be a closed convex subset of a real Hilbert space $H$. Let $S$ be a nonexpansive mapping of $C$ into itself and let $T$ be a strictly pseudocontractive mapping with constant $k$ of $C$ into itself such that $F(S) \cap F(T) \neq \emptyset$. For $x_1 = u \in C$, let the sequence $\{x_n\}$ be given by*

$$\begin{cases} y_n = (1-\lambda)x_n + \lambda T x_n \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S((1-\lambda)y_n + \lambda T y_n), \end{cases}$$

*for all $n \in \mathbf{N} \cup \{0\}$, where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are three sequences in $[0,1]$ and $\lambda \in [0, 1-k]$. If $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\lambda$ are chosen so that $\lambda \in [a,b]$ for some $a, b$ with $0 < a < b < 1-k$ and*

(i) $\alpha_n + \beta_n + \gamma_n = 1$,
(ii) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

*then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)}u$.*

*Proof.* Setting $A = I - T$, we obtain $A$ is $\frac{1-k}{2}-$inverse-strongly monotone. We observe that $F(T)$ is the solution set of $VI(A,C)$ i.e., $F(T) = VI(A,C)$. Moreover, since $P_C(x_n - \lambda A x_n) = (1-\lambda)x_n + \lambda T x_n$ and $P_C(y_n - \lambda A y_n) = (1-\lambda)y_n + \lambda T y_n$, by Theorem 4.1, the conclusion follows. $\square$

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