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# FIXED POINT THEORY IN GENERALIZED APPROXIMATE NEIGHBORHOOD EXTENSION SPACES

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**Abstract.** Several new fixed point results for compact self maps in new classes of spaces are presented in this paper.
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## 1. Introduction

In Sections 2 we present new results on fixed point theory in extension type spaces. In particular we discuss compact self-maps on $GNES$ (generalized neighborhood extension spaces) and $GANES$ (generalized approximate neighborhood extension spaces) spaces. These spaces are generalization of spaces considered in [8, 9, 15, 16]. Our results were motivated in part from ideas in [1, 2, 9, 11, 12, 15, 16].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose $X$ and $Y$ are topological spaces. Given a class $\mathcal{X}$ of maps, $\mathcal{X}(X,Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of $Y$) belonging to $\mathcal{X}$, and $\mathcal{X}_c$ the set of finite compositions of maps in $\mathcal{X}$. We let

$$\mathcal{F}(\mathcal{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathcal{X}(Z,Z)\}$$

where $Fix\, F$ denotes the set of fixed points of $F$.

The class $\mathcal{A}$ of maps is defined by the following properties:
(i). $\mathcal{A}$ contains the class $\mathcal{C}$ of single valued continuous functions;
(ii). each $F \in \mathcal{A}_c$ is upper semicontinuous and closed valued; and
(iii). $B^n \in \mathcal{F}(\mathcal{A}_c)$ for all $n \in \{1, 2, ....\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

**Remark 1.1.** The class $\mathcal{A}$ is essentially due to Ben-El-Mechaiekh and Deguire [6]. $\mathcal{A}$ includes the class of maps $\mathcal{U}$ of Park ($\mathcal{U}$ is the class of maps defined by (i), (iii) and (iv). each $F \in \mathcal{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathcal{A}_c$ is compact valued the class $\mathcal{A}$ and $\mathcal{U}$ coincide and this is what occurs in Section 2 since our maps will be compact.

We next consider the class $\mathcal{U}_c^{\kappa}(X,Y)$ (respectively $\mathcal{A}_c^{\kappa}(X,Y)$) of maps $F : X \to 2^Y$ such that for each $F$ and each nonempty compact subset $K$ of $X$ there exists a map $G \in \mathcal{U}_c(K,Y)$ (respectively $G \in \mathcal{A}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

Notice [14] that $\mathcal{U}_c^\kappa$ is closed under compositions. The class $\mathcal{U}_c^\kappa$ include (see [3]) the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible with respect to Gorniewicz.

For a subset $K$ of a topological space $X$, we denote by $Cov_X\,(K)$ the set of all coverings of $K$ by open sets of $X$ (usually we write $Cov\,(K) = Cov_X\,(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov\,(X)$, a point $x \in X$ is said to be an $\alpha$-fixed point of $F$ if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, $f$ and $g$ are said to be $\alpha$-close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say $f$ and $g$ are $\alpha$-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \leq t \leq 1)$ joining $f$ and $g$ such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$.

The following results can be found in [4, Lemma 1.2 and 4.7].

**Theorem 1.1.** *Let $X$ be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X\,(\overline{F(X)})$ such that $F$ has an $\alpha$-fixed point for every $\alpha \in \theta$. Then $F$ has a fixed point.*

**Remark 1.2.** From Theorem 1.1 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [5 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set $A$ admit refinements of the form $\{U[x] : x \in A\}$ where $U$ is a member of the uniformity [13 pp. 199] so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular) [7 pp. 431] (see also [7 pp. 434]). Note in Theorem 1.1 if $F$ is compact valued then the assumption that $X$ is regular can be removed. For convenience in this paper we will apply Theorem 1.1 only when the space is uniform.

Let $X$, $Y$ and $\Gamma$ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic

(ii). $p$ is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where $p$ is a Vietoris map and $q$ is continuous. We will denote every such diagram by $(p,q)$. Given two diagrams $(p,q)$ and $(p',q')$, where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from $X$ to $Y$. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}\,(x)$ where $\phi = [(p,q)]$ is called an image of $x$ under a morphism $\phi$.

Consider vector spaces over a field $K$. Let $E$ be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the $n^{th}$ iterate of $f$, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call $f$ admissible if $dim\,\tilde{E} < \infty$; for such $f$ we define the generalized trace $Tr(f)$ of $f$ by putting $Tr(f) = tr(\tilde{f})$ where $tr$ stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call $f$ a Leray endomorphism if (i). all $f_q$ are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such $f$ we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

Let $H$ be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $K$ from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the $q$-dimensional Čech homology group with compact carriers of $X$. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [10 pp. 364]) we have the following well known result (note the homology functor $H$ extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H\,(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [8 pp. 227].

**Theorem 1.2.** *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here $X$, $Y$ and $Z$ are Hausdorff topological spaces) then*

$$(\psi \,\circ\, \phi)_\star = \psi_\star \,\circ\, \phi_\star.$$

Two morphisms $\phi,\, \psi \in M(X, Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x, 0) = \phi(x)$, $\chi(x, 1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$). Recall the following result [9, pp. 231]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of $Y$). A pair $(p, q)$ of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of $\phi$ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i). $p$ is a Vietoris map

and

(ii). $q\,(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

**Definition 1.1.** An upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [9, 10] (and we write $\phi \in Ads(X,Y)$) provided there exists a selected pair $(p,q)$ of $\phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$.

**Definition 1.2.** A map $\phi \in Ads(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ with $\phi(x) = q\,(p^{-1}(x))$ for $x \in X$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

When we talk about $\phi \in Ads$ it is assumed that we are also considering a specified selected pair $(p,q)$ of $\phi$ with $\phi(x) = q\,(p^{-1}(x))$.

**Remark 1.3.** In fact since we specify the pair $(p,q)$ of $\phi$ it is enough to say $\phi$ is a Lefschetz map if $\phi_\star = q_\star\, p_\star^{-1} : H(X) \to H(X)$ is a Leray endomorphism. However for the examples of $\phi$, $X$ known in the literature [9] the more restrictive condition in Definition 1.2 works. We note [9, pp 227] that $\phi_\star$ does not depend on the choice of diagram from $[(p,q)]$, so in fact we could specify the morphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [9, 10]) $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \Lambda(q_\star\, p_\star^{-1}).$$

If we do not wish to specify the selected pair $(p,q)$ of $\phi$ then we would consider the Lefschetz set $\mathbf{\Lambda}\,(\phi) = \{\Lambda(q_\star\, p_\star^{-1}) : \phi = q\,(p^{-1})\}$.

**Definition 1.3.** A Hausdorff topological space $X$ is said to be a Lefschetz space (for the class $Ads$) provided every compact $\phi \in Ads(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies $\phi$ has a fixed point.

**Definition 1.4.** An upper semicontinuous map $\phi : X \to Y$ with closed values is said to be admissible (and we write $\phi \in Ad(X,Y)$) provided there exists a selected pair $(p,q)$ of $\phi$.

**Definition 1.5.** A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star\, p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}\,(\phi)$ (or $\mathbf{\Lambda}_X\,(\phi)$) by

$$\mathbf{\Lambda}\,(\phi) = \left\{\Lambda(q_\star\, p_\star^{-1}) : \ (p,q) \subset \phi\right\}.$$

**Definition 1.6.** A Hausdorff topological space $X$ is said to be a Lefschetz space (for the class $Ad$) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies $\phi$ has a fixed point.

**Remark 1.4.** Many examples of Lefschetz spaces (for the class $Ad$ or $Ads$) can be found in [1, 2, 8, 9, 10, 11, 12, 15, 16].

**Definition 1.7.** A multivalued map $F : X \to K(Y)$ ($K(Y)$ denotes the class of nonempty compact subsets of $Y$) is in the class $\mathcal{A}_m(X,Y)$ if (i). $F$ is continuous, and (ii). for each $x \in X$ the set $F(x)$ consists of one or $m$ acyclic components; here $m$ is a positive integer. We say $F$ is of class $\mathcal{A}_0(X,Y)$ if $F$ is upper semicontinuous and for each $x \in X$ the set $F(x)$ is acyclic.

**Definition 1.8.** A decomposition $(F_1, ..., F_n)$ of a multivalued map $F : X \to 2^Y$ is a sequence of maps

$$X = X_0 \overset{F_1}{\to} X_1 \overset{F_2}{\to} X_2 \overset{F_3}{\to} \quad ..... \quad \overset{F_{n-1}}{\to} X_{n-1} \overset{F_n}{\to} X_n = Y,$$

where $F_i \in \mathcal{A}_{m_i}(X_{i-1}, X_i)$, $F = F_n \circ .... \circ F_1$. One can say that the map $F$ is determined by the decomposition $(F_1, ..., F_n)$. The number $n$ is said to be the length of the decomposition $(F_1, ..., F_n)$. We will denote the class of decompositions by $\mathcal{D}(X, Y)$.

**Definition 1.9.** An upper semicontinuous map $F : X \to K(Y)$ is permissible provided it admits a selector $G : X \to K(Y)$ which is determined by a decomposition $(G_1, ..., G_n) \in \mathcal{D}(X, Y)$. We denote the class of permissible maps from $X$ into $Y$ by $\mathcal{P}(X, Y)$.

Let $X$ be a Hausdorff topological space and let a map $\Phi$ be determined by $(\Phi_1, ...., \Phi_k) \in \mathcal{D}(X, X)$. Then $\Phi$ is said to be a Lefschetz map if the induced homology homomorphism [9, pp 262, 263] $(\Phi_1, ..., \Phi_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [9]) $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\mathbf{\Lambda}(\Phi) = \Lambda((\Phi_1, ..., \Phi_k)_\star).$$

A Hausdorff topological space $X$ is said to be a Lefschetz space (for the class $\mathcal{D}$) provided every compact $\Phi : X \to K(X)$ determined by a decomposition $(\Phi_1, ...., \Phi_k) \in \mathcal{D}(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq 0$ implies $\Phi$ has a fixed point.

A map $\Phi \in \mathcal{P}(X, X)$ is said to be a Lefschetz map provided every selector $G : X \to K(X)$ of $\Phi$ which is determined by $(G_1, ..., G_k) \in \mathcal{D}(X, X)$ is such that $(G_1, ..., G_k)_\star : H(X) \to H(X)$ is a Leray endomorphism.

If $\Phi \in \mathcal{P}(X, X)$ is a Lefschetz map as described above then we define the Lefschetz set $\mathbf{\Lambda}(\Phi)$ (or $\mathbf{\Lambda}_X(\Phi)$) by

$$\begin{aligned}\mathbf{\Lambda}(\Phi) \quad = \quad & \{\Lambda((G_1, ..., G_k)_\star) : \ (G_1, ..., G_k) \in \mathcal{D}(X, X) \\ & \text{and } \ (G_1, ...., G_k) \ \text{ determines a selection of } \ \Phi\}.\end{aligned}$$

A Hausdorff topological space $X$ is said to be a Lefschetz space (for the class $\mathcal{P}$) provided every compact $\Phi \in \mathcal{P}(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies $\Phi$ has a fixed point.

## 2. Fixed point theory

By a space we mean a Hausdorff topological space. Let $X$ be a space and $F \in \mathcal{U}_c^\kappa(X, X)$. We say $X \in GNES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) if there exists a space $U$, a single valued continuous map $r : U \to X$ and a compact valued upper semicontinuous map $\Phi \in \mathcal{U}_c^\kappa(K, U)$ with $r\,\Phi = id_K$ (here $K = \overline{F(X)}$).

**Remark 2.1.** Examples of $GNES$ spaces can be found in [16, Section 4]. The space $U$ will be discussed below for particular classes of $\mathcal{U}_c^\kappa$ maps.

Now assume $X \in GNES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) and $F \in \mathcal{U}_c^\kappa(X, X)$ a compact map. Let $U$, $r$ and $\Phi$ be as described above and let $G = \Phi\, F\, r$. Notice $G \in \mathcal{U}_c^\kappa(U, U)$

is a compact map (note the image of a compact set under $\Phi$ is compact). We now assume

$$G \in \mathcal{U}_c^\kappa(U,U) \text{ has a fixed point.} \tag{2.1}$$

Then there exists $x \in U$ with $x \in G\,x$. Let $y = r(x)$, so $y \in r\,\Phi\,F\,(y)$ i.e. $y \in r\,\Phi\,(q)$ for some $q \in F\,(y)$. Note $q \in K = \overline{F(X)}$. Now since $r\,\Phi = id_K$ we have $y \in F\,(y)$.

**Theorem 2.1.** *Let $X \in GNES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact map. Also assume* (2.1) *holds with $U$, $r$ and $\Phi$ as described above. Then $F$ has a fixed point.*

We discuss Theorem 2.1 for the class $Ad(X,X)$. Let $X$ be a space and $F \in Ad(X,X)$. We say $X \in GNES$ (w.r.t. $Ad$ and $F$) if there exists a Lefschetz space (for the class $Ad$) $U$, a single valued continuous map $r : U \to X$ and a compact valued map $\Phi \in Ad(K,U)$ with $r\,\Phi = id_K$ (here $K = \overline{F(X)}$).

Now assume $X \in GNES$ (w.r.t. $Ad$ and $F$) and $F \in Ad(X,X)$ a compact map. Let $U$, $r$ and $\Phi$ be as described above and let $G = \Phi\,F\,r$. Notice $G \in Ad(U,U)$ is a compact map. Let $(p,q)$ be a selected pair for $F$ and $(p_1,q_1)$ be a selected pair of $\Phi$. Now since $F\,r \in Ad(U,X)$ then [9, Section 40] guarantees that there exists a selected pair $(p',q')$ of $F\,r$ with

$$(q')_\star\,(p')_\star^{-1} = q_\star\,p_\star^{-1}\,r_\star. \tag{2.2}$$

Also there exists [9, Section 40] a selected pair $(\overline{p},\overline{q})$ of $G$ with

$$(\overline{q})_\star\,(\overline{p})_\star^{-1} = (q_1)_\star\,(p_1)_\star^{-1}\,(q')_\star\,(p')_\star^{-1} \tag{2.3}$$

so (2.2) and (2.3) imply

$$(\overline{q})_\star\,(\overline{p})_\star^{-1} = (q_1)_\star\,(p_1)_\star^{-1}\,q_\star\,p_\star^{-1}\,r_\star. \tag{2.4}$$

Notice as well that

$$q_\star\,p_\star^{-1}\,r_\star\,(q_1)_\star\,(p_1)_\star^{-1} = q_\star\,p_\star^{-1} \tag{2.5}$$

since $r\,\Phi = id_K$ (here $K = \overline{F(X)}$). Now $U$ is a Lefschetz space (for the class $Ad$) so $(\overline{q})_\star\,(\overline{p})_\star^{-1}$ is a Leray endomorphism. Now [8, page 214, see (1.3)] (here $E' = U'$, $E'' = X'$, $u = (q')_\star\,(p')_\star^{-1}$, $v = (q_1)_\star\,(p_1)_\star^{-1}$, $f' = (\overline{q})_\star\,(\overline{p})_\star^{-1}$ and $f'' = q_\star\,p_\star^{-1}$ and note (2.2), (2.4) and (2.5)) guarantees that $q_\star\,p_\star^{-1}$ is a Leray endomorphism and $\Lambda\,(q_\star\,p_\star^{-1}) = \Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1})$. Thus $\mathbf{\Lambda}\,(F)$ is well defined.

Next suppose $\mathbf{\Lambda}\,(F) \neq \{0\}$. Then there exists a selected pair $(p,q)$ as described above with $\Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Let $\overline{p}$ and $\overline{q}$ be as described above with $\Lambda\,((\overline{q})_\star\,(\overline{p})_\star^{-1}) = \Lambda\,(q_\star\,p_\star^{-1}) \neq 0$. Now since $U$ is a Lefschetz space (for the class $Ad$) there exists $x \in U$ with $x \in \overline{q}\,(\overline{p})^{-1}(x)$ i.e. $x \in G(x)$ so (2.1) is satisfied. Combining with Theorem 2.1 we have the following result.

**Theorem 2.2.** *Let $X \in GNES$ (w.r.t. $Ad$ and $F$) and $F \in Ad(X,X)$ a compact map. Then $\mathbf{\Lambda}\,(F)$ is well defined and if $\mathbf{\Lambda}\,(F) \neq \{0\}$ then $F$ has a fixed point.*

**Remark 2.2.** One could also discuss $Ads$ maps. Let $X$ be a space and $F \in Ads(X,X)$. We say $X \in GNES$ (w.r.t. $Ads$ and $F$) if there exists a Lefschetz space (for the class $Ads$) $U$, a single valued continuous map $r : U \to X$ and a map $\Phi \in Ads(K,U)$ with $r\,\Phi = id_K$ (here $K = \overline{F(X)}$). Essentially the same reasoning as in Theorem 2.2 establishes: Let $X \in GNES$ (w.r.t. $Ads$ and $F$) and

$F \in Ads(X,X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then $F$ has a fixed point.

**Remark 2.3.** One could also obtain a result for the class $\mathcal{P}$ (of course the results here can trivially be adjusted for the class $\mathcal{D}$). Let $X$ be a space and $F \in \mathcal{P}(X,X)$. We say $X \in GNES$ (w.r.t. $\mathcal{P}$ and $F$) if there exists a Lefschetz space (for the class $\mathcal{P}$) $U$, a single valued continuous map $r : U \to X$ and a map $\Phi \in \mathcal{P}(K,U)$ with $r\,\Phi = id_K$ (here $K = \overline{F(X)}$). Let $X \in GNES$ (w.r.t. $\mathcal{P}$ and $F$) and $F \in \mathcal{P}(X,X)$ a compact map. Let $U$, $r$ and $\Phi$ be as described above and let $G = \Phi\,F\,r$. Notice $G \in \mathcal{P}(U,U)$ is a compact map. We now assume the following condition:

$$\begin{cases} \text{for every selector } H : X \to K(X) \text{ of } F \text{ which is} \\ \text{determined by } (H_1,...,H_n) \in \mathcal{D}(X,X) \text{ and every selector} \\ \Psi : \overline{F(X)} \to K(U) \text{ of } \Phi \text{ which is determined by} \\ (\Psi_1,...,\Psi_m) \in \mathcal{D}(\overline{F(X)},U) \text{ there exists a} \\ (S_1,...,S_k) \in \mathcal{D}(U,X) \text{ which determines } F\,r \text{ with} \\ (S_1,...,S_k)_\star = (H_1,...,H_n)_\star\, r_\star \text{ and there exists a} \\ (G_1,...,G_l) \in \mathcal{D}(U,U) \text{ which determines a selection } G \text{ with} \\ (H_1,...,H_n)_\star\, r_\star\,(\Psi_1,...,\Psi_m)_\star = (H_1,...,H_n)_\star \text{ and} \\ (\Psi_1,...,\Psi_m)_\star\,(S_1,...,S_k)_\star = (G_1,...,G_l)_\star\,. \end{cases} \tag{2.6}$$

Fix a selector $H$ of $F$ and a selector $\Psi$ of $\Phi$ and let $(H_1,...,H_n)$, $(\Psi_1,...,\Psi_m)$ and $(S_1,...,S_k)$ be as described in (2.6). Notice

$$(\Psi_1,...,\Psi_m)_\star\,(S_1,...,S_k)_\star = (G_1,...,G_l)_\star$$

and

$$(S_1,...,S_k)_\star\,(\Psi_1,...,\Psi_m)_\star = (H_1,...,H_n)_\star\,r_\star\,(\Psi_1,...,\Psi_m)_\star = (H_1,...,H_n)_\star.$$

Now $U$ is a Lefschetz space (for the class $\mathcal{P}$) so $(G_1,...,G_l)_\star$ is a Leray endomorphism. Now [8, page 214, see (1.3)] (here $E' = U'$, $E'' = X'$, $u = (S_1,...,S_k)_\star$, $v = (\Psi_1,...,\Psi_m)_\star$, $f' = (G_1,...,G_l)_\star$ and $f'' = (H_1,...,H_n)_\star$) guarantees that $(H_1,...,H_n)_\star$ is a Leray endomorphism and $\Lambda((H_1,...,H_n)_\star) = \Lambda((G_1,...,G_l)_\star)$. Thus $\mathbf{\Lambda}(F)$ is well defined. Also as in Theorem 2.2 it is easy to see that if $\mathbf{\Lambda}(F) \neq \{0\}$ then $F$ has a fixed point.

Let $X$ be a space and $F \in \mathcal{U}_c^\kappa(X,X)$. We say $X \in GANES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) if for each $\alpha \in Cov_X(K)$ (here $K = \overline{F(X)}$) there exists a space $U_\alpha$, a single valued continuous map $r_\alpha : U_\alpha \to X$ and a compact valued upper semicontinuous map $\Phi_\alpha \in \mathcal{U}_c^\kappa(K,U_\alpha)$ such that $r_\alpha\,\Phi_\alpha : K \to 2^X$ and $i : K \to X$ are strongly $\alpha$-close (by this we mean for each $x \in K$ there exists $V_x \in \alpha$ with $r_\alpha\,\Phi_\alpha(x) \subseteq V_x$ and $x = i(x) \in V_x$).

Now assume $X \in GANES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) is a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ is a compact upper semicontinuous map with closed values. Let $\alpha \in Cov_X(\overline{F(X)})$ and let $U_\alpha$, $r_\alpha$ and $\Phi_\alpha$ be as described above. Now let $G_\alpha = \Phi_\alpha\,F\,r_\alpha$. Notice $G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha,U_\alpha)$ is a compact upper semicontinuous map with closed values. We now assume

$$G_\alpha \in \mathcal{U}_c^\kappa(U_\alpha,U_\alpha) \text{ has a fixed point (for each } \alpha \in Cov_X(\overline{F(X)})). \tag{2.7}$$

Then there exists $x \in U_\alpha$ with $x \in G_\alpha(x)$. Let $y = r_\alpha(x)$, so $y \in r_\alpha \Phi_\alpha F(y)$ i.e. $y \in r_\alpha \Phi_\alpha(q)$ for some $q \in F(y)$. Note $q \in K = \overline{F(X)}$. Now since $r_\alpha \Phi_\alpha : K \to 2^X$ and $i : K \to X$ are strongly $\alpha$-close there exists $V \in \alpha$ with

$$r_\alpha \Phi_\alpha(q) \subseteq V \quad \text{and} \quad q \in V.$$

Thus $y \in V$ since $y \in r_\alpha \Phi_\alpha(q)$, $q \in F(y)$ and $q \in V$, so

$$y \in V \quad \text{and} \quad F(y) \cap V \neq \emptyset.$$

As a result $F$ has an $\alpha$-fixed point so Theorem 1.1 guarantees that $F$ has a fixed point.

**Theorem 2.3.** *Let $X \in GANES$ (w.r.t. $\mathcal{U}_c^\kappa$ and $F$) be a uniform space and $F \in \mathcal{U}_c^\kappa(X,X)$ a compact upper semicontinuous map with closed values. Also assume (2.7) holds with $U_\alpha$, $r_\alpha$ and $\Phi_\alpha$ as described above. Then $F$ has a fixed point.*

We discuss Theorem 2.3 for the class $Ad(X,X)$. Let $X$ be a space and $F \in Ad(X,X)$. We say $X \in GANES$ (w.r.t. $Ad$ and $F$) if for each $\alpha \in Cov_X(K)$ (here $K = \overline{F(X)}$) there exists a Lefschetz space (for the class $Ad$) $U_\alpha$, a single valued continuous map $r_\alpha : U_\alpha \to X$ and a compact valued map $\Phi_\alpha \in Ad(K, U_\alpha)$ such that $r_\alpha \Phi_\alpha : K \to X$ and $i : K \to X$ are $\alpha$-close (by this we mean for each $x \in K$ there exists $V_x \in \alpha$ with $r_\alpha \Phi_\alpha(x) \in V_x$ and $x = i(x) \in V_x$) and $\alpha$-homotopic.

Now assume $X \in GANES$ (w.r.t. $Ad$ and $F$) is a uniform space and $F \in Ad(X,X)$ is a compact map. Let $\alpha \in Cov_X(\overline{F(X)})$ and let $U_\alpha$, $r_\alpha$ and $\Phi_\alpha$ be as described above. Let $G_\alpha = \Phi_\alpha F r_\alpha$. Let $(p,q)$ be a selected pair for $F$ and $(p_\alpha^0, q_\alpha^0)$ be a selected pair of $\Phi_\alpha$. Now since $F r_\alpha \in Ad(U_\alpha, X)$ then [9, Section 40] guarantees that there exists a selected pair $(p'_\alpha, q'_\alpha)$ of $F r_\alpha$ with

$$(q'_\alpha)_\star (p'_\alpha)_\star^{-1} = q_\star p_\star^{-1} (r_\alpha)_\star. \tag{2.8}$$

Also there exists [9, Section 40] a selected pair $(\overline{p}_\alpha, \overline{q}_\alpha)$ of $G_\alpha$ with

$$(\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1} = (q_\alpha^0)_\star (p_\alpha^0)_\star^{-1} (q'_\alpha)_\star (p'_\alpha)_\star^{-1} \tag{2.9}$$

so (2.8) and (2.9) imply

$$(\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1} = (q_\alpha^0)_\star (p_\alpha^0)_\star^{-1} q_\star p_\star^{-1} (r_\alpha)_\star. \tag{2.10}$$

Notice as well that

$$q_\star p_\star^{-1} (r_\alpha)_\star (q_\alpha^0)_\star (p_\alpha^0)_\star^{-1} = q_\star p_\star^{-1} \tag{2.11}$$

since $r_\alpha \Phi_\alpha$ and $i$ are $\alpha$-homotopic (so [9 pp. 202] guarantees that $(r_\alpha \Phi_\alpha)_\star = i_\star$ and so for any selected pair $(p_\alpha^1, q_\alpha^1)$ of $\Phi_\alpha$ there exists a selected pair $(p_\alpha^2, q_\alpha^2)$ of $i$ with $i_\star = (q_\alpha^2)_\star (p_\alpha^2)_\star^{-1} = (r_\alpha)_\star (q_\alpha^1)_\star (p_\alpha^1)_\star^{-1}$). Now $U_\alpha$ is a Lefschetz space (for the class $Ad$) so $(\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}$ is a Leray endomorphism. Now [8, page 214, see (1.3)] (here $E' = U'_\alpha$, $E'' = X'$, $u = (q'_\alpha)_\star (p'_\alpha)_\star^{-1}$, $v = (q_\alpha^0)_\star (p_\alpha^0)_\star^{-1}$, $f' = (\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}$ and $f'' = q_\star p_\star^{-1}$ and note (2.8), (2.10) and (2.11)) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\Lambda(q_\star p_\star^{-1}) = \Lambda((\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1})$. Thus $\mathbf{\Lambda}(F)$ is well defined.

Next suppose $\mathbf{\Lambda}(F) \neq \{0\}$. Then there exists a selected pair $(p,q)$ as described above with $\Lambda(q_\star p_\star^{-1}) \neq 0$. Let $\overline{p}_\alpha$ and $\overline{q}_\alpha$ be as described above with $\Lambda((\overline{q}_\alpha)_\star (\overline{p}_\alpha)_\star^{-1}) = \Lambda(q_\star p_\star^{-1}) \neq 0$. Now since $U_\alpha$ is a Lefschetz space (for the class $Ad$) there exists $x \in U_\alpha$ with $x \in \overline{q}_\alpha (\overline{p}_\alpha)^{-1}(x)$ i.e. $x \in G_\alpha(x)$ so (2.7) is satisfied. Note (2.7) here implies $F$ has an $\alpha$-fixed point since $r_\alpha \Phi_\alpha$ and $i$ are automatically strongly $\alpha$-close (since $r_\alpha \Phi_\alpha : K \to X$ and $i : K \to X$ are $\alpha$-close). Thus we have the following result.

**Theorem 2.4.** *Let $X \in GANES$ (w.r.t. $Ad$ and $F$) be a uniform space and $F \in Ad(X, X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then $F$ has a fixed point.*

**Remark 2.4.** In the definition of $GANES$ (w.r.t. $Ad$ and $F$) it is easy to see that one could replace the assumption that $r_\alpha \Phi_\alpha : K \to X$ and $i : K \to X$ are $\alpha$-close and $\alpha$-homotopic with the assumption that $r_\alpha \Phi_\alpha : K \to 2^X$ and $i : K \to X$ are strongly $\alpha$-close and $(r_\alpha)_\star (q_\alpha^1)_\star (p_\alpha^1)_\star^{-1} = i_\star$ for any selected pair $(p_\alpha^1, q_\alpha^1)$ of $\Phi_\alpha$.

**Remark 2.5.** One could also discuss $Ads$ maps. Let $X$ be a space and $F \in Ads(X, X)$. We say $X \in GANES$ (w.r.t. $Ads$ and $F$) if for each $\alpha \in Cov_X(K)$ (here $K = \overline{F(X)}$) there exists a Lefschetz space (for the class $Ads$) $U_\alpha$, a single valued continuous map $r_\alpha : U_\alpha \to X$ and a map $\Phi_\alpha \in Ads(K, U_\alpha)$ such that $r_\alpha \Phi_\alpha : K \to X$ and $i : K \to X$ are $\alpha$-close and $\alpha$-homotopic. Essentially the same reasoning as in Theorem 2.4 establishes: Let $X \in GANES$ (w.r.t. $Ads$ and $F$) be a uniform space and $F \in Ads(X, X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq 0$ then $F$ has a fixed point. There is also an analogue of Remark 2.4 in this situation.

**Remark 2.6.** One could also obtain a result for the class $\mathcal{P}$ (of course the results here can trivially be adjusted for the class $\mathcal{D}$). Let $X$ be a space and $F \in \mathcal{P}(X, X)$. We say $X \in GANES$ (w.r.t. $\mathcal{P}$ and $F$) if for each $\alpha \in Cov_X(K)$ (here $K = \overline{F(X)}$) there exists a Lefschetz space (for the class $\mathcal{P}$) $U_\alpha$, a single valued continuous map $r_\alpha : U_\alpha \to X$ and a map $\Phi_\alpha \in \mathcal{P}(K, U_\alpha)$ such that $r_\alpha \Phi_\alpha : K \to 2^X$ and $i : K \to X$ are strongly $\alpha$-close. Now assume $X \in GANES$ (w.r.t. $\mathcal{P}$ and $F$) is a uniform space and $F \in \mathcal{P}(X, X)$ is a compact map. Let $\alpha \in Cov_X(\overline{F(X)})$ and let $U_\alpha$, $r_\alpha$ and $\Phi_\alpha$ be as described above. Let $G_\alpha = \Phi_\alpha F r_\alpha$. We now assume the following condition (for each $\alpha \in Cov_X(\overline{F(X)})$):

$$\begin{cases} \text{for every selector } H : X \to K(X) \text{ of } F \text{ which is} \\ \text{determined by } (H_1, ..., H_n) \in \mathcal{D}(X, X) \text{ and every selector} \\ \Psi_\alpha : \overline{F(X)} \to K(U_\alpha) \text{ of } \Phi_\alpha \text{ which is determined by} \\ (\Psi_{1,\alpha}, ..., \Psi_{m,\alpha}) \in \mathcal{D}(\overline{F(X)}, U_\alpha) \text{ there exists a} \\ (S_{1,\alpha}, ..., S_{k,\alpha}) \in \mathcal{D}(U_\alpha, X) \text{ which determines } F r_\alpha \text{ with} \\ (S_{1,\alpha}, ..., S_{k,\alpha})_\star = (H_1, ..., H_n)_\star (r_\alpha)_\star \text{ and there exists a} \\ (G_{1,\alpha}, ..., G_{l,\alpha}) \in \mathcal{D}(U_\alpha, U_\alpha) \text{ which determines a selection } G_\alpha \text{ with} \\ (H_1, ..., H_n)_\star (r_\alpha)_\star (\Psi_{1,\alpha}, ..., \Psi_{m,\alpha})_\star = (H_1, ..., H_n)_\star \text{ and} \\ (\Psi_{1,\alpha}, ..., \Psi_{m,\alpha})_\star (S_{1,\alpha}, ..., S_{k,\alpha})_\star = (G_{1,\alpha}, ..., G_{l,\alpha})_\star . \end{cases}$$

Reasoning as above establishes $\Lambda((H_1, ..., H_n)_\star) = \Lambda((G_{1,\alpha}, ..., G_{l,\alpha})_\star)$, so $\mathbf{\Lambda}(F)$ is well defined. Also it is easy to see that if $\mathbf{\Lambda}(F) \neq \{0\}$ then $F$ has a fixed point.

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