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ON CERTAIN STABILITY RESULTS OF BARBET AND NACHI

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Abstract. Stability results for a sequence of mappings in a Hausdorff uniform space are proved. The results obtained here in are the generalizations of the recent results of Barbet and Nachi.
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1. Introduction

The problem of investigating sufficient conditions under which the convergence of a sequence of self mappings on a metric space (X, d) implies the convergence of the sequence of their fixed points has been of continuing interest. In fixed point theory, this problem is known as stability (or continuity) of fixed points. The first result in this direction for contraction mappings stated below is due to Bonsall [7].

Theorem 1.1. *Let (X, d) be a complete metric space, and T and $T_n(n = 1, 2, ...)$ be contraction mappings of X into itself with the same Lipschitz constant $k < 1$, and with fixed points u and $u_n(n = 1, 2, ...)$ respectively. Suppose that $\lim_n T_n x = Tx$ for every $x \in X$. Then $\lim_n u_n = u$.*

The above result also appears in Sonnenschein [14] with a different proof. Moreover, as an application a periodic solution of a nonlinear differential equation is obtained.

The following remarks can be made with respect to Theorem 1.1:

- (a) the condition that all the contraction mappings T_n $(n = 1, 2, ...)$ have the same Lipschitz constant k is too restrictive as one can easily see by the remarks and example given in Nadler [11].
- (b) the assumption that T is a contraction mapping is superfluous as this follows from the fact that $T_n(n = 1, 2, ...)$ is a contraction mapping.
- (c) the completeness condition may be replaced by the assumption of the existence of fixed points for the mappings T and T_n $(n = 1, 2, ...)$.

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Under uniform convergence of the sequence $\{T_n\}$ to T and retaining the essence of (a), (b) and (c) the following stability result was obtained by Nadler [11, Theorem 1].

Theorem 1.2. *Let (X,d) be a metric space, let $T_n : X \to X$ be a mapping with at least one fixed point a_n for each $n = 1, 2, ...$and let $T_0 : X \to X$ be a contraction mapping with fixed point a_0. If the sequence $\{T_n\}_{n=1}^{\infty}$ converges uniformly to T_0, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to a_0.*

It is well known that fixed points can be viewed as solutions of various operator equations and in many cases a localized version (where the domain of definition of a given operator is a nonempty subset of the given space) of a particular fixed point theorem becomes more useful. In respect of stability results, uniform convergence (resp. pointwise convergence) plays a crucial role. However, when the domain of definition of all mappings T_n in question is not the same space X or a unique nonempty subset M of it, the above notions do not work. This difficulty has recently been overcome by Barbet and Nachi [6] (see also, Barbet and Nachi [5] and Nachi [10]) where a number of new notions of convergence have been introduced. Interesting examples presented there in illustrate the generality of their notions over the existing ones. Subsequently, these notions are utilized to generalize Theorem 1.2 above and, in addition, a number of other supporting results are also obtained. In this paper we present a double generalization of the results of Bonsall [7] and Nadler [11] in the sense that the underlying space has been freed to a non-metrizable setting and the nature of convergence is generalized after the style of Barbet and Nachi [6].

2. Preliminaries

Let (X,u) be a uniform space. A family $P = \{p_\alpha : \alpha \in I\}$ of pseudometrics on X, where I is an indexing set is called an associated family for the uniformity u if the family $\beta = \{V(\alpha,r) : \alpha \in I, r > 0\}$, where $V(\alpha,r) = \{(x,y) \in X \times X : p_\alpha(x,y) < r\}$ is a subbase for the uniformity u. We may assume β itself to be a base for u by adjoining finite intersections of members of β if necessary. The corresponding family of pseudometrics is called an augmented associated family for u. An augmented associated family for u will be denoted by P^*. (cf. Mishra [9] and Thron [16]). In view of Kelley [8], we note that each member $V(\alpha,r)$ of β is symmetric and p_α is uniformly continuous on $X \times X$ for each $\alpha \in I$. Further, the uniformity u is not necessarily pseudometrizable (resp. metrizable) unless β is countable and in that case u may be generated by a single pseudometric (resp. a metric) p on X For an interesting motivation we refer to Reilly [12, Example 2] (see also, Kelley [8, Example C, page 204]).

For further details on uniform spaces and a systematic account of fixed points in uniform spaces and their applications, we refer to Weil [17] and Angelov [3] respectively (see also [2]).

Now onwards, unless stated otherwise, X will denote a uniform space (X,u) defined by P^*. $\mathbb{N}$ will denote the set of natural numbers while $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Definition 2.1. ([15]) Let X be a uniform space and let $\{p_\alpha : \alpha \in I\} = P^*$. A mapping $f : X \to X$ is called a P^*−contraction if for each $\alpha \in I$, there exists a real $k(\alpha), 0 < k(\alpha) < 1$ such that $p_\alpha(f(x), f(y)) \leq k(\alpha)p_\alpha(x,y)$ for all $x, y \in X$.

It is well known that $f : X \to X$ is P^*-contraction if and only if it is $P-$contraction (see Tarafdar [15, Remark 1]). Hence, now onward, we shall simply use the term $k-$contraction to mean either of them. In case the above condition is satisfied for any $k = k(\alpha) > 0$, f will be called $k-$Lipschitz.

The following result due to Tarafdar [15, Theorem 1.1] (see also, Acharya [1, Theorem 3.1]) presents an analogue of the well-known Banach contraction principle [4].

Theorem 2.2. *Let X be a Hausdorff complete uniform space and let $\{p_\alpha : \alpha \in I\} = P^*$. Let f be a contraction on X. Then f has a unique fixed point $a \in X$ such that $f^n(x) \to a$ in τ_u (the uniform topology) for each $x \in X$.*

Definition 2.3. Let $\{X_n\}_{n\in\overline{\mathbb{N}}}$ be a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then:

T_∞ is called a $(G)-$limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G), if the following condition holds:

- (G) $Gr(T_\infty) \subset \liminf Gr(T_n)$: for every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$ such that for any $\alpha \in I$, we have $\lim_n p_\alpha(x_n, x) = 0$ and $\lim_n p_\alpha(T_n x_n, T_\infty x) = 0$, where $Gr(T)$ stands for the graph of T.

T_∞ is called a $(G^-)-$limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (G^-), if the following condition holds:

- (G^-) $Gr(T_\infty) \subset \limsup Gr(T_n)$: for every $x \in X_\infty$, there exists a sequence $\{x_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$ and an $s \in S$ such that for any $\alpha \in I$, we have $\lim_n p_\alpha(x_{s(n)}, x) = 0$ and $\lim_n p_\alpha(T_{s(n)} x_{s(n)}, T_\infty x) = 0$, where S denotes the set of all increasing functions $s : \mathbb{N} \to \mathbb{N}$.

Further, T_∞ is called an $(H)-$limit of the sequence $\{T_n\}_{n\in\mathbb{N}}$ or, equivalently $\{T_n\}_{n\in\overline{\mathbb{N}}}$ satisfies the property (H) if the following condition holds:

- (H) For all sequences $\{x_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$, there exists a sequence $\{y_n\}$ in X_∞ such that for any $\alpha \in I$, we have $\lim_n p_\alpha(x_n, y_n) = 0$ and $\lim_n(T_n x_n, T_n y_n) = 0$.

  If X is a metrizable uniform space (i.e., the uniformity u is generated by a metric d), we get the corresponding definitions due to Barbet and Nachi [6].

Remark 2.4. We note the following properties of the above limits. For details we refer the reader to [6].

- (i) a $(G)-$limit need not be unique. However, if T_n is a $k-$contraction (resp. $k-$Lipschitz) for each $n \in \mathbb{N}$, then it is so.
- (ii) a $(H)-$limit need not be unique.
- (iii) when T_∞ is continuous and the condition $X_\infty \subset \liminf X_n$ is satisfied, then the following implications hold: $(H) \Rightarrow (G) \Rightarrow (G^-)$.

  However, without the two restrictions above, we have the relationship $(G) \Rightarrow (G^-)$, $(H) \Rightarrow (G^-)$, whereas a counter example in [6] shows that $(G)-$limit is not necessarily an $(H)-$limit.
- (iv) pointwise convergence $\Rightarrow (G)-$convergence. However, the above implication is not reversible unless $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous on the common domain of definition.
- (v) the interrelationship between the (H) convergence and uniform convergence is captured in Theorem 3.12.

3. Results

Our main results of this section are the two stability results (namely, Theorem 3.3 and Theorem 3.13) which generalize the corresponding results of Barbet and Nachi [6, Theorem 2 and Theorem 11] and which in turn include the results of Bonsall [7] and Nadler [11, Theorem 1]. Other supporting results of this section present the uniform space version of the corresponding results in [6]. For the sake of brevity, we shall present the detailed proofs of Proposition 3.1 and Theorem 3.3 only as far as the analysis in uniform spaces is concerned. For others, we give only outlines that parallel the respective proof techniques of [6].

Proposition 3.1. *Let X be a Hausdorff uniform space,$\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $T_n : X_n \to X$ a $k-$contraction mapping for each $n \in \mathbb{N}$. If $T_\infty : X_\infty \to X$ is a $(G)-$limit of $\{T_n\}_{n\in\mathbb{N}}$, then T_∞ is unique.*

Proof. Let $U \in u$ be an arbitrary entourage. Then, since β is a base for u, there exists $V(\alpha, r) \in \beta, \alpha \in I, r > 0$ such that $V(\alpha, r) \subset U$. Assume that $T_\infty : X_\infty \to X$ and $S_\infty : X_\infty \to X$ are two different $(G)-$limits of the sequence $\{T_n\}_{n\in\mathbb{N}}$. Now by the property (G), for any $x \in X_\infty$, there exist sequences $\{x_n\}$ and $\{y_n\}$ in $\Pi_{n\in\ \mathbb{N}}\ X_n$ such that for any $\alpha \in I$,

$$\lim_n p_\alpha(x_n, x) = 0 \text{ and } \lim_n p_\alpha(T_n x_n, T_\infty x) = 0,$$
$$\lim_n p_\alpha(y_n, x) = 0 \text{ and } \lim_n p_\alpha(T_n y_n, S_\infty x) = 0.$$

Further, since T_n is a $k-$contraction, for any $\alpha \in I$, there exists a constant $k(\alpha)$ such that $p_\alpha(T_n x_n, T_n y_n) \le k(\alpha) p_\alpha(x_n, y_n) \le k(\alpha) [p_\alpha(x_n, x) + p_\alpha(x, y_n)] \to 0$ as $n \to \infty$.

Now for any $n \in \mathbb{N}$ and $\alpha \in I$,

$$\begin{aligned} p_\alpha(T_\infty x, S_\infty x) &\le p_\alpha(T_\infty x, T_n x_n) + p_\alpha(T_n x_n, T_n y_n) + p_\alpha(T_n y_n, S_\infty x) \\ &\le p_\alpha(T_\infty x, T_n x_n) + k(\alpha) p_\alpha(x_{n,}\ y_n) + p_\alpha(T_n y_n, S_\infty x). \end{aligned}$$

The R.H.S. of the above expression tends to 0 as $n \to \infty$. Hence $p_\alpha\,(T_\infty x, S_\infty x) < r$ for all $n \ge N(\alpha, r)$. Therefore $(T_\infty x, S_\infty x) \in V(\alpha, r) \subset U$ and since X is Hausdorff, it follows that $T_\infty x = S_\infty x$. □

Remark 3.2. The above proposition still remains true if each T_n is $k-$Lipschitz in stead of $k-$contraction for each $n \in \mathbb{N}$. Further, if X is metrizable, then we obtain a result of Barbet and Nachi [6, Proposition 1] in this case.

The following theorem presents our first stability result.

Theorem 3.3. *Let X be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and T_n is a $k-$contraction for each $n \in \mathbb{N}$. If x_n is a fixed point of T_n for each $n \in \overline{\mathbb{N}}$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. Let $W \in u$ be arbitrary. Then there exists $V(\lambda, r) \in \beta, \lambda \in I, r > 0$ such that $V(\lambda, r) \subset W$. By the property (G) and the fact that $x_\infty \in X_\infty$, there exists a sequence $\{y_n\}$ in X_n for all $n \in \mathbb{N}$ such that for any $\lambda \in I$,

$$\lim_n p_\lambda\,(y_n, x_\infty) = 0 \text{ and } \lim_n p_\lambda\,(T_n y_n, T_\infty x_\infty) = 0.$$

Hence by using the contractive condition for T_n, for any $\lambda \in I$, we have

$$\begin{aligned} p_\lambda (x_n, x_\infty) &\le p_\lambda (T_n x_n, T_n y_n) + p_\lambda (T_n y_n, T_\infty x_\infty) \\ &\le k(\lambda) p_\lambda (x_n, y_n) + p_\lambda (T_n\, y_n, T_\infty x_\infty) \\ &\le k(\lambda) [p_\lambda (x_n, x_\infty) + p_\lambda (x_\infty, y_n)] + p_\lambda (T_n y_n, T_\infty x_\infty) \end{aligned}$$

Thus, $(1 - k(\lambda)) p_\lambda (x_n, x_\infty) \le k(\lambda) p_\lambda (x_\infty, y_n) + p_\lambda (T_n y_n, T_\infty x_\infty) \to 0$ as $n \to \infty$ and since $k(\lambda) < 1$, it follows that $p_\lambda (x_n, x_\infty) \to 0$ as $n \to \infty$. Hence $p_\lambda (x_n, x_\infty) < r$ for all $n \ge N(\lambda, r)$ and so $(x_n, x_\infty) \in V(\lambda, r) \subset W$ and the conclusion follows. □

In case X is metrizable we obtain Theorem 2 of Barbet and Nachi [6] which in turn includes the result of Bonsall [7] (where X is complete and $X_n = X$ for all $n \in \overline{\mathbb{N}}$).

Proposition 3.4. *Let X be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and that T_n is a contraction mapping with contraction constant k_n for each $n \in \mathbb{N}$ such that $\lim_n k_n = k < 1$. Then T_∞ is $k-$contraction.*

Proof. By the property (G), for any two points $x, y \in X_\infty$, there exist sequences $\{x_n\}, \{y_n\}$ in $\Pi_n X_n$ such that for any $\alpha \in I$

$$\begin{aligned} &\lim_n p_\alpha(x_n, x) = 0 \text{ and } \lim_n p_\alpha (T_n x_n, T_\infty x) = 0, \\ &\lim_n p_\alpha (y_n, y) = 0 \text{ and } \lim_n p_\alpha (T_n y_n, T_\infty y) = 0 \end{aligned}$$

Further, for any $n \in \mathbb{N}$ and for $\alpha \in I$,

$$\begin{aligned} p_\alpha (T_\infty x, T_\infty y) &\le p_\alpha (T_\infty x, T_n x_n) + p_\alpha (T_n x_n, T_n y_n) + p_\alpha (T_n y_n, T_\infty y) \\ &\le p_\alpha (T_\infty x, T_n x_n) + k_n p_\alpha (x_n, y_n) + p_\alpha (T_n y_n, T_\infty y) \end{aligned}$$

Now making $n \to \infty$, the conclusion follows. □

The following result that generalizes Proposition 4 of Barbet and Nachi [6] can be easily proved by following the proof techniques of Proposition 3.4.

Proposition 3.5. *Let X be a Hausdorff uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and that T_n is a Lipscitz mapping with Lipscitz constant k_n for each $n \in \mathbb{N}$ and $\{k_n\}$ is bounded (resp. convergent). Then T_∞ is $k-$Lipschitz with $k = \lim_n \sup k_n$ (resp. $\lim_n k_n = k$).*

As noted in Remark 2.4, in general uniform convergence and $(G)-$convergence are not equivalent (see [6], the Example on page 53). When the domain of definition is a unique nonempty subset of the space X, the following theorem establishes the required equivalence.

Proposition 3.6. *Let M be a nonempty subset of a uniform space X and $\{T_n : M \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that the sequence $\{T_n\}_{n\in\overline{\mathbb{N}}}$ is equicontinuous on M. Then $\{T_n\}_{n\in\overline{\mathbb{N}}}$ converges to T_∞.*

Proof. The necessary part follows from Theorem 3.3. To prove the sufficiency, assume that the sequence $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous on M with its $(G)-$limit T_∞. Then given any $x \in X_\infty$, there exists a sequence $\{x_n\}$ in M such that for any $\alpha \in I$ $\lim_n p_\alpha (x_n, x\ ) = 0$ and $\lim_n p_\alpha (T_n x_n, T_\infty x\ ) = 0$. Since $\{T_n\}_{n\in\mathbb{N}}$ is equicontinuous on M, we have for any $\alpha \in I$, $p_\alpha(T_n x_n, T_n x) \to 0$ as $n \to \infty$. Hence $p_\alpha(T_n x, T_\infty x) \to 0$ as $n \to \infty$ and the conclusion follows. □

Corollary 3.7. *Let X be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for any $n \in \mathbb{N}$, T_n is a $k-$contraction. Assume that x_n is a fixed point of T_n for each $n \in \mathbb{N}$. Then, T_∞ admits a fixed point $\Leftrightarrow \{x_n\}$ converges to a point in $X_\infty \Leftrightarrow \{x_n\}$ admits a subsequence converging to a point in X_∞.*

Remark 3.8. Under the assumptions of Corollary 3.7 and if

(*i*) $\liminf X_n \subset X_\infty$(i.e., the limit of any convergent sequence $\{x_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$ is in X_∞), then

T_∞ admits a fixed point $\Leftrightarrow \{x_n\}$ converges to a point in X_∞;

(*ii*) $\limsup X_n \subset X_\infty$ (i.e., any cluster point of a sequence $\{x_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$ is in X_∞), then

T_∞ admits a fixed point $\Leftrightarrow \{x_n\}$ admits a subsequence converging to a point in X_∞.

The following theorem, which is a simple consequence of the Remark 3.8 and Theorem 3.3, ensures the existence of a fixed point of the $(G)-$limit map from the existence of fixed points of the contraction map T_n under certain compactness condition.

Theorem 3.9. *Let X be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G) and such that for each $n \in \mathbb{N}$, T_n is a $k-$contraction mapping. Assume that $\limsup X_n \subset X_\infty$ and $\bigcup_{n\in\mathbb{N}} X_n$ is relatively compact. If for any $n \in \mathbb{N}$, T_n admits a fixed point x_n, then the $(G)-$limit map T_∞ admits a fixed point x_∞ and the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Theorem 3.10. *Let X be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (G^-). If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then x_∞ is a cluster point of the sequence $\{x_n\}_{n\in\mathbb{N}}$.*

Proof. By the property (G^-), there exists a sequence $\{y_n\}$ in $\Pi_{n\in\mathbb{N}} X_n$ with a subsequence $\{y_{s(n)}\}$ such that for any $\alpha \in I$, we have that $\lim_n p_\alpha(y_{s(n)}, x_\infty) = 0$ and $\lim_n p_\alpha(T_{s(n)} y_{s(n)}, T_\infty x_\infty) = 0$. Therefore for each $\alpha \in I$ and for each $n \in \mathbb{N}$, we have

$$\begin{aligned} p_\alpha(x_{n(s)}, x_\infty) = p_\alpha(T_{s(n)} x_{s(n)}, T_\infty x_\infty) &\le p_\alpha(T_{s(n)} x_{s(n)}, T_{s(n)} y_{s(n)}) \\ &+ p_\alpha(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \\ &\le k(\alpha) p_\alpha(x_{s(n)}, y_{s(n)}) + p_\alpha(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \\ &\le k(\alpha)\left[p_\alpha(x_{s(n)}, x_\infty) + p_\alpha(y_{s(n)}, x_\infty)\right] \\ &+ p_\alpha(T_{s(n)} y_{s(n)}, T_\infty x_\infty) \end{aligned}$$

Thus $(1 - k(\alpha))\, p_\alpha(x_{n(s)}, x_\infty) \le k(\alpha) p_\alpha(y_{s(n)}, x_\infty + p_\alpha(T_{s(n)} y_{s(n)}, T_\infty x_\infty)$. Since $1 - k(\alpha) < 1$, we have that $\{x_{s(n)}\}$ converges to x_∞ as $n \to \infty$. □

The following result reveals a relationship between the $(G)-$ convergence and $(H)-$convergence.

Proposition 3.11. *Let X be a uniform space, $\{X_n\}$ a family of nonempty subsets of X such that $X_\infty \subset \liminf X_n$. Let $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ be a family of mappings such that T_∞ is continuous on X_∞. If T_∞ is a $(H)-$limit of $\{T_n\}_{n\in\mathbb{N}}$, then T_∞ is a $(G)-$limit of $\{T_n\}$.*

Proof. Let $x \in X_\infty$. Then since $X_\infty \subset \liminf X_n$, there exists a sequence $\{x_n\}$ in X such that $x_n \in X_n$ and $x_n \to x$ as $n \to \infty$. Further, by the property (H),there exists a sequence $\{y_n\}$ in X_∞ such that for each $\alpha \in I$ and $n \in \mathbb{N}$, $p_\alpha(x_n, y_n) \to 0$ and $p_\alpha(T_n x_n, T_\infty y_n) \to 0$ as $n \to \infty$. Consequently, $y_n \to x$ as $n \to \infty$. Hence by the continuity of T_∞, we have $T_\infty y_n \to T_\infty x$ as $n \to \infty$. Now by the triangle inequality

$$p_\alpha(T_n x_n, T_\infty x) \leq p_\alpha(T_n x_n, T_\infty y_n) + p_\alpha(T_\infty y_n, T_\infty x_n)$$

we conclude that $T_n x_n \to T_\infty x$ as $n \to \infty$ and the property (G) holds. □

When $X_n = M \neq \phi$ for all $n \in \overline{\mathbb{N}}$, the following proposition presents a comparision between the uniform convergence and $(H)-$ convergence. However, as noted in [6, Example p. 56], in general the $(H)-$convergence need not imply uniform convergence.

Proposition 3.12. *Let X be a uniform space, M a nonempty subset of X and $\{T_n : M \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings. Then:*

- (a) *$\{T_n\}_{n\in\mathbb{N}}$ converges uniformly to T_∞ on $M \Rightarrow T_\infty$ is a $(H)-$limit of $\{T_n\}_{n\in\mathbb{N}}$.*
- (b) *the converse holds when T_∞ is uniformly continuous on M.*

Proof. We shall prove (b) as (a) is obvious. Suppose (b) does not hold. So, let the limit map T_∞ be uniformly continuous on M and $\{T_n\}$ does not converge uniformly to T_∞. Hence there exists a sequence $\{x_n\}$ in M such that for any $\alpha \in I$, $\lim_n p_\alpha(T_n x_n, T_\infty x_n) \neq 0$. Now, if the property (H) holds, then there exists a sequence $\{y_n\}$ in M such that for any $\alpha \in I$ $\lim_n p_\alpha(x_n, y_n) = 0$ and $\lim_n p_\alpha(T_n x_n, T_\infty y_n) = 0$, so that $p_\alpha(T_n x_n, T_\infty x_n) \to 0$ as $n \to \infty$, a contradiction. Hence (b) must hold. □

The following theorem presents our second stability result.

Theorem 3.13. *Let X be a uniform space, $\{X_n\}_{n\in\overline{\mathbb{N}}}$ a family of nonempty subsets of X and $\{T_n : X_n \to X\}_{n\in\overline{\mathbb{N}}}$ a family of mappings satisfying the property (H) and such that T_∞ is a $k_\infty-$contraction. If for any $n \in \overline{\mathbb{N}}$, x_n is a fixed point of T_n, then $\{x_n\}_{n\in\mathbb{N}}$ converges to x_∞.*

Proof. By the property (H), there exists a sequence $\{y_n\}$ in X_∞ such that for all $\alpha \in I$ we have that $\lim_n p_\alpha(x_n, y_n) = 0$ and $\lim_n p_\alpha(T_n x_n, T_\infty y_n) = 0$.

Using the triangle inequality, it can be easily verified that for any $\alpha \in I$,

$$p_\alpha(x_n, x_\infty) \leq \frac{1}{(1 - k_\infty(\alpha))}\left[p_\alpha(T_n x_n, T_\infty y_n) + k_\infty(\alpha) p_\alpha(y_n, x_n)\right]$$

and making $n \to \infty$, we obtain that $\{x_n\}$ converges to x_∞.

Remark 3.14. If X is metrizable, then we get a stability result of Barbet and Nach [6, Theorem 11} which in turn includes a result of Nadler [11, Theorem 1].

Remark 3.15. Every locally convex topological vector space X is uniformzable being completely regular (cf. Kelley [8], Shaefer [13]) where the family of pseudometric $\{p_\alpha, \alpha \in I\}$is induced by a family of seminorm $\{\rho_\alpha, \alpha \in I\}$ so that $p_\alpha(x, y) = \rho_\alpha(x-y)$ for all x, $y \in X$. Therefore all the results proved previously for uniform spaces also apply to locally convex spaces.

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