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FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS OF FENG-LIU TYPE

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Abstract. This paper presents some fixed point theorems for multi-valued contraction mappings that generalize recent results proved by Y. Feng and S. Liu [*Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings*, J. Math. Anal. Appl., **317**(2006), 103-112], D. Klim and D. Wardowski [*Fixed point theorems for set-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **334**(2007), 132-139] or L. Ćirić [*Fixed point theorems for multi-valued contractions in complete metric spaces*, J. Math. Anal. Appl., **348**(2008), 499-507; *Multi-valued nonlinear contraction mappings*, Nonlinear Anal., **71**(2009), 2716-2723].
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1. Introduction and preliminaries

Let (X, d) be a metric space. We consider the following families of sets

$$\mathcal{P}(X) = \{A \subseteq X : A \text{ is nonempty}\},$$

$$\mathcal{P}_{cl}(X) = \{A \subseteq X : A \text{ is nonempty and closed}\},$$

$$\mathcal{P}_{cp}(X) = \{A \subseteq X : A \text{ is nonempty and compact}\}.$$

A multi-valued mapping $T : X \to \mathcal{P}(X)$ is said to have a fixed point if there exists $x \in X$ such that $x \in Tx$.

In [7], Nadler proved a well-known fixed point result which states that every multi-valued contraction defined on a complete metric space with bounded and closed values has a fixed point. This result was extended in various directions. Such generalizations include the ones given in [6, 8, 9, 10]. Another more recent extension of Nadler's [7] result was proved by Feng and Liu in [3]. Before stating the result we recall the following notions.

A function $f : X \to \mathbb{R}$ is lower semi-continuous if for all $x \in X$ and for all $(x_n) \subseteq X$ with $x_n \to x$ we have that $f(x) \leq \liminf_{n\to\infty} f(x_n)$.

We define the distance of a point $x \in X$ to $A \subseteq X$ by

$$D(x, A) = \inf\{d(x, a) : a \in A\}.$$

Theorem 1.1. (Feng, Liu [3]) *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. If there exist $b, c \in (0, 1)$ with $c < b$ such that for any $x \in X$ there is $y \in Tx$ satisfying*

$$bd(x, y) \leq f(x) \text{ and } f(y) \leq cd(x, y),$$

then T has a fixed point.

Other results of this type were obtained by Klim and Wardowski in [5].

Theorem 1.2. (Klim, Wardowski [5]) *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $b \in (0, 1)$ and a function $\varphi\colon [0, \infty) \to [0, b)$ satisfying*

$$\limsup_{r \to t+} \varphi(r) < b \text{ for each } t \in [0, \infty)$$

and for any $x \in X$ there is $y \in Tx$ such that

$$bd(x, y) \leq f(x) \text{ and } f(y) \leq \varphi(d(x, y))d(x, y).$$

Then T has a fixed point.

Theorem 1.3. (Klim, Wardowski [5]) *Let (X, d) be a complete metric space, $T\colon X \to P_{cp}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there is a function $\varphi\colon [0, \infty) \to [0, 1)$ satisfying*

$$\limsup_{r \to t+} \varphi(r) < 1 \text{ for each } t \in [0, \infty)$$

and is such that for any $x \in X$ there is $y \in Tx$ such that

$$d(x, y) = f(x) \text{ and } f(y) \leq \varphi(d(x, y))d(x, y).$$

Then T has a fixed point.

Motivated by these results, Ćirić proved the following theorems in [1, 2].

Theorem 1.4. (Ćirić [2]) *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there is a function $\varphi\colon [0, \infty) \to [a, 1), a \in (0, 1)$, satisfying*

$$\limsup_{r \to t+} \varphi(r) < 1 \text{ for each } t \in [0, \infty)$$

and is such that for any $x \in X$ there is $y \in Tx$ such that

$$\sqrt{\varphi(f(x))}d(x, y) \leq f(x) \text{ and } f(y) \leq \varphi(f(x))d(x, y).$$

Then T has a fixed point.

Theorem 1.5. (Ćirić [2]) *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there is a function $\varphi\colon [0, \infty) \to [a, 1), a \in (0, 1)$, satisfying*

$$\limsup_{r \to t+} \varphi(r) < 1 \text{ for each } t \in [0, \infty)$$

and is such that for any $x \in X$ there is $y \in Tx$ such that

$$\sqrt{\varphi(d(x, y))}d(x, y) \leq f(x) \text{ and } f(y) \leq \varphi(d(x, y))d(x, y).$$

Then T has a fixed point.

Theorem 1.6. (Ćirić [1]) *Let (X,d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. If there exist the functions $\varphi\colon [0,\infty) \to (0,1)$ and $\eta : [0,\infty) \to [b,1), b > 0$ such that η is non-decreasing,*

$$\varphi(t) < \eta(t), \quad \limsup_{r\to t+} \varphi(r) < \limsup_{r\to t+} \eta(r) \text{ for each } t \in [0,\infty)$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x,y))d(x,y) \le f(x) \text{ and } f(y) \le \varphi(d(x,y))d(x,y),$$

then T has a fixed point.

The purpose of this paper is to present some results that aim to extend or rewrite conditions in the very recent theorems due to Ćirić [1, 2].

2. Fixed point theorems for multi-valued mappings

Analyzing Theorem 1.1, a first natural question which arises is whether we can substitute the existence of $b, c \in (0,1)$ with the existence of a single constant $b \in (0,1)$ such that for any $x \in X$ there is $y \in Tx$ with the property

$$D(y,Ty) \le bd(x,y) \le D(x,Tx), \tag{2.1}$$

and the conclusion of the theorem still stands. The answer to this problem is negative as the following example shows.

Example 2.1. *Let $X = [1,\infty)$ with the usual metric and let $T\colon X \to P_{cl}(X)$ be defined by*

$$Tx = \left\{ x + 2 + \frac{1}{x}, x + 4 + \frac{2}{x} - \frac{1}{x^2} \right\}.$$

Then there exists $b \in (0,1)$ such that for every $x \in X$ there is $y \in Tx$ satisfying (2.1), but T is fixed point free.

Proof. Obviously, X is complete. The mapping $f\colon X \to \mathbb{R}$,

$$f(x) = D(x,Tx) = 2 + \frac{1}{x},$$

is lower semi-continuous and T has no fixed points.
Let $b = \frac{1}{2}$. For $x \in [1,\infty)$ take $y = x + 4 + \frac{2}{x} - \frac{1}{x^2}$. Then

$$f(y) = 2 + \frac{x^2}{x^3 + 4x^2 + 2x - 1} \text{ and } d(x,y) = 4 + \frac{2}{x} - \frac{1}{x^2}.$$

It can be easily seen that $f(y) < bd(x,y) < f(x)$ proving that the inequalities in (2.1) hold even strictly. □

Now we move our attention to Theorem 1.4 due to Ćirić [2] which requires a condition involving the square root of φ. As in Theorem 1.6 we can formulate this condition in a more general way.

Theorem 2.2. *Let (X,d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there exist the functions $\varphi\colon [0,\infty) \to [0,1), \eta\colon [0,\infty) \to [b,1], b \in (0,1)$ such that*

$$\varphi(t) < \eta(t), \quad \limsup_{r\to t+} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for all } t \in [0,\infty),$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(f(x))d(x,y) \le f(x) \tag{2.2}$$

and

$$f(y) \le \varphi(f(x))d(x,y). \tag{2.3}$$

Then T has a fixed point.

Proof. The proof follows similar patterns as in Theorem 1.4. Let $x_0 \in X$. We can choose $x_1 \in Tx_0$ such that

$$\eta(f(x_0))d(x_0,x_1) \le f(x_0) \text{ and } f(x_1) \le \varphi(f(x_0))d(x_0,x_1).$$

Then,

$$f(x_1) \le \frac{\varphi(f(x_0))}{\eta(f(x_0))}\eta(f(x_0))d(x_0,x_1) \le \frac{\varphi(f(x_0))}{\eta(f(x_0))}f(x_0).$$

In this manner we can build the sequence $(x_n) \subseteq X$ such that for $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$,

$$\eta(f(x_n))d(x_n,x_{n+1}) \le f(x_n) \tag{2.4}$$

and

$$f(x_{n+1}) \le \frac{\varphi(f(x_n))}{\eta(f(x_n))}f(x_n). \tag{2.5}$$

From (2.5) it follows that $(f(x_n))$ is a decreasing sequence of positive real numbers, so there exists $\delta \ge 0$ such that $\lim\limits_{n\to\infty} f(x_n) = \delta$.

Let $\beta = \limsup\limits_{n\to\infty} \dfrac{\varphi(f(x_n))}{\eta(f(x_n))} < 1$. Then, for $q = \frac{\beta+1}{2} < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{\varphi(f(x_n))}{\eta(f(x_n))} < q \text{ for all } n \ge n_0.$$

Thus,

$$f(x_{n+1}) \le q^{n-n_0+1}f(x_{n_0}) \text{ for all } n \ge n_0 \tag{2.6}$$

and so

$$d(x_n,x_{n+1}) \le \frac{1}{\eta(f(x_n))}f(x_n) \le \frac{1}{b}q^{n-n_0}f(x_{n_0}) \text{ for all } n \ge n_0.$$

Hence, (x_n) is Cauchy so there exists $z \in X$ such that $\lim\limits_{n\to\infty} x_n = z$. Letting $n \to \infty$ in (2.6) we obtain that $\delta = 0$. The lower semi-continuity of f yields that $0 \le f(z) \le \liminf\limits_{n\to\infty} f(x_n) = 0$. Thus, $D(z,Tz) = 0$. Since Tz is closed, $z \in Tz$ which completes the proof. □

Next we give an example of a mapping which satisfies the hypotheses of Theorem 2.2 but does not fulfill the conditions of Theorem 1.4.

Example 2.3. *Let* $X = [0, 10]$ *with the usual metric and consider the mapping* $T\colon X \to P_{cl}(X)$,

$$Tx = \begin{cases} \{3, 4\} & \textit{if } x = 6, \\ \{\frac{x}{2}\} & \textit{if } x \in [0, 10] \setminus \{6\}. \end{cases}$$

Then Theorem 2.2 can be applied for T*, but the hypotheses in Theorem 1.4 are not fulfilled.*

Proof. The function

$$f(x) = D(x, Tx) = \begin{cases} 2 & \text{if } x = 6, \\ \frac{x}{2} & \text{if } x \in [0, 10] \setminus \{6\} \end{cases}$$

is lower semi-continuous. We now prove that T does not satisfy the conditions of Theorem 1.4. Suppose there is $a \in (0, 1)$ and there exists a function $\varphi\colon [0, \infty) \to [a, 1)$ such that for any $x \in [0, 10]$ there is $y \in Tx$ satisfying

$$\sqrt{\varphi(f(x))}d(x, y) \leq f(x) \tag{2.7}$$

and

$$f(y) \leq \varphi(f(x))d(x, y). \tag{2.8}$$

For $x = 6, T6 = \{3, 4\}, f(6) = 2$.
If $y = 3, f(3) = \frac{3}{2}$. Relation (2.8) yields $\varphi(2) \geq \frac{1}{2}$ while (2.7) requests $\varphi(2) \leq \frac{4}{9}$ which is a contradiction.
If $y = 4, f(4) = 2$ and (2.8) implies $\varphi(2) \geq 1$ which is false. Therefore, we cannot apply Theorem 1.4 for T.
However, it is a simple exercise to show that for $\varphi(x) = \frac{1}{2}$ and $\eta(x) = \frac{2}{3}$ for every $x \geq 0$, the mapping T satisfies the hypotheses of Theorem 2.2. □

Theorem 1.5 also makes use of the square root of φ while Theorem 1.6 imposes a monotonicity condition on η. We rewrite the original assumptions to obtain the following result.

Theorem 2.4. *Let* (X, d) *be a complete metric space,* $T\colon X \to P_{cl}(X)$ *and* $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ *lower semi-continuous. Suppose there exist the functions* $\varphi\colon [0, \infty) \to [0, 1), \eta\colon [0, \infty) \to [b, 1], b \in (0, 1)$ *such that*

$$\varphi(t) < \eta(t) \textit{ for all } t \in [0, \infty), \tag{2.9}$$

$$\limsup_{r \to t} \frac{\varphi(r)}{\eta(r)} < 1 \textit{ for all } t \in [0, \infty), \tag{2.10}$$

and for any $x \in X$ *there is* $y \in Tx$ *satisfying*

$$\eta(d(x, y))d(x, y) \leq f(x) \tag{2.11}$$

and

$$f(y) \leq \varphi(d(x, y))d(x, y). \tag{2.12}$$

Then T *has a fixed point.*

Proof. As in the proof of Theorem 2.2 we can build a sequence $(x_n) \subseteq X$ such that for every $n \in \mathbb{N}, x_{n+1} \in Tx_n$,

$$\eta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \leq f(x_n) \tag{2.13}$$

and

$$f(x_{n+1}) \leq \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} f(x_n). \tag{2.14}$$

Relations (2.9) and (2.14) yield that the sequence $(f(x_n))$ is decreasing. Because it is also bounded below by 0 it follows that it converges to some $\delta \geq 0$. Suppose $\delta > 0$. Then,

$$\frac{f(x_{n+1})}{f(x_n)} \leq \frac{\varphi(d(x_n, x_{n+1}))}{\eta(d(x_n, x_{n+1}))} < 1 \text{ for any } n \in \mathbb{N}.$$

Letting here $n \to \infty$ we obtain

$$\lim_{n\to\infty} \frac{\varphi}{\eta}\left(d(x_n, x_{n+1})\right) = 1. \tag{2.15}$$

Using (2.13) we have that

$$\delta \leq f(x_n) \leq d(x_n, x_{n+1}) \leq \frac{f(x_n)}{b} \leq \frac{f(x_0)}{b} \text{ for any } n \in \mathbb{N}. \tag{2.16}$$

Hence, the sequence $(d(x_n, x_{n+1}))$ is bounded. Therefore, it contains a convergent subsequence. But then condition (2.10) contradicts (2.15). Thus, $\delta = 0$. Because of (2.16) we have that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Now we can go on as in the proof of Theorem 2.2 to finally show that T has a fixed point. □

Another approach worth investigating would be to try to generalize conditions (2.2), (2.3) and (2.11), (2.12) respectively even further. In this direction we state the next results.

Theorem 2.5. *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0,\infty) \to [0,\infty), \eta\colon [0,\infty) \to (0,\infty)$ such that $\varphi(t) < \eta(t)$ for all $t > 0$, φ is non-decreasing,*

$$\limsup_{r\to t+} \frac{\varphi(r)}{\eta(r)} < 1 \textit{ for all } t \in [0,\infty), \tag{2.17}$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x, y)) \leq f(x) \tag{2.18}$$

and

$$f(y) \leq \varphi(f(x)). \tag{2.19}$$

Then T has a fixed point.

Proof. Let $x_0 \in X$. We can choose $x_1 \in Tx_0$ such that

$$\eta(d(x_0, x_1)) \leq f(x_0) \text{ and } f(x_1) \leq \varphi(f(x_0)).$$

In this way we build the sequence $(x_n) \subseteq X$ such that for $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$,

$$\eta(d(x_n, x_{n+1})) \leq f(x_n) \tag{2.20}$$

and

$$f(x_{n+1}) \leq \varphi(f(x_n)). \tag{2.21}$$

Since

$$\begin{aligned}\varphi\left(d(x_{n+1}, x_{n+2})\right) <& \eta\left(d(x_{n+1}, x_{n+2})\right) \leq f(x_{n+1}) \leq \varphi\left(f(x_n)\right) \\ \leq& \varphi\left(d(x_n, x_{n+1})\right),\end{aligned}$$

it follows that $\varphi\left(d(x_{n+1}, x_{n+2})\right) < \varphi\left(d(x_n, x_{n+1})\right)$. Because φ is non-decreasing, the sequence $(d(x_n, x_{n+1}))$ is decreasing. Because it is also bounded below, it converges to some positive value. Using (2.20) and (2.21) we have that

$$f(x_{n+1}) \leq \frac{\varphi\left(d(x_n, x_{n+1})\right)}{\eta\left(d(x_n, x_{n+1})\right)} f(x_n).$$

Because of (2.17) there exist $q \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$\frac{\varphi\left(d(x_n, x_{n+1})\right)}{\eta\left(d(x_n, x_{n+1})\right)} < q \text{ for all } n \geq n_0.$$

Thus,

$$f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0.$$

For $n \geq n_0 + 1$,

$$\begin{aligned}\varphi\left(d(x_n, x_{n+1})\right) <& \eta\left(d(x_n, x_{n+1})\right) \leq f(x_n) \leq \varphi\left(f(x_{n-1})\right) \\ \leq& \varphi\left(q^{n-n_0-1} f(x_{n_0})\right).\end{aligned}$$

Since φ is non-decreasing, $d(x_n, x_{n+1}) \leq q^{n-n_0-1} f(x_{n_0})$. It is easy to see that (x_n) is a Cauchy sequence and its limit is a fixed point for T. □

Theorem 2.6. *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0,\infty) \to [0,\infty), \eta\colon [0,\infty) \to (0,\infty)$ such that $\varphi(t) < \eta(t)$ for all $t > 0$, η is non-decreasing,*

$$\limsup_{r \to t+} \frac{\varphi(r)}{\eta(r)} < 1 \textit{ for all } t \in [0,\infty),$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x,y)) \leq f(x) \textit{ and } f(y) \leq \varphi(f(x)).$$

Then T has a fixed point.

Proof. We build the sequence $(x_n) \subseteq X$ as in the proof of Theorem 2.5. Since η is non-decreasing we obtain that for $n \in \mathbb{N}$,

$$f(x_{n+1}) \leq \frac{\varphi(f(x_n))}{\eta(f(x_n))} f(x_n).$$

Hence, $(f(x_n))$ is decreasing. Because it is also bounded below, it converges to some positive value. Again there exist $q \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0.$$

For $n \geq n_0 + 1$,

$$\eta\left(d(x_n, x_{n+1})\right) \leq f(x_n) \leq \varphi\left(f(x_{n-1})\right) < \eta\left(f(x_{n-1})\right) \leq \eta\left(q^{n-n_0-1} f(x_{n_0})\right).$$

But η is non-decreasing, so $d(x_n, x_{n+1}) \leq q^{n-n_0-1} f(x_{n_0})$. As above we can show that T has a fixed point. $\square$

In the sequel we prove two related theorems.

Theorem 2.7. *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0,\infty) \to [0,\infty), \eta\colon [0,\infty) \to (0,\infty)$ such that $\varphi(t) < \eta(t) \leq t$ for all $t > 0$, φ is continuous and non-decreasing,*

$$\limsup_{r \to 0+} \frac{\varphi(r)}{\eta(r)} < 1,$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x, y)) \leq f(x) \text{ and } f(y) \leq \varphi(f(x)).$$

Then T has a fixed point.

Proof. Again we build the sequence (x_n) with $x_{n+1} \in Tx_n$ such that (2.20) and (2.21) hold. We can assume that for $n \in \mathbb{N}$, $d(x_n, x_{n+1}) > 0$ and $f(x_n) > 0$ because otherwise we obtain a fixed point.
Let $t > 0$. Because $0 \leq \varphi(t) < t$, $(\varphi^n(t))$ is a decreasing sequence which is bounded below by 0. Suppose its limit is $\epsilon > 0$. Then,

$$\epsilon = \lim_{n\to\infty} \varphi^n(t) = \varphi\left(\lim_{n\to\infty} \varphi^{n-1}(t)\right) = \varphi(\epsilon) < \epsilon$$

which is a contradiction. Therefore, $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. From (2.21) it follows that for $n \in \mathbb{N}$,

$$f(x_{n+1}) \leq \varphi\left(f(x_n)\right) \leq \ldots \leq \varphi^{n+1}\left(f(x_0)\right).$$

Now it is clear that $\lim\limits_{n\to\infty} f(x_n) = 0$.
Since φ is non-decreasing and

$$\begin{aligned}\varphi\left(d(x_{n+1}, x_{n+2})\right) <& \eta\left(d(x_{n+1}, x_{n+2})\right) \leq f(x_{n+1}) \leq \varphi\left(f(x_n)\right) \\ \leq& \varphi\left(d(x_n, x_{n+1})\right),\end{aligned}$$

the sequence $(d(x_n, x_{n+1}))$ is decreasing. Assume $\alpha > 0$ is its limit. Letting $n \to \infty$ in $\varphi\left(d(x_n, x_{n+1})\right) < f(x_n)$ we obtain that $\varphi(\alpha) = 0$. But since $\lim\limits_{n\to\infty} f(x_n) = 0$, there exists $n_1 \in \mathbb{N}$ such that $f(x_{n_1}) < \alpha$. Then $\varphi\left(f(x_{n_1})\right) = 0$ which means that $f(x_{n_1+1}) = 0$. In this way we obtain a fixed point. Therefore, we may consider $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = 0$. Continuing as in the proof of Theorem 2.5 one can show that T is not fixed point free. $\square$

Using a similar argument as before we can prove the following result.

Theorem 2.8. *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0,\infty) \to [0,\infty), \eta\colon [0,\infty) \to (0,\infty)$ such that $\varphi(t) < \eta(t) < t$ for all $t > 0$, η is continuous and non-decreasing,*

$$\limsup_{r \to 0+} \frac{\varphi(r)}{\eta(r)} < 1,$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x,y)) \leq f(x) \text{ and } f(y) \leq \varphi(f(x)).$$

Then T has a fixed point.

Proof. Using the fact that $\lim_{n\to\infty} \eta^n(t) = 0$ for every $t \geq 0$ and

$$\begin{aligned} f(x_{n+1}) \leq& \varphi\left(f(x_n)\right) < \eta\left(f(x_n)\right) \leq \eta\left(\varphi\left(f(x_{n-1})\right)\right) \leq \eta^2\left(f(x_{n-1})\right) \\ &\leq \ldots \leq \eta^{n+1}\left(f(x_0)\right), \end{aligned}$$

we have that $\lim_{n\to\infty} f(x_n) = 0$. As in Theorem 2.6 we can prove that T has a fixed point. □

In the above results it would be interesting to replace condition (2.19) by $f(y) \leq \varphi\left(d(x,y)\right)$. Pursuing this idea we can give the following two theorems.

Theorem 2.9. *Let (X,d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x,Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0,\infty) \to [0,\infty), \eta\colon [0,\infty) \to (0,\infty)$ such that $\varphi(t) < \eta(t)$ for all $t > 0$, φ is non-decreasing and subadditive,*

$$\limsup_{r\to t+} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for every } t \in [0,\infty),$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x,y)) \leq f(x) \text{ and } f(y) \leq \varphi(d(x,y)).$$

Then T has a fixed point.

Proof. Similarly as before we can build the sequence $(x_n) \subseteq X$ such that for $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$,

$$\eta(d(x_n,x_{n+1})) \leq f(x_n) \text{ and } f(x_{n+1}) \leq \varphi(d(x_n,x_{n+1})).$$

The sequence $(d(x_n,x_{n+1}))$ is decreasing since φ is non-decreasing and

$$\varphi\left(d(x_{n+1},x_{n+2})\right) < \eta\left(d(x_{n+1},x_{n+2})\right) \leq f(x_{n+1}) \leq \varphi\left(d(x_n,x_{n+1})\right).$$

Thus, it converges to some positive value. Then there exist $q \in (0,1)$ and $n_0 \in \mathbb{N}$ such that

$$f(x_{n+1}) \leq q^{n-n_0+1} f(x_{n_0}) \text{ for all } n \geq n_0.$$

For $n \geq n_0$,

$$\varphi\left(d(x_n,x_{n+1})\right) < \eta\left(d(x_n,x_{n+1})\right) \leq f(x_n) \leq q^{n-n_0} f(x_{n_0}).$$

Then for $n \geq n_0$ and $p \in \mathbb{N}$,

$$\begin{aligned} \varphi\left(d(x_n,x_{n+p})\right) \leq& \varphi\left(\sum_{k=0}^{p-1} d(x_{n+k},x_{n+k+1})\right) \leq \sum_{k=0}^{p-1} \varphi\left(d(x_{n+k},x_{n+k+1})\right) \\ &\leq \sum_{k=0}^{p-1} q^{n-n_0+k} f(x_{n_0}) \leq \frac{q^{n-n_0}}{1-q} f(x_{n_0}). \end{aligned}$$

Since we may assume that $\varphi(t) > 0$ for $t > 0$ (otherwise $\varphi(t) = 0$ for every $t \geq 0$ and the existence of a fixed point is immediate) we can prove by contradiction that (x_n) is Cauchy and its limit is a fixed point for T. $\square$

In the same manner we can prove the next result.

Theorem 2.10. *Let (X, d) be a complete metric space, $T\colon X \to P_{cl}(X)$ and $f\colon X \to \mathbb{R}, f(x) = D(x, Tx)$ lower semi-continuous. Suppose there exist $\varphi\colon [0, \infty) \to [0, \infty), \eta\colon [0, \infty) \to (0, \infty)$ such that $\varphi(t) < \eta(t)$ for all $t > 0$, η is non-decreasing and subadditive,*

$$\limsup_{r \to t+} \frac{\varphi(r)}{\eta(r)} < 1 \text{ for every } t \in [0, \infty),$$

and for any $x \in X$ there is $y \in Tx$ satisfying

$$\eta(d(x, y)) \leq f(x) \text{ and } f(y) \leq \varphi(d(x, y)).$$

Then T has a fixed point.

Remark 2.11. *For further developments, we can consider the framework given in* [4].

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