FIXED POINT THEOREMS
FOR MULTI-VALUED MAPPINGS OF FENG-LIU TYPE

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1. Introduction and preliminaries

Let \((X, d)\) be a metric space. We consider the following families of sets

\[
\mathcal{P}(X) = \{A \subseteq X : A \text{ is nonempty}\}, \\
\mathcal{P}_c(X) = \{A \subseteq X : A \text{ is nonempty and closed}\}, \\
\mathcal{P}_{cp}(X) = \{A \subseteq X : A \text{ is nonempty and compact}\}.
\]

A multi-valued mapping \(T : X \to \mathcal{P}(X)\) is said to have a fixed point if there exists \(x \in X\) such that \(x \in Tx\).

In [7], Nadler proved a well-known fixed point result which states that every multi-valued contraction defined on a complete metric space with bounded and closed values has a fixed point. This result was extended in various directions. Such generalizations include the ones given in [6, 8, 9, 10]. Another more recent extension of Nadler’s [7] result was proved by Feng and Liu in [3]. Before stating the result we recall the following notions.

A function \(f : X \to \mathbb{R}\) is lower semi-continuous if for all \(x \in X\) and for all \((x_n) \subseteq X\) with \(x_n \to x\) we have that \(f(x) \leq \liminf_{n \to \infty} f(x_n)\).

We define the distance of a point \(x \in X\) to \(A \subseteq X\) by

\[
D(x, A) = \inf\{d(x, a) : a \in A\}.
\]
Theorem 1.1. (Feng, Liu [3]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. If there exist $b,c \in (0,1)$ with $c < b$ such that for any $x \in X$ there is $y \in Tx$ satisfying
\[ bd(x,y) \leq f(x) \text{ and } f(y) \leq cd(x,y), \]
then $T$ has a fixed point.

Other results of this type were obtained by Klim and Wardowski in [5].

Theorem 1.2. (Klim, Wardowski [5]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there exist $b \in (0,1)$ and a function $\varphi : [0, \infty) \to [0,b)$ satisfying
\[ \limsup_{r \to t^+} \varphi(r) < b \text{ for each } t \in [0,\infty) \]
and for any $x \in X$ there is $y \in Tx$ such that
\[ bd(x,y) \leq f(x) \text{ and } f(y) \leq \varphi(d(x,y))d(x,y). \]
Then $T$ has a fixed point.

Theorem 1.3. (Klim, Wardowski [5]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there is a function $\varphi : [0, \infty) \to [0,1)$ satisfying
\[ \limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0,\infty) \]
and is such that for any $x \in X$ there is $y \in Tx$ such that
\[ d(x,y) = f(x) \text{ and } f(y) \leq \varphi(d(x,y))d(x,y). \]
Then $T$ has a fixed point.

Motivated by these results, Ćirić proved the following theorems in [1, 2].

Theorem 1.4. (Ćirić [2]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there is a function $\varphi : [0, \infty) \to [a,1), a \in (0,1)$, satisfying
\[ \limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0,\infty) \]
and is such that for any $x \in X$ there is $y \in Tx$ such that
\[ \sqrt{\varphi(f(x))d(x,y)} \leq f(x) \text{ and } f(y) \leq \varphi(f(x))d(x,y). \]
Then $T$ has a fixed point.

Theorem 1.5. (Ćirić [2]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there is a function $\varphi : [0, \infty) \to [a,1), a \in (0,1)$, satisfying
\[ \limsup_{r \to t^+} \varphi(r) < 1 \text{ for each } t \in [0,\infty) \]
and is such that for any $x \in X$ there is $y \in Tx$ such that
\[ \sqrt{\varphi(d(x,y))d(x,y)} \leq f(x) \text{ and } f(y) \leq \varphi(d(x,y))d(x,y). \]
Then $T$ has a fixed point.

**Theorem 1.6.** (Čirić [1]) Let $(X, d)$ be a complete metric space, $T : X \to P_{cl}(X)$ and $f : X \to \mathbb{R}$, $f(x) = D(x, Tx)$ lower semi-continuous. If there exist the functions $\varphi : [0, \infty) \to (0, 1)$ and $\eta : [0, \infty) \to [b, 1), b > 0$ such that $\eta$ is non-decreasing, 

$$
\varphi(t) < \eta(t), \quad \limsup_{r \to t^+} \varphi(r) < \limsup_{r \to t^+} \eta(r) \quad \text{for each } t \in [0, \infty)
$$

and for any $x \in X$ there is $y \in Tx$ satisfying 

$$
\eta(d(x, y))d(x, y) \leq f(x) \quad \text{and} \quad f(y) \leq \varphi(d(x, y))d(x, y),
$$

then $T$ has a fixed point.

The purpose of this paper is to present some results that aim to extend or rewrite conditions in the very recent theorems due to Ćirić [1, 2].

2. Fixed point theorems for multi-valued mappings

Analyzing Theorem 1.1, a first natural question which arises is whether we can substitute the existence of $b, c \in (0, 1)$ with the existence of a single constant $b \in (0, 1)$ such that for any $x \in X$ there is $y \in Tx$ with the property

$$D(y, Ty) \leq bd(x, y) \leq D(x, Tx), \quad (2.1)$$

and the conclusion of the theorem still stands. The answer to this problem is negative as the following example shows.

**Example 2.1.** Let $X = [1, \infty)$ with the usual metric and let $T : X \to P_{cl}(X)$ be defined by

$$Tx = \left\{ x + 2 + \frac{1}{x}, x + 4 + \frac{2}{x} - \frac{1}{x^2} \right\}.
$$

Then there exists $b \in (0, 1)$ such that for every $x \in X$ there is $y \in Tx$ satisfying (2.1), but $T$ is fixed point free.

**Proof.** Obviously, $X$ is complete. The mapping $f : X \to \mathbb{R}$,

$$f(x) = D(x, Tx) = 2 + \frac{1}{x},
$$

is lower semi-continuous and $T$ has no fixed points.

Let $b = \frac{1}{2}$. For $x \in [1, \infty)$ take $y = x + 4 + \frac{2}{x} - \frac{1}{x^2}$. Then

$$f(y) = 2 + \frac{x^2}{x^3 + 4x^2 + 2x - 1} \quad \text{and} \quad d(x, y) = 4 + \frac{2}{x} - \frac{1}{x^2}.
$$

It can be easily seen that $f(y) < bd(x, y) < f(x)$ proving that the inequalities in (2.1) hold even strictly. \qed

Now we move our attention to Theorem 1.4 due to Ćirić [2] which requires a condition involving the square root of $\varphi$. As in Theorem 1.6 we can formulate this condition in a more general way.
Theorem 2.2. Let \((X,d)\) be a complete metric space, \(T: X \to P_{cl}(X)\) and \(f: X \to \mathbb{R}, f(x) = D(x,Tx)\) lower semi-continuous. Suppose there exist the functions \(\varphi: [0,\infty) \to [0,1], \eta: [0,\infty) \to [b,1], b \in (0,1)\) such that
\[
\varphi(t) < \eta(t), \quad \limsup_{r \to +t+} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for all } t \in [0,\infty),
\]
and for any \(x \in X\) there is \(y \in Tx\) satisfying
\[
\eta(f(x))d(x,y) \leq f(x) \tag{2.2}
\]
and
\[
f(y) \leq \varphi(f(x))d(x,y). \tag{2.3}
\]
Then \(T\) has a fixed point.

Proof. The proof follows similar patterns as in Theorem 1.4. Let \(x_0 \in X\). We can choose \(x_1 \in Tx_0\) such that
\[
\eta(f(x_0))d(x_0,x_1) \leq f(x_0) \text{ and } f(x_1) \leq \varphi(f(x_0))d(x_0,x_1).
\]
Then,
\[
f(x_1) \leq \frac{\varphi(f(x_0))}{\eta(f(x_0))} \eta(f(x_0))d(x_0,x_1) \leq \frac{\varphi(f(x_0))}{\eta(f(x_0))} f(x_0).
\]
In this manner we can build the sequence \((x_n) \subseteq X\) such that for \(n \in \mathbb{N}, x_{n+1} \in Tx_n\),
\[
\eta(f(x_n))d(x_n,x_{n+1}) \leq f(x_n) \tag{2.4}
\]
and
\[
f(x_{n+1}) \leq \frac{\varphi(f(x_n))}{\eta(f(x_n))} f(x_n). \tag{2.5}
\]
From (2.5) it follows that \((f(x_n))\) is a decreasing sequence of positive real numbers, so there exists \(\delta \geq 0\) such that \(\lim_{n \to \infty} f(x_n) = \delta\).

Let \(\beta = \limsup_{n \to \infty} \frac{\varphi(f(x_n))}{\eta(f(x_n))} < 1\). Then, for \(q = \frac{\beta+1}{2} < 1\), there exists \(n_0 \in \mathbb{N}\) such that
\[
\frac{\varphi(f(x_n))}{\eta(f(x_n))} < q \text{ for all } n \geq n_0.
\]
Thus,
\[
f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0 \tag{2.6}
\]
and so
\[
d(x_n,x_{n+1}) \leq \frac{1}{\eta(f(x_n))} f(x_n) \leq \frac{1}{b} q^{n-n_0} f(x_{n_0}) \text{ for all } n \geq n_0.
\]
Hence, \((x_n)\) is Cauchy so there exists \(z \in X\) such that \(\lim_{n \to \infty} x_n = z\). Letting \(n \to \infty\) in (2.6) we obtain that \(\delta = 0\). The lower semi-continuity of \(f\) yields that \(0 \leq f(z) \leq \liminf_{n \to \infty} f(x_n) = 0\). Thus, \(D(z,Tz) = 0\). Since \(Tz\) is closed, \(z \in Tz\) which completes the proof. \(\square\)

Next we give an example of a mapping which satisfies the hypotheses of Theorem 2.2 but does not fulfill the conditions of Theorem 1.4.
Example 2.3. Let $X = [0, 10]$ with the usual metric and consider the mapping $T: X \to P_{cl}(X)$,

$$Tx = \begin{cases} 
\{3, 4\} & \text{if } x = 6, \\
\{\frac{x}{2}\} & \text{if } x \in [0, 10] \setminus \{6\}.
\end{cases}$$

Then Theorem 2.2 can be applied for $T$, but the hypotheses in Theorem 1.4 are not fulfilled.

Proof. The function

$$f(x) = D(x, Tx) = \begin{cases} 
2 & \text{if } x = 6, \\
\frac{x}{2} & \text{if } x \in [0, 10] \setminus \{6\}
\end{cases}$$

is lower semi-continuous. We now prove that $T$ does not satisfy the conditions of Theorem 1.4. Suppose there is $a \in (0, 1)$ and there exists a function $\varphi: [0, \infty) \to [a, 1)$ such that for any $x \in [0, 10]$ there is $y \in Tx$ satisfying

$$\sqrt{\varphi(f(x))d(x, y)} \leq f(x) \quad (2.7)$$

and

$$f(y) \leq \varphi(f(x))d(x, y). \quad (2.8)$$

For $x = 6, T6 = \{3, 4\}, f(6) = 2$. If $y = 3, f(3) = \frac{3}{2}$. Relation (2.8) yields $\varphi(2) \geq \frac{1}{2}$ while (2.7) requests $\varphi(2) \leq \frac{4}{5}$ which is a contradiction.

If $y = 4, f(4) = 2$ and (2.8) implies $\varphi(2) \geq 1$ which is false. Therefore, we cannot apply Theorem 1.4 for $T$.

However, it is a simple exercise to show that for $\varphi(x) = \frac{1}{2}$ and $\eta(x) = \frac{2}{3}$ for every $x \geq 0$, the mapping $T$ satisfies the hypotheses of Theorem 2.2. $\square$

Theorem 1.5 also makes use of the square root of $\varphi$ while Theorem 1.6 imposes a monotonicity condition on $\eta$. We rewrite the original assumptions to obtain the following result.

Theorem 2.4. Let $(X, d)$ be a complete metric space, $T: X \to P_{cl}(X)$ and $f: X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist the functions $\varphi: [0, \infty) \to [0, 1), \eta: [0, \infty) \to [b, 1], b \in (0, 1)$ such that

$$\varphi(t) < \eta(t) \text{ for all } t \in [0, \infty), \quad (2.9)$$

$$\limsup_{r \to t} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for all } t \in [0, \infty), \quad (2.10)$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x, y))d(x, y) \leq f(x) \quad (2.11)$$

and

$$f(y) \leq \varphi(d(x, y))d(x, y). \quad (2.12)$$

Then $T$ has a fixed point.
Proof. As in the proof of Theorem 2.2 we can build a sequence \((x_n) \subseteq X\) such that for every \(n \in \mathbb{N}, x_{n+1} \in Tx_n\),
\[
\eta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq f(x_n) \tag{2.13}
\]
and
\[
f(x_{n+1}) \leq \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} f(x_n). \tag{2.14}
\]
Relations (2.9) and (2.14) yield that the sequence \((f(x_n))\) is decreasing. Because it is also bounded below by 0 it follows that it converges to some \(\delta \geq 0\). Suppose \(\delta > 0\). Then,
\[
\frac{f(x_{n+1})}{f(x_n)} \leq \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} < 1 \quad \text{for any } n \in \mathbb{N}.
\]
Letting here \(n \to \infty\) we obtain
\[
\lim_{n \to \infty} \frac{\varphi}{\eta}(d(x_n, x_{n+1})) = 1. \tag{2.15}
\]
Using (2.13) we have that
\[
\delta \leq f(x_n) \leq d(x_n, x_{n+1}) \leq \frac{f(x_n)}{b} \leq \frac{f(x_0)}{b} \quad \text{for any } n \in \mathbb{N}. \tag{2.16}
\]
Hence, the sequence \((d(x_n, x_{n+1}))\) is bounded. Therefore, it contains a convergent subsequence. But then condition (2.10) contradicts (2.15). Thus, \(\delta = 0\). Because of (2.16) we have that \(\lim_{n \to \infty} d(x_n, x_{n+1}) = 0\). Now we can go on as in the proof of Theorem 2.2 to finally show that \(T\) has a fixed point. \(\square\)

Another approach worth investigating would be to try to generalize conditions (2.2), (2.3) and (2.11), (2.12) respectively even further. In this direction we state the next results.

**Theorem 2.5.** Let \((X, d)\) be a complete metric space, \(T: X \to P_{cl}(X)\) and \(f: X \to \mathbb{R}, f(x) = D(x, Tx)\) lower semi-continuous. Suppose there exist \(\varphi: [0, \infty) \to [0, \infty), \eta: [0, \infty) \to (0, \infty)\) such that \(\varphi(t) < \eta(t)\) for all \(t > 0\), \(\varphi\) is non-decreasing,
\[
\limsup_{r \to t^+} \frac{\varphi(r)}{\eta(r)} < 1 \quad \text{for all } t \in [0, \infty), \tag{2.17}
\]
and for any \(x \in X\) there is \(y \in Tx\) satisfying
\[
\eta(d(x, y)) \leq f(x) \tag{2.18}
\]
and
\[
f(y) \leq \varphi(f(x)). \tag{2.19}
\]
Then \(T\) has a fixed point.

**Proof.** Let \(x_0 \in X\). We can choose \(x_1 \in Tx_0\) such that
\[
\eta(d(x_0, x_1)) \leq f(x_0) \quad \text{and} \quad f(x_1) \leq \varphi(f(x_0)).
\]
In this way we build the sequence \((x_n) \subseteq X\) such that for \(n \in \mathbb{N}, x_{n+1} \in Tx_n\),
\[
\eta(d(x_n, x_{n+1})) \leq f(x_n) \tag{2.20}
\]
and
\[ f(x_{n+1}) \leq \varphi(f(x_n)). \quad (2.21) \]

Since
\[ \varphi(d(x_{n+1}, x_{n+2})) < \eta(d(x_{n+1}, x_{n+2})) \leq f(x_{n+1}) \leq \varphi(f(x_n)) \leq \varphi(d(x_n, x_{n+1})), \]
it follows that \( \varphi(d(x_{n+1}, x_{n+2})) < \varphi(d(x_n, x_{n+1})) \). Because \( \varphi \) is non-decreasing, the sequence \( d(x_n, x_{n+1}) \) is decreasing. Because it is also bounded below, it converges to some positive value. Using (2.20) and (2.21) we have that
\[ f(x_{n+1}) \leq \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} f(x_n). \]

Because of (2.17) there exist \( q \in (0, 1) \) and \( n_0 \in \mathbb{N} \) such that
\[ \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} < q \text{ for all } n \geq n_0. \]

Thus,
\[ f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0. \]

For \( n \geq n_0 + 1, \)
\[ \varphi(d(x_n, x_{n+1})) < \eta(d(x_n, x_{n+1})) \leq f(x_n) \leq \varphi(f(x_{n-1})) \leq \varphi(q^{n-n_0-1} f(x_{n_0})). \]

Since \( \varphi \) is non-decreasing, \( d(x_n, x_{n+1}) \leq q^{n-n_0-1} f(x_{n_0}) \). It is easy to see that \( (x_n) \) is a Cauchy sequence and its limit is a fixed point for \( T \).

**Theorem 2.6.** Let \( (X, d) \) be a complete metric space, \( T : X \to P_d(X) \) and \( f : X \to \mathbb{R}, f(x) = D(x, Tx) \) lower semi-continuous. Suppose there exist \( \varphi : [0, \infty) \to [0, \infty), \eta : [0, \infty) \to (0, \infty) \) such that \( \varphi(t) < \eta(t) \) for all \( t > 0, \eta \) is non-decreasing,
\[ \limsup_{r \to t^+} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for all } t \in [0, \infty), \]

and for any \( x \in X \) there is \( y \in Tx \) satisfying
\[ \eta(d(x, y)) \leq f(x) \text{ and } f(y) \leq \varphi(f(x)). \]

Then \( T \) has a fixed point.

**Proof.** We build the sequence \( (x_n) \subseteq X \) as in the proof of Theorem 2.5. Since \( \eta \) is non-decreasing we obtain that for \( n \in \mathbb{N}, \)
\[ f(x_{n+1}) \leq \frac{\varphi(f(x_n))}{\eta(f(x_n))} f(x_n). \]

Hence, \( (f(x_n)) \) is decreasing. Because it is also bounded below, it converges to some positive value. Again there exist \( q \in (0, 1) \) and \( n_0 \in \mathbb{N} \) such that
\[ f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0. \]

For \( n \geq n_0 + 1, \)
\[ \eta(d(x_n, x_{n+1})) \leq f(x_n) \leq \varphi(f(x_{n-1})) < \eta(f(x_{n-1})) \leq \eta(q^{n-n_0-1} f(x_{n_0})). \]
But $\eta$ is non-decreasing, so $d(x_n, x_{n+1}) \leq q^{n-n_0-1} f(x_{n_0})$. As above we can show that $T$ has a fixed point.

In the sequel we prove two related theorems.

**Theorem 2.7.** Let $(X, d)$ be a complete metric space, $T: X \rightarrow P_d(X)$ and $f: X \rightarrow \mathbb{R}$, $f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi: [0, \infty) \rightarrow [0, \infty)$, $\eta: [0, \infty) \rightarrow (0, \infty)$ such that $\varphi(t) < \eta(t) \leq t$ for all $t > 0$, $\varphi$ is continuous and non-decreasing,

$$\limsup_{r \rightarrow 0^+} \frac{\varphi(r)}{\eta(r)} < 1,$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x, y)) \leq f(x) \text{ and } f(y) \leq \varphi(f(x)).$$

Then $T$ has a fixed point.

**Proof.** Again we build the sequence $(x_n)$ with $x_{n+1} \in Tx_n$ such that (2.20) and (2.21) hold. We can assume that for $n \in \mathbb{N}$, $d(x_n, x_{n+1}) > 0$ and $f(x_n) > 0$ because otherwise we obtain a fixed point.

Let $t > 0$. Because $0 \leq \varphi(t) < t$, $(\varphi^n(t))$ is a decreasing sequence which is bounded below by 0. Suppose its limit is $\epsilon > 0$. Then,

$$\epsilon = \lim_{n \rightarrow \infty} \varphi^n(t) = \varphi\left( \lim_{n \rightarrow \infty} \varphi^{n-1}(t) \right) = \varphi(\epsilon) < \epsilon$$

which is a contradiction. Therefore, $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for all $t > 0$. From (2.21) it follows that for $n \in \mathbb{N}$,

$$f(x_{n+1}) \leq \varphi(f(x_n)) \leq \ldots \leq \varphi^{n+1}(f(x_0)).$$

Now it is clear that $\lim_{n \rightarrow \infty} f(x_n) = 0$.

Since $\varphi$ is non-decreasing and

$$\varphi\left( d(x_{n+1}, x_{n+2}) \right) \leq \eta \left( d(x_{n+1}, x_{n+2}) \right) \leq f(x_{n+1}) \leq \varphi\left( f(x_{n+1}) \right)$$

the sequence $(d(x_n, x_{n+1}))$ is decreasing. Assume $\alpha > 0$ is its limit. Letting $n \rightarrow \infty$ in $\varphi(d(x_n, x_{n+1})) < f(x_n)$ we obtain that $\varphi(\alpha) = 0$. But since $\lim_{n \rightarrow \infty} f(x_n) = 0$, there exists $n_1 \in \mathbb{N}$ such that $f(x_{n_1}) < \alpha$. Then $\varphi(f(x_{n_1})) = 0$ which means that $f(x_{n_1+1}) = 0$. In this way we obtain a fixed point. Therefore, we may consider $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. Continuing as in the proof of Theorem 2.5 one can show that $T$ is not fixed point free. □

Using a similar argument as before we can prove the following result.

**Theorem 2.8.** Let $(X, d)$ be a complete metric space, $T: X \rightarrow P_d(X)$ and $f: X \rightarrow \mathbb{R}$, $f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi: [0, \infty) \rightarrow [0, \infty)$, $\eta: [0, \infty) \rightarrow (0, \infty)$ such that $\varphi(t) < \eta(t) < t$ for all $t > 0$, $\eta$ is continuous and non-decreasing,

$$\limsup_{r \rightarrow 0^+} \frac{\varphi(r)}{\eta(r)} < 1,$$
and for any \( x \in X \) there is \( y \in Tx \) satisfying

\[
\eta(d(x,y)) \leq f(x) \quad \text{and} \quad f(y) \leq \varphi(f(x)).
\]

Then \( T \) has a fixed point.

**Proof.** Using the fact that \( \lim_{n \to \infty} \eta^n(t) = 0 \) for every \( t \geq 0 \) and

\[
\begin{align*}
f(x_{n+1}) \leq & \varphi(f(x_n)) < \eta(f(x_n)) \leq \eta(\varphi(f(x_{n-1}))) \\
& \leq \ldots \leq \eta^{n+1}(f(x_0)),
\end{align*}
\]

we have that \( \lim_{n \to \infty} f(x_n) = 0 \). As in Theorem 2.6 we can prove that \( T \) has a fixed point. \( \square \)

In the above results it would be interesting to replace condition (2.19) by \( f(y) \leq \varphi(d(x,y)) \). Pursuing this idea we can give the following two theorems.

**Theorem 2.9.** Let \((X,d)\) be a complete metric space, \( T \colon X \to P_{cl}(X) \) and \( f \colon X \to \mathbb{R}, f(x) = D(x,Tx) \) lower semi-continuous. Suppose there exist \( \varphi \colon [0, \infty) \to [0, \infty), \eta \colon [0, \infty) \to (0, \infty) \) such that \( \varphi(t) < \eta(t) \) for all \( t > 0 \), \( \varphi \) is non-decreasing and subadditive,

\[
\limsup_{r \to t^+} \frac{\varphi(r)}{\eta(r)} < 1 \quad \text{for every } t \in [0, \infty),
\]

and for any \( x \in X \) there is \( y \in Tx \) satisfying

\[
\eta(d(x,y)) \leq f(x) \quad \text{and} \quad f(y) \leq \varphi(d(x,y)).
\]

Then \( T \) has a fixed point.

**Proof.** Similarly as before we can build the sequence \( (x_n) \subseteq X \) such that for \( n \in \mathbb{N}, x_{n+1} \in Tx_n \),

\[
\eta(d(x_n,x_{n+1})) \leq f(x_n) \quad \text{and} \quad f(x_{n+1}) \leq \varphi(d(x_n,x_{n+1})).
\]

The sequence \( (d(x_n,x_{n+1})) \) is decreasing since \( \varphi \) is non-decreasing and

\[
\varphi(d(x_{n+1},x_{n+2})) < \eta(d(x_{n+1},x_{n+2})) \leq f(x_{n+1}) \leq \varphi(d(x_n,x_{n+1})).
\]

Thus, it converges to some positive value. Then there exist \( q \in (0,1) \) and \( n_0 \in \mathbb{N} \) such that

\[
f(x_n) \leq q^{n-n_0+1}f(x_{n_0}) \quad \text{for all } n \geq n_0.
\]

For \( n \geq n_0 \),

\[
\varphi(d(x_n,x_{n+1})) < \eta(d(x_n,x_{n+1})) \leq f(x_n) \leq q^{n-n_0}f(x_{n_0}).
\]

Then for \( n \geq n_0 \) and \( p \in \mathbb{N} \),

\[
\varphi(d(x_n,x_{n+p})) \leq \varphi \left( \sum_{k=0}^{p-1} d(x_{n+k},x_{n+k+1}) \right) \leq \sum_{k=0}^{p-1} \varphi(d(x_{n+k},x_{n+k+1})) \leq \sum_{k=0}^{p-1} \varphi \left( d(x_{n+k},x_{n+k+1}) \right) \leq \sum_{k=0}^{p-1} q^{n-n_0+k}f(x_{n_0}) \leq \frac{q^{n-n_0}}{1-q} f(x_{n_0}).
\]
Since we may assume that \( \varphi(t) > 0 \) for \( t > 0 \) (otherwise \( \varphi(t) = 0 \) for every \( t \geq 0 \) and the existence of a fixed point is immediate) we can prove by contradiction that \( (x_n) \) is Cauchy and its limit is a fixed point for \( T \). □

In the same manner we can prove the next result.

**Theorem 2.10.** Let \( (X,d) \) be a complete metric space, \( T: X \to P_{cl}(X) \) and \( f: X \to \mathbb{R}, f(x) = D(x,Tx) \) lower semi-continuous. Suppose there exist \( \varphi: [0, \infty) \to [0, \infty), \eta: [0, \infty) \to (0, \infty) \) such that \( \varphi(t) < \eta(t) \) for all \( t > 0 \), \( \eta \) is non-decreasing and subadditive,

\[
\limsup_{r \to t^+} \frac{\varphi(r)}{\eta(r)} < 1 \quad \text{for every} \quad t \in [0, \infty),
\]

and for any \( x \in X \) there is \( y \in Tx \) satisfying

\[
\eta(d(x,y)) \leq f(x) \quad \text{and} \quad f(y) \leq \varphi(d(x,y)).
\]

Then \( T \) has a fixed point.

**Remark 2.11.** For further developments, we can consider the framework given in [4].

3. **Acknowledgment**

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**References**


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