WEAK AND STRONG MEAN CONVERGENCE THEOREMS FOR SUPER HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first introduce a class of nonlinear mappings called extended hybrid in a Hilbert space containing the class of generalized hybrid mappings. The class is different from the class of super hybrid mappings which was defined by Kocourek, Takahashi and Yao [12]. We prove a fixed point theorem for generalized hybrid nonself-mapping in a Hilbert space. Next, we prove a nonlinear ergodic theorem of Baillon’s type for super hybrid mappings in a Hilbert space. Finally, we deal with two strong convergence theorems of Halpern’s type for these nonlinear mappings in a Hilbert space.

Key Words and Phrases: Hilbert space, nonexpansive mapping, nonspreading mapping, hybrid mapping, fixed point, mean convergence.

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1. Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. Then a mapping $T : C \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. The set of fixed points of $T$ is denoted by $F(T)$. From Baillon [2] we know the following first nonlinear ergodic theorem in a Hilbert space.

Theorem 1.1. Let $C$ be a nonempty closed convex subset of $H$ and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

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converges weakly to an element \( z \in F(T) \).

The following strong convergence theorem of Halpern’s type [7] was proved by Wittmann [26]; see also [19].

**Theorem 1.2.** Let \( C \) be a nonempty closed convex subset of \( H \) and let \( T : C \to C \) be a nonexpansive mapping with \( F(T) \neq \emptyset \). For any \( x_1 = x \in C \), define a sequence \( \{x_n\} \) in \( C \) by

\[
x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n, \quad \forall n = 1, 2, ...,
\]

where \( \{\alpha_n\} \subset [0, 1] \) satisfies \( \alpha_n \to 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and \( \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \). Then \( \{x_n\} \) converges strongly to a fixed point of \( T \).

An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. Let \( C \) be a nonempty subset of \( H \). A mapping \( F : C \to H \) is said to be firmly nonexpansive [17] if

\[
\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy \rangle
\]

for all \( x, y \in C \); see, for instance, Browder [4] and Goebel and Kirk [6]. It is known that a firmly nonexpansive mapping \( F \) can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [3] and [5]. Recently, Kohsaka and Takahashi [14], and Takahashi [21] introduced the following nonlinear mappings which are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping \( T : C \to H \) is called nonspreading [14] if

\[
2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2
\]

for all \( x, y \in C \). They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [13] and Iemoto and Takahashi [10]. Very recently, Kocourek, Takahashi and Yao [12] introduced a broad class of mappings \( T : C \to H \) called generalized hybrid such that for some \( \alpha, \beta \in \mathbb{R} \),

\[
\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
\]

for all \( x, y \in C \). Such a class contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Further, they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings in a Hilbert space. Such a class is called a class of super hybrid mappings. A generalized hybrid mapping with a fixed point is quasi-nonexpansive. However, a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point.

In this paper, we first introduce a class of nonlinear mappings called extended hybrid in a Hilbert space containing the class of generalized hybrid mappings. The class is different from the class of super hybrid mappings which was defined by Kocourek, Takahashi and Yao [12]. We prove a fixed point theorem for generalized hybrid nonself-mapping in a Hilbert space. Next, we prove a nonlinear ergodic theorem of Baillon’s type for super hybrid mappings in a Hilbert space. Finally, we deal with two strong convergence theorems of Halpern’s type for these nonlinear mappings in a Hilbert space.
2. Preliminaries

Throughout this paper, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{R} \) the set of real numbers. Let \( H \) be a (real) Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \), respectively. We denote the strong convergence and the weak convergence of \( \{x_n\} \) to \( x \in H \) by \( x_n \to x \) and \( x_n \rightharpoonup x \), respectively. From [20], we know the following basic equality: For \( x, y, u, v \in H \) and \( \lambda \in \mathbb{R} \), we have

\[
\| \lambda x + (1 - \lambda)y \|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}
\]

Further, we know that for \( x, y, u, v \in H \)

\[
2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}
\]

From (2.2), we have also the following equality.

\[
\|x - y + u - v\|^2 = \|x - y\|^2 + \|u - v\|^2 + 2 \langle x - y, u - v \rangle = \|x - y\|^2 + \|u - v\|^2 + \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.3}
\]

Let \( C \) be a nonempty closed convex subset of \( H \) and let \( T \) be a mapping from \( C \) into itself. Then, we denote by \( F(T) \) the set of fixed points of \( T \). A mapping \( T: C \to H \) is said to be nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \) for all \( x, y \in C \). A mapping \( T: C \to H \) with \( F(T) \neq \emptyset \) is called quasi-nonexpansive if \( \|x - Ty\| \leq \|x - y\| \) for all \( x \in F(T) \) and \( y \in C \). It is well-known that the set \( F(T) \) of fixed points of a quasi-nonexpansive mapping \( T \) is closed and convex; see Ito and Takahashi [11]. Let \( C \) be a nonempty closed convex subset of \( H \) and \( x \in H \). Then, we know that there exists a unique nearest point \( z \in C \) such that \( \|x - z\| = \inf_{y \in C} \|x - y\| \). We denote such a correspondence by \( z = P_C x \). \( P_C \) is called the metric projection of \( H \) onto \( C \).

It is known that \( P_C \) is nonexpansive and

\[
\langle x - P_Cx, P_Cx - u \rangle \geq 0
\]

for all \( x \in H \) and \( u \in C \). Further, we know that

\[
\|P_Cx - P_Cy\|^2 \leq \langle x - y, P_Cx - P_Cy \rangle \tag{2.4}
\]

for all \( x, y \in H \); see [20] for more details. The following lemma was proved by Takahashi and Toyoda [23].

**Lemma 2.1.** Let \( D \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( P \) be the metric projection of \( H \) onto \( D \) and let \( \{x_n\} \) be a sequence in \( H \). If \( \|x_{n+1} - u\| \leq \|x_n - u\| \) for all \( u \in D \) and \( n \in \mathbb{N} \), then \( \{Px_n\} \) converges strongly.

Let \( C \) be a nonempty subset of \( H \). Then, a nonself-mapping \( T: C \to H \) is called generalized hybrid [12] if there are \( \alpha, \beta \in \mathbb{R} \) such that

\[
\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.5}
\]

for all \( x, y \in C \). We call such a mapping an \((\alpha, \beta)\)-generalized hybrid mapping. We observe that the mapping above covers several well-known mappings. For example, an \((\alpha, \beta)\)-generalized hybrid mapping is nonexpansive for \( \alpha = 1 \) and \( \beta = 0 \), nonspreading
for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,
\[
\alpha \|x - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|x - y\|^2 + (1 - \beta)\|x - y\|^2
\]
and hence $\|x - Ty\| \leq \|x - y\|$. This means that an $(\alpha, \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive.

Let $C$ be a nonempty subset of a Hilbert space $H$. A mapping $S : C \to H$ is called super hybrid [12, 25] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\[
\alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \leq \\
(\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\
+ (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2
\]
for all $x, y \in C$. We call such a mapping an $(\alpha, \beta, \gamma)$-super hybrid mapping. An $(\alpha, \beta, 0)$-super hybrid mapping is $(\alpha, \beta)$-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. Let us consider a super hybrid mapping $S$ with $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. Then, we have
\[
\|Sx - Sy\|^2 + \|x - Sy\|^2 \leq -\|Sx - y\|^2 + 3\|x - y\|^2
\]
for all $x, y \in C$. This is equivalent to
\[
\|Sx - Sy\|^2 + 2(x - y, Sx - Sy) \leq 3\|x - y\|^2
\]
for all $x, y \in C$. In the case of $H = \mathbb{R}$, consider $Sx = 2\cos x - x$ for all $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Then, we have
\[
|Sx - Sy|^2 + 2(x - y, Sx - Sy) \\
= |2\cos x - x - (2\cos y - y)|^2 + 2(x - y, 2\cos x - x - (2\cos y - y)) \\
= 4(\cos x - \cos y)^2 - 2(x - y, 2\cos x - 2\cos y) + (x - y)^2 \\
- (x - y)^2 + 2(x - y, 2\cos x - 2\cos y) \\
\leq 4(x - y)^2 - (x - y)^2 \\
= 3(x - y)^2
\]
and hence $S$ is super hybrid. However, $S$ is not quasi-nonexpansive. Further, we have that
\[
Tx = \frac{1}{2}(2\cos x - x) + \frac{1}{2}x = \cos x
\]
and hence $T$ is a nonexpansive mapping with a fixed point. The following theorem was proved in [25] and [12].

**Theorem 2.2.** Let $C$ be a nonempty subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $S$ and $T$ be mappings of $C$ into $H$ such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, $S$ is $(\alpha, \beta, \gamma)$-super hybrid if and only if $T$ is $(\alpha, \beta)$-generalized hybrid. In this case, $F(S) = F(T)$. In particular, let $C$ be a nonempty closed and convex subset of $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. If a mapping $S : C \to C$ is $(\alpha, \beta, \gamma)$-super hybrid, then the mapping $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$ is an $(\alpha, \beta)$-generalized hybrid mapping of $C$ into itself.
Kocourek, Takahashi and Yao [12] also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

**Theorem 2.3.** Let $C$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an $(\alpha, \beta, \gamma)$-super hybrid mapping. Then, $S$ has a fixed point in $C$. In particular, if $S : C \to C$ be an $(\alpha, \beta)$-generalized hybrid mapping, then $S$ has a fixed point in $C$.

To prove one of our main results, we need the following lemma [1]:

**Lemma 2.4.** Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n \to \infty} \gamma_n \leq 0$. Suppose that

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n \to \infty} s_n = 0$.

### 3. Fixed Point Theorem for Non-Self Mappings

In this section, we prove a fixed point theorem for generalized hybrid nonself-mappings in a Hilbert space. Before proving it, we need the following lemma.

**Lemma 3.1.** Let $H$ be a Hilbert space and let $C$ be a nonempty subset of $H$. Let $\alpha$ and $\beta$ be in $\mathbb{R}$. Then, a nonself-mapping $T : C \to H$ is $(\alpha, \beta)$-generalized hybrid if and only if it satisfies that

$$\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)\langle x - Tx, y - Ty \rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

for all $x, y \in C$.

**Proof.** We have that for all $x, y \in C$,

$$\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)\langle x - Tx, y - Ty \rangle - (\alpha - \beta - 1)\|y - Tx\|^2$$

$$\iff \|Tx - Ty\|^2 \leq (1 - \beta)\|x - y\|^2 + (\alpha - 1)\|x - y\|^2 + (\alpha - 1)\|Tx - Ty\|^2$$

$$\iff \alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2.$$

Using Lemma 3.1, we have the following result.

**Lemma 3.2.** Let $H$ be a Hilbert space and let $C$ be a nonempty bounded subset of $H$. If a nonself-mapping $T : C \to H$ is generalized hybrid, then $TC$ is bounded.
Proof. Since $T : C \to H$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \quad (3.1)$$

for all $x, y \in C$. We have from Lemma 3.1 that

$$\|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2$$

$$+ 2(\alpha - 1)(x - Tx, y - Ty) - (\alpha - \beta - 1)\|y - Tx\|^2$$

for all $x, y \in C$. Fix $z \in C$. Then, we have that for any $y \in C$,

$$\|Tz - Ty\|^2 \leq (\alpha - \beta)\|z - y\|^2$$

$$+ 2(\alpha - 1)(z - Tz, y - Ty) - (\alpha - \beta - 1)\|y - Tz\|^2$$

$$\leq |\alpha - \beta|\|z - y\|^2$$

$$+ 2|\alpha - 1|\|z - Tz\\|\|y - Ty\| + |\alpha - \beta - 1|\|y - Tz\|^2$$

$$= |\alpha - \beta|\|z - y\|^2$$

$$+ |\alpha - 1|\|z - Tz\|(\|y - Tz\| + \|Tz - Ty\|) + |\alpha - \beta - 1|\|y - Tz\|^2.$$  

So, $\{\|Tz - Ty\| : y \in C\}$ is bounded and hence $TC$ is bounded. \hfill \Box

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers. Then, $U : C \to H$ is called an $(\alpha, \beta, \gamma)$-extended hybrid mapping if

$$\alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2$$

$$\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2$$

$$- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2$$

for all $x \in C$.

**Theorem 3.3.** Let $C$ be a nonempty closed convex subset of a Hilbert space $H$ and let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$. Let $T$ and $U$ be mappings of $C$ into $H$ such that $U = \frac{1}{1 + \gamma}T + \frac{\gamma}{1 + \gamma}I$, where $Ix = x$ for all $x \in H$. Then, for $1 + \gamma > 0$, $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping if and only if $U : C \to H$ is an $(\alpha, \beta, \gamma)$-extended hybrid mapping.

**Proof.** Since $U = \frac{1}{1 + \gamma}T + \frac{\gamma}{1 + \gamma}I$, we have $T = (1 + \gamma)U - \gamma I$. So, we have from Theorem 3.1 that for any $x, y \in C$,

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2$$

$$\leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

$$\iff \|Tx - Ty\|^2 \leq (\alpha - \beta)\|x - y\|^2$$

$$+ 2(\alpha - 1)(x - Tx, y - Ty) - (\alpha - \beta - 1)\|y - Tx\|^2$$

$$\iff \|(1 + \gamma)(x - Tx, y - Ty) - (\alpha - \beta - 1)\|y - Tx\|^2$$

$$\leq (\alpha - \beta)\|x - y\|^2$$

$$+ 2(\alpha - 1)((1 + \gamma)(x - Ux), (1 + \gamma)(y - Uy))$$

$$- (\alpha - \beta - 1)\|y - (1 + \gamma)Ux + \gamma x\|^2.$$
\[ \Leftrightarrow \| (1 + \gamma)(Ux - Uy) - \gamma(x - y) \|^2 \leq (\alpha - \beta)\|x - y\|^2 + 2(\alpha - 1)(1 + \gamma)^2\langle x - Ux, y - Uy \rangle \\
- (\alpha - \beta - 1)\|y - Ux + \gamma(x - Ux)\|^2 \]
\[ \Leftrightarrow \alpha(1 + \gamma)^2\|Ux - Uy\|^2 + (1 + \gamma)(1 - \alpha(1 + \gamma))\|x - Uy\|^2 \leq (1 + \gamma)(\beta + \alpha\gamma)\|Ux - y\|^2 + (1 + \gamma)(1 - \beta - \alpha\gamma)\|x - y\|^2 \\
- (1 + \gamma)\gamma(\alpha - \beta)\|x - Sx\|^2 - \gamma(1 + \gamma)\|y - Uy\|^2 \]
\[ \Leftrightarrow \alpha(1 + \gamma)^2\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - \beta - \alpha\gamma)\|x - y\|^2 \\
- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2. \]

This completes the proof. \( \square \)

**Theorem 3.4.** Let \( C \) be a nonempty bounded closed convex subset of a Hilbert space \( H \) and let \( \alpha \) and \( \beta \) be real numbers. Let \( T \) be an \((\alpha, \beta)\)-generalized hybrid mapping with \( \alpha - \beta \geq 0 \) of \( C \) into \( H \). Suppose that there exists \( m > 1 \) such that for any \( x \in C \), \( Tx = x + t(y - x) \) for some \( y \in C \) and \( t \) with \( 1 \leq t \leq m \). Then, \( T \) has a fixed point in \( C \).

**Proof.** By the assumption, we have that for any \( x \in C \), there are \( y \in C \) and \( t \) with \( 1 \leq t \leq m \) such that \( Tx = x + t(y - x) \). We have \( Tx = ty + (1 - t)x \) and hence \( y = \frac{1}{t}Tx + \frac{1 - t}{t}x. \) Define \( Ux \in C \) as follows:

\[ Ux = (1 - \frac{t}{m})x + \frac{t}{m}(\frac{1}{t}Tx + \frac{1 - t}{t}x). \]

So, we have \( Ux = \frac{1}{m}Tx + \frac{\gamma - 1}{m}x. \) Taking \( \gamma > 0 \) with \( m = 1 + \gamma \), we have

\[ Ux = \frac{1}{1 + \gamma}Tx + \frac{\gamma}{1 + \gamma}x. \quad (3.2) \]

Thus, we can define a mapping \( U \) of \( C \) into itself satisfying (3.2). Since \( T : C \to H \) is an \((\alpha, \beta)\)-generalized hybrid mapping with \( \alpha - \beta \geq 0 \), from Theorem 3.3 \( U \) is an \((\alpha, \beta, \gamma)\)-extended hybrid mapping of \( C \) into itself, i.e.,

\[ \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\
- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2 \]

for all \( x \in C \). From \( \alpha - \beta \geq 0 \) and \( \gamma > 0 \), we have

\[ \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \]

for all \( x \in C \). This implies that \( U \) is an \((\alpha(1 + \gamma), \beta + \alpha\gamma)\)-generalized hybrid mapping of \( C \) into itself. So, we have a fixed point from Theorem 2.3. This completes the proof. \( \square \)
Let us give an example of mappings $T : C \to H$ such that for any $x \in C$, there are $y \in C$ and $t$ with $1 \leq t \leq m$ such that $Tx = x + t(y - x)$. In the case of $H = \mathbb{R}$, consider a mapping $T : [0, \frac{\pi}{2}] \to \mathbb{R}$:

$$Tx = (1 + 2x)\cos x - 2x^2, \quad \forall x \in [0, \frac{\pi}{2}].$$

Then, we have

$$Tx = (1 + 2x)(\cos x - x) + x, \quad \forall x \in [0, \frac{\pi}{2}].$$

For any $x \in [0, \frac{\pi}{2}]$, take $t = 1 + 2x$, $y = \cos x$ and $m = 1 + \pi$. Then, we have $Tx = t(y - x) + x, y = \cos x \in [0, \frac{\pi}{2}]$ and $1 \leq t = 1 + 2x \leq 1 + \pi$.

4. Nonlinear Ergodic Theorem

In this section, using the technique developed by Takahashi [17], we prove a nonlinear ergodic theorem of Baillon’s type [2] for super hybrid mappings in a Hilbert space. Before proving it, we need the following lemma.

**Lemma 4.1.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping from $C$ into itself. Suppose that $\{T^n x\}$ is bounded for some $x \in C$. Define $S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$. Then, $\lim_{n \to \infty} \|S_n x - T S_n x\| = 0$. In particular, if $C$ is bounded, then

$$\lim_{n \to \infty} \sup_{x \in C} \|S_n x - T S_n x\| = 0.$$

**Proof.** Since $T : C \to C$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. For any $y \in C$ and $k \in \mathbb{N}$, we have that

$$0 \leq \beta \|T^{k+1} x - y\|^2 + (1 - \beta)\|T^k x - y\|^2$$

$$- \alpha \|T^{k+1} x - Ty\|^2 - (1 - \alpha)\|T^k x - Ty\|^2$$

$$= \beta \left\{ \|T^{k+1} x - Ty\|^2 + 2 \langle T^{k+1} x - Ty, Ty - y \rangle + \|Ty - y\|^2 \right\}$$

$$+ (1 - \beta) \left\{ \|T^k x - Ty\|^2 + 2 \langle T^k x - Ty, Ty - y \rangle + \|Ty - y\|^2 \right\}$$

$$- \alpha \|T^{k+1} x - Ty\|^2 - (1 - \alpha)\|T^k x - Ty\|^2$$

$$= \|Ty - y\|^2 + 2 \langle \beta T^{k+1} x + (1 - \beta)T^k x, Ty - y \rangle$$

$$+ (\beta - \alpha) \{ \|T^{k+1} x - Ty\|^2 - \|T^k x - Ty\|^2 \}$$

$$= \|Ty - y\|^2 + 2 \langle T^k x - Ty + \beta (T^{k+1} x - T^k x), Ty - y \rangle$$

$$+ (\beta - \alpha) \{ \|T^{k+1} x - Ty\|^2 - \|T^k x - Ty\|^2 \}.$$
Deviding this inequality by \( n \), we have
\[
0 \leq \|Ty - y\|^2 + 2(S_n x - Ty, Ty - y) + \frac{1}{n}2\beta(T^{n+1}x - Tx, Ty - y) \\
+ \frac{1}{n}(\beta - \alpha)(\|T^{n+1}x - Ty\|^2 - \|Tx - Ty\|^2).
\]
where \( S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x \). Replacing \( y \) by \( S_n x \), we obtain
\[
0 \leq \|TS_n x - S_n x\|^2 \\
+ 2(S_n x - TS_n x, TS_n x - S_n x) + \frac{1}{n}2\beta(T^{n+1}x - Tx, TS_n x - S_n x) \\
+ \frac{1}{n}(\beta - \alpha)(\|T^{n+1}x - TS_n x\|^2 - \|Tx - TS_n x\|^2)
\]
and hence
\[
\|TS_n x - S_n x\|^2 \leq \frac{1}{n}2\beta(T^{n+1}x - Tx, TS_n x - S_n x) \\
+ \frac{1}{n}(\beta - \alpha)(\|T^{n+1}x - TS_n x\|^2 - \|Tx - TS_n x\|^2).
\]
By the assumption, \( \{T^n x\} \) is bounded. So, \( \{S_n x\} \) is also bounded. By Lemma 3.2, \( \{TS_n x\} \) is bounded. So, we have \( \limsup_{n \to \infty} \|S_n x - TS_n x\| \leq 0 \) and hence \( \lim_{n \to \infty} \|S_n x - TS_n x\| = 0 \). In particular, if \( C \) is bounded, then we have
\[
\limsup_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| \leq 0
\]
and hence \( \limsup_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0 \). This completes the proof. \( \square \)

**Theorem 4.2.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha, \beta \) and \( \gamma \) be real numbers with \( \gamma \geq 0 \) and let \( S : C \to C \) be an \( (\alpha, \beta, \gamma) \)-super hybrid mapping with \( F(S) \neq \emptyset \) and let \( P \) be the metric projection of \( H \) onto \( F(T) \). Then, for any \( x \in C \),
\[
S_n x = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \right)^k x
\]
converges weakly to \( z \in F(S) \), where \( z = \lim_{n \to \infty} PT^n x \) and \( T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \).

**Proof.** Put \( T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I \). From Theorem 2.2, we have that \( T : C \to C \) is an \( (\alpha, \beta) \)-generalized hybrid mapping, i.e.,
\[
\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
\]
for all \( x, y \in C \). Since \( T \) is a generalized hybrid mapping and \( F(T) = F(S) \neq \emptyset \), \( T \) is quasi-nonexpansive. So, \( F(T) \) is closed and convex. Let \( x \in C \) and \( u \in F(T) \). Then, we have \( \|T^{n+1}x - u\| \leq \|T^n x - u\| \). Putting \( D = F(T) \) in Lemma 2.1, we have that \( \lim_{n \to \infty} PT^n x \) converges strongly. Put \( z = \lim_{n \to \infty} PT^n x \). Let us show \( S_n x \to z \). Since \( \{T^n x\} \) is bounded, so is \( \{S_n x\} \). Let \( \{S_n x\} \) be a subsequence of \( \{S_n x\} \) such
that $S_n x \to v$. By Lemma 4.1, we know $\lim_{n \to \infty} \|S_n x - TS_n x\| = 0$. If $v \neq Tv$, we have from Opial’s theorem and Lemma 3.1 that
\[
\liminf_{i \to \infty} \|S_n x - v\|^2
\leq \liminf_{i \to \infty} \|S_n x - Tv\|^2
= \liminf_{i \to \infty} (\|S_n x - TS_n x\|^2 + \|TS_n x - Tv\|^2
+ 2\langle S_n x - TS_n x, TS_n x - Tv \rangle)
= \liminf_{i \to \infty} \|TS_n x - Tv\|^2
\leq \liminf_{i \to \infty} ((\alpha - \beta)\|S_n x - v\|^2 + 2(\alpha - 1)\langle S_n x - TS_n x, v - Tv \rangle
- (\alpha - \beta - 1)\|v - TS_n x\|^2
\leq \liminf_{i \to \infty} ((\alpha - \beta)\|S_n x - v\|^2 - (\alpha - \beta - 1)\|v - TS_n x\|^2)
\leq \liminf_{i \to \infty} ((\alpha - \beta)\|S_n x - v\|^2 - (\alpha - \beta - 1)\|v - S_n x + S_n x - TS_n x\|^2)
\leq \liminf_{i \to \infty} ((\alpha - \beta)\|S_n x - v\|^2 - (\alpha - \beta - 1)\|v - S_n x\|^2)
= \liminf_{i \to \infty} \|S_n x - v\|^2,
\]
which is a contradiction. Therefore, we have $v \in F(T)$. To show $S_n x \to z$, it is sufficient to prove $z = v$. From $v \in F(T)$, we have
\[
\langle v - z, T^k x - PT^k x \rangle = \langle v - PT^k x, T^k x - PT^k x \rangle + \langle PT^k x - z, T^k x - PT^k x \rangle
\leq \langle PT^k x - z, T^k x - PT^k x \rangle
\leq \|PT^k x - z\| \|T^k x - PT^k x\|
\leq \|PT^k x - z\| L
\]
for all $k \in \mathbb{N}$, where $L = \sup \{\|T^k x - PT^k x\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to $n_i$ and dividing by $n_i$, we have
\[
\left\langle v - z, S_n x - \frac{1}{n_i} \sum_{k=1}^{n_i} PT^k x \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|PT^k x - z\| L.
\]
Since $S_n x \to v$ as $i \to \infty$ and $PT^k x \to z$ as $n \to \infty$, we have $\langle v - z, v - z \rangle \leq 0$. This implies $z = v$. Therefore, $\{S_n x\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} PT^k x$. So, we get the desired result.

\section{5. Strong Convergence Theorems}

In this section, we first prove a strong convergence theorem of Halpern’s type \[7\] for super hybrid nonself-mappings in a Hilbert space.

\textbf{Theorem 5.1.} Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\gamma$ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be a mapping such that
\[
\|Sx - Sy\|^2 + 2\gamma(x - y, Sx - Sy) \leq (1 + 2\gamma)\|x - y\|^2
\]
for all \( x, y \in C \). Let \( \{ \alpha_n \} \subset [0,1] \) be a sequence of real numbers such that \( \alpha_n \to 0 \), \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and \( \sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty \). Suppose \( \{ x_n \} \) is a sequence generated by \( x_1 = x \in C \), \( u \in C \) and

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n) P_C \left\{ \frac{1}{1 + \gamma} S x_n + \frac{\gamma}{1 + \gamma} x_n \right\}, \quad n = 1, 2, \ldots
\]

If \( F(S) \neq \emptyset \), then the sequence \( \{ x_n \} \) converges strongly to an element \( v \) of \( F(S) \), where \( v = P_{F(S)}u \) and \( P_{F(S)} \) is the metric projection of \( H \) onto \( F(S) \).

**Proof.** We have that for any \( x, y \in C \),

\[
\| Sx - Sy \|^2 + 2\gamma \langle x - y, Sx - Sy \rangle \leq (1 + 2\gamma) \| x - y \|^2
\]

\[
\iff \| Sx - Sy \|^2 + \gamma (\| x - Sx \|^2 + \| Sx - y \|^2 - \| Sx - x \|^2 - \| y - Sx \|^2) \\
\leq (1 + 2\gamma) \| x - y \|^2
\]

\[
\iff \| Sx - Sy \|^2 + \gamma \| x - Sx \|^2 \\
\leq -\gamma \| Sx - y \|^2 + (1 + 2\gamma) \| x - y \|^2 + \gamma \| Sx - x \|^2 + \gamma \| y - Sx \|^2.
\]

So, \( S \) is a \((1,0,\gamma)\)-super hybrid mapping of \( C \) into \( H \). Put \( T = \frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I \). Then, we have from Theorem 2.2 that \( T \) is a \((1,0)\)-generalized hybrid mapping of \( C \) into \( H \), i.e., \( T \) is a nonexpansive mapping of \( C \) into \( H \). Furthermore, we have \( F(S) = F(T) \). From Wittmann’s theorem [26], we obtain \( x_n \to P_{F(P_C T)}u \); see also Takahashi [19]. Let us show \( F(P_C T) = F(T) = F(S) \). We know \( F(T) = F(S) \). It is obvious that \( F(T) \subseteq F(P_C T) \). We show \( F(P_C T) \subseteq F(T) \). If \( P_C T v = v \), we have from the property of \( P_C \) (2.4) that for \( u \in F(T) \),

\[
2\| v - u \|^2 = 2\| P_C T v - u \|^2 \\
\leq 2\| TV - u, P_C T v - u \| \\
= \| TV - u \|^2 + \| P_C T v - u \|^2 - \| TV - P_C T v \|^2
\]

and hence

\[
2\| v - u \|^2 \leq \| v - u \|^2 + \| v - u \|^2 - \| TV - v \|^2.
\]

So, we have \( 0 \leq -\| TV - v \|^2 \) and hence \( TV = v \). This completes the proof. \( \square \)

**Remark 5.2.** We do not know whether a strong convergence theorem of Halpern’s type for generalized hybrid mappings holds or not.

Next, using an idea of mean convergence, we prove a strong convergence theorem of Halpern’s type for super hybrid mappings in a Hilbert space.

**Theorem 5.3.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and let \( \alpha, \beta \) and \( \gamma \) be real numbers with \( \gamma \geq 0 \). Let \( S : C \to C \) be a \((\alpha, \beta, \gamma)\)-super hybrid mapping with \( F(S) \neq \emptyset \) and let \( P \) be the metric projection of \( H \) onto \( F(S) \). Suppose \( \{ x_n \} \) is a sequence generated by \( x_1 = x \in C \), \( u \in C \) and

\[
\begin{align*}
x_{n+1} &= \alpha_n u + (1 - \alpha_n) z_n, \\
z_n &= \frac{1}{n} \sum_{k=1}^{n} (\frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I)^k x_n
\end{align*}
\]
for all \( n = 1, 2, \ldots \), where \( 0 \leq \alpha_n \leq 1 \), \( \alpha_n \to 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then \( \{x_n\} \) converges strongly to \( Pu \).

**Proof.** For a \((\alpha, \beta, \gamma)\)-super hybrid mapping \( S : C \to C \), define

\[
T = \frac{1}{1 + \gamma} S + \frac{\gamma}{1 + \gamma} I.
\]

Then, from Theorem 2.2 \( T : C \to C \) is an \((\alpha, \beta)\)-generalized hybrid mapping such that \( F(T) = F(S) \). Since \( F(T) \) is nonempty, we take \( q \in F(T) \). Put \( r = \|u - q\| \). We define

\[
D = \{ y \in H : \|y - q\| \leq r \} \cap C.
\]

Then \( D \) is a nonempty bounded closed convex subset of \( C \). \( D \) is \( T \)-invariant and contains \( u \). Thus we assume that \( C \) is bounded without loss of generality. \( T \) is quasi-nonexpansive. So, we have that for all \( q \in F(T) \) and \( n = 1, 2, 3, \ldots \),

\[
\|z_n - q\| = \left\| \frac{1}{n} \sum_{k=1}^{n} T^k x_n - q \right\| \leq \frac{1}{n} \sum_{k=1}^{n} \|T^k x_n - q\| \leq \frac{1}{n} \sum_{k=1}^{n} \|x_n - q\|.
\]

Let us show \( \limsup_{n \to \infty} \langle u - Pu, z_n - Pu \rangle \leq 0 \). Since \( \{z_n\} \) is bounded, there exists a subsequence \( \{z_{n_k}\} \) of \( \{z_n\} \) with \( z_{n_k} \to v \). We may assume without loss of generality

\[
\limsup_{n \to \infty} \langle u - Pu, z_n - Pu \rangle = \lim_{i \to \infty} \langle u - Pu, z_{n_i} - Pu \rangle.
\]

By Lemma 4.1, we have \( \lim_{n \to \infty} \|z_n - Tz_n\| = 0 \). Using this equality and Opial’s theorem, we have \( v \in F(T) \). In fact, if \( v \neq Tv \), we have

\[
\liminf_{i \to \infty} \|z_{n_i} - v\|^2 < \liminf_{i \to \infty} \|z_{n_i} - Tv\|^2 = \liminf_{i \to \infty} (\|z_{n_i} - Tz_{n_i}\|^2 + \|Tz_{n_i} - v\|^2 + 2 \langle z_{n_i} - Tz_{n_i}, Tz_{n_i} - v \rangle) = \liminf_{i \to \infty} \|Tz_{n_i} - v\|^2 \leq \liminf_{i \to \infty} ((\alpha - \beta) \|z_{n_i} - v\|^2 + 2(\alpha - 1) \langle z_{n_i} - Tz_{n_i}, v - Tz_{n_i} \rangle - (\alpha - \beta - 1) \|v - Tz_{n_i}\|^2)
\]

\[
\leq \liminf_{i \to \infty} ((\alpha - \beta) \|z_{n_i} - v\|^2 - (\alpha - \beta - 1) \|v - z_{n_i} + z_{n_i} - Tz_{n_i}\|^2)
\]

\[
\leq \liminf_{i \to \infty} \|z_{n_i} - v\|^2 - (\alpha - \beta - 1) \|v - z_{n_i}\|^2
\]

\[
= \liminf_{i \to \infty} \|z_{n_i} - v\|^2
\]
which is a contradiction. Therefore, we have \( v \in F(T) \). Since \( P \) is the metric projection of \( H \) onto \( F(T) \), we have
\[
\lim_{i \to \infty} \langle u - Pu, z_n - Pu \rangle = \langle u - Pu, v - Pu \rangle \leq 0.
\]
This implies
\[
\limsup_{n \to \infty} \langle u - Pu, z_n - Pu \rangle \leq 0. \tag{5.2}
\]
Since \( x_{n+1} - Pu = (1 - \alpha_n)(z_n - Pu) + \alpha_n(u - Pu) \), from (5.1) we have
\[
\|x_{n+1} - Pu\|^2 = \|(1 - \alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2
\leq (1 - \alpha_n)^2\|z_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu \rangle
\leq (1 - \alpha_n)\|x_n - Pu\|^2 + 2\alpha_n\langle u - Pu, x_{n+1} - Pu \rangle.
\]
Putting \( s_n = \|x_n - Pu\|^2 \), \( \beta_n = 0 \) and \( \gamma_n = 2\langle u - Pu, x_{n+1} - Pu \rangle \) in Lemma 2.4, from \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and (5.2) we have
\[
\lim_{n \to \infty} \|x_n - Pu\| = 0.
\]
\[\Box\]

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