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A FIXED POINT THEOREM OF KRASNOSELSKII-SCHAEFER TYPE AND ITS APPLICATIONS IN CONTROL AND PERIODICITY OF INTEGRAL EQUATIONS

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Abstract. In this paper, we prove a fixed point theorem for the sum of a nonlinear contraction mapping and compact operator. The fixed point theorem obtained here resembles that of Krasnoselskii in which the mapping function is a combination of contraction and compact operators. It also takes the form of Schaefer's fixed point theorem of continuation type. Criteria on periodicity and control in integral equations are obtained by applying the fixed point theorem established.

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1. Introduction

Motivated by a number of continuation theorems on Leray-Schauder Principle, we prove a fixed point theorem of Krasnoselskii-Schaefer type, which is a combination of nonlinear contraction mapping theorem and Schauder's fixed point theorem, enabling us to establish criteria on existence of periodic solutions and attractivity in integral control equations with infinite delay.

Fixed point theory has undergone rapid development in the last several decades. The growth has been strongly promoted by the large number of applications in applied mathematics, engineering, natural sciences, global economics, and population models. Many problems in applied sciences are treated using differential and integral equations. A common method of applying fixed point theory is to write the differential equation as an integral equation which then defines a mapping; if the mapping has a fixed point, then it is a solution of the differential equation. Moreover, crucial

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properties of a solution may be derived from the mapping. For the historical background and discussion of applications, we refer to the reader to, for example, the work of Agarwal, Meahan, and O'Regan [1], Burton [6], Smart [27], Zeidler [29], Zhang [31], and the references contained therein.

In order to put the problem into its historical context, we state the Banach Contraction Mapping Principle, Schauder's Fixed Point Theorem, and a generalized version of Rothe's Theorem (Smart [27, p. 2, 25-27]).

Theorem 1.1. (Banach) Let (S, ρ) be a complete metric space. If $P : S \to S$ is a contraction mapping, i.e., there is a constant $\alpha < 1$ such that for each pair $\phi_1, \phi_2 \in S$, we have $\rho(P\phi_1, P\phi_2) \leq \alpha\rho(\phi_1, \phi_2)$, then there is a unique point $\phi \in S$ with $P\phi = \phi$.

Theorem 1.2. (Schauder) Let M be a non-empty convex subset of a Banach space X. Let $T : M \to M$ be continuous and compact. Then T has a fixed point.

Theorem 1.3. (Rothe) Let M be a closed convex subset of a normed space X with ∂M the boundary of M in X. Let $T : M \to X$ be continuous and compact such that $T(\partial M) \subset M$. Then T has a fixed point.

According to Smart [27, p. 31], Krasnoselskii studied a 1932 paper of Schauder [26] on partial differential equations and formulated the working hypothesis: The inversion of a perturbed differential operator yields the sum of a contraction and compact map. Accordingly, he formulated the following theorem (Krasnoselskii [17] or Smart [27, p. 31]).

Theorem 1.4. (Krasnoselskii) Let M be a closed convex non-empty subset of a Banach space V. Suppose that A and B map M into V and that

(i) $Bx + Ay \in M \quad (\forall x, y \in M)$,
(ii) A is compact and continuous,
(iii) B is a contraction mapping.

Then there exists y in M such that $\quad By + Ay = y$.

In dealing with a perturbed differential operator, we may find that the perturbation leads to a contraction mapping while inversion of the differential operator gives a compact mapping. Also, in the study of a neutral differential equation, say

$$\frac{d}{dt}\Big(x(t) - g(t, x_t)\Big) = G(t, x_t) \tag{1.1}$$

investigators often convert the equation to an integral equation, say

$$x(t) = h(t, x_t) + \int_0^t G(s, x_s)ds \tag{1.2}$$

with a view of proving $h(t, x_t)$ is a contraction and the integral term is compact. Note that the initial functions for the differential equation are absorbed into the term $h(t, x_t)$.

We may consider the right-hand side of (1.2) as a mapping $Px = Bx + Ax$ on an appropriate Banach space V. To apply Krasnoselskii' theorem, we must require $Bx + Ay \in M$ for all $x, y \in M$ for a closed convex subset M of V. However, P, in general, does not satisfy this condition unless the growth of $h(t, x_t)$ and $G(t, x_t)$ in x_t is restricted. This presents a significant challenge to investigators. A modern approach to such a problem is to use topological degree theory or transversality method to derive the existence of fixed points of a non-self map P; that is, P may not necessarily map M into M (Zeidler [29], Küpper, Li, and Zhang [19], Wu, Xia, and Zhang [28], Zhang [30]). This method requires the construction of a homotopy. We will follow this direction to prove our fixed point theorem (Theorem 2.2).

Various attempts have been made to replace the Leray-Schauder degree theory (Smart [27, p. 82]) by theorems in which the degree is not used. These theorems use conditions on a homotopy U_λ which may be less general, but more easily established in applications. One of the most useful results is that of Schaefer [25], a theorem of continuation-type. Schaefer's theorem has been used in a variety of areas in differential equations and control theory (Balachandran and Sakthivel [2], Burton [8], Gao and Zhang [13]).

Theorem 1.5. (Schaefer) Let V be a normed space, T a continuous mapping of V into V which is compact on each bounded subset X of V. Then either

(i) the equation $x = \lambda T(x)$ has a solution for $\lambda = 1$, or
(ii) the set of all such solutions x, for $0 < \lambda < 1$, is unbounded.

If we view $U_\lambda(x) = \lambda T(x)$ in Schaefer's theorem as a homotopy, then it can be restated in the form of Leray-Schauder Principle (Zeidler [29, p. 245]). It is often used in application.

Browder [5, p. 106] recognized that the restriction on Schaefer's homotopy $U_\lambda = \lambda U_1$ could be removed, and his argument was adapted by Potter [24] to prove a more general theorem. Let M be a region in a normed space V and consider a family of mappings U_λ of M into V such that U_λ has no fixed point on the boundary ∂M. This means that as λ changes, fixed points cannot "escape" from M through ∂M. Thus if U_0 satisfies suitable conditions (which ensure a fixed point for U_0), we expect that U_1 must have a fixed point.

Theorem 1.6. (Browder-Potter) Let M be a closed convex subset of a normed space V. Let $U(\lambda, x)$ be a continuous mapping of $[0, 1] \times M$ into a compact subset of V such that

(i) $U_0(\partial M) \subset M$,
(ii) for $0 \le \lambda \le 1$, U_λ has no fixed point on ∂M (where $U_\lambda(x) = U(\lambda, x)$).

Then U_1 has a fixed point in M.

Observe that by Rothe's Theorem, (i) and (ii) imply that U_0 has a fixed point $x^* \in M/\partial M$. In many applications, U_0 is a constant map sending M to a point $p \in M/\partial M$. In this case, U_0 is an "essential" map. If $U_\lambda(\phi)$ is fixed point free on ∂M for all $\lambda \in (0, 1]$, then $U_1(\phi)$ is essential having a fixed point property in M (Granas and Dugundji [14, p. 120-123]). This fact is often written in the form of Leray-Schauder Principle or its nonlinear alternatives (Agarwal, Meahan, and O'Regan [1, p. 48], Granas and Dugundji [14, p. 123-124]). The following formulation is from Wu, Xia, and Zhang [28].

Theorem 1.7. (Nonlinear Alternative) Let M be a closed subset of a normed space V, $U_\lambda(x) = U(\lambda, x)$ a continuous mapping of $[0, 1] \times M$ into a compact subset of V such that $U_0(x) = p \in M/\partial M$ for all $x \in M$. Then either

(i) U_1 has a fixed point in $M/\partial M$ or
(ii) there exists $x \in \partial M$ and $\lambda \in (0,1]$ such that $x = U(\lambda, x)$.

Working on an integral equation, Burton and Kirk [10] proved a fixed point theorem which is a combination of the contraction mapping theorem and Schaefer's theorem to show the existence of a T-periodic solution of an integral equation. The theorem may be viewed as a continuation theorem of Krasnoselskii-Schaefer type.

Theorem 1.8. (Burton-Kirk) Let V be a Banach space, A, $B : V \to V$, B a contraction with contraction number $\alpha < 1$, and A continuous with A mapping bounded sets into compact sets. Then either

(i) $x = \lambda B(x/\lambda) + \lambda Ax$ has a solution in V for $\lambda = 1$, or
(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Y. Liu and Z. Liu [22] showed that Theorem 1.8 is still valid if B is replaced by a separate contraction or large contraction (see Remark 2.1). These contractions belong to a general class of nonlinear contractions, but possess additional properties essential in applications. Using the theory of measure of noncompactness and condensing maps, O'Regan [23] obtains the following fixed point theorem of continuation type.

Theorem 1.9. (O'Regan) Let U be an open set in a closed, convex set C of a Banach space $(E, \|\cdot\|)$ with $0 \in U$. Suppose that $F : \overline{U} \to C$ is given by $F = F_1 + F_2$ and $F(\overline{U})$ is a bounded set in C. In addition, assume that $F_1 : \overline{U} \to C$ is continuous and completely continuous and for $F_2 : \overline{U} \to C$, there exists a continuous, nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(t) < t$ for $t > 0$ such that $\|F_2(x) - F_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \overline{U}$. Then either

(A1) F has a fixed point in $\overline{U}$, or
(A2) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$.

Note that Theorem 1.9 asks a less restrictive condensing condition on F_2, but requires $F(\overline{U})$ be bounded. To show that F has a fixed point in $\overline{U}$, one must prove that the alternative (A2) does not hold by showing that the homotopy $U_\lambda(x) = \lambda F(x)$ is fixed point free on ∂U for $\lambda \in (0,1)$. These continuation theorems, without actually calculating degrees, are nonlinear alternatives of Leray-Schauder degree theory and require a less restrictive growth condition on the functions involved.

The goal of this paper is to prove a fixed point theorem of Krasnoselskii-Schaefer type with a nonlinear contraction and study its applications in integral equations. The paper is organized as follows. In Section 2, we prove a new fixed point theorem. Its applications in periodicity and control of integral equations will be given in Section 3 and 4, respectively. The examples are shown in simple forms for illustrative purpose, and they can easily be generalized. Let R^-, R^+, R denote the intervals $(-\infty, 0], [0,\infty)$, and $(-\infty,\infty)$ respectively.

2. A Fixed Point Theorem of Krasnoselskii-Schaefer Type

In this section, we prove a fixed point theorem in which the mapping function is the combination of a general nonlinear contraction and compact operator. It takes certain forms of Theorem 1.8 and Theorem 1.9. Without asking $F(\overline{U})$ be bounded or having a λ term in B, our theorem very much resembles those of Krasnoselskii and Schaefer.

Definition 2.1. Let (X, ρ) be a metric space and $T : X \to X$. T is said to be a nonlinear contraction if there exists a function $\psi : R^+ \to R^+$ such that

$$\rho(Tx, Ty) \leq \psi(\rho(x, y)) \tag{2.1}$$

for all $x, y \in X$, where $\psi(r) < r$ for all $r > 0$.

Theorem 2.1. (Boyd and Wong [4]) Let (X, ρ) be a complete metric space, and let $T : X \to X$ be a nonlinear contraction, where ψ satisfies

$$\limsup_{s \to r^+} \psi(s) < r \tag{2.2}$$

for all $r > 0$. Then T has a unique fixed point x_0, and $T^n x \to x_0$ for each $x \in X$.

Remark 2.1. It is clear that (2.2) is satisfied if the function ψ is upper semi-continuous from the right; that is, $\limsup_{s \to t^+} \psi(s) \leq \psi(t)$. In particular, if $\psi(r) = \alpha(r)r$, where $\alpha(r)$ is decreasing or increasing, and $0 \leq \alpha(r) < 1$ for $r > 0$, then ψ is a nonlinear contraction satisfying (2.2). Once ψ is found, Theorem 2.1 can be implemented by applying an iterative method to find the fixed point of a contractive map: it produces approximations of any required accuracy, and moreover, the fixed point is unique. We only need the notion of completeness and a cleverly chosen metric space to work with. When working with applied problems, investigators have found many nonlinear contractions such as large contraction (Burton [7]) and separate contraction (Y. Liu and Z. Liu [22]). These contractions are all satisfying (2.2), but they yield strong properties such as $(I - T)^{-1}$ being continuous which is often used in solving operator equations with implicit function forms. Finding the function ψ in (2.2) proves to be challenging even in the space of bounded continuous functions. In Example 2.1 below, T is almost a local contraction, but fails near $x = 0$. To save space, we omit the details here.

Example 2.1. Let $X = BC(R, R)$ be the Banach space of bounded continuous functions $\phi : R \to R$ with the supremum norm $\|\cdot\|$. Let $T : X \to X$ be defined by

$$T(x)(t) = x(t) - \frac{x^3(t)}{4(1 + x^2(t))} \tag{2.3}$$

for each $x \in X$. Then T is a nonlinear contraction on X with

$$\psi(r) = \begin{cases} \frac{61}{64}\, r, & r \geq \sqrt{3} \\ (1 - \frac{1}{64} r^2)\, r, & 0 \leq r < \sqrt{3}. \end{cases} \tag{2.4}$$

It is clear that ψ satisfies (2.2).

Lemma 2.1. Let $(S, \|\cdot\|)$ be a normed space. If $T : S \to S$ is a nonlinear contraction with

$$\liminf_{s \to r} (s - \psi(s)) > 0 \text{ for } 0 < r \leq \infty, \tag{2.5}$$

then $(I - T)$ is a homeomorphism of S onto $(I - T)S$.

Remark 2.2. Condition (2.5) can be rewritten as

$$\limsup_{s\to r}\ \psi(s) < r \text{ for all } r > 0 \text{ and } \liminf_{s\to\infty}\,(s-\psi(s)) > 0.$$

We see that (2.2) is satisfied whenever (2.5) holds. Moreover, (2.5) ensures that $(I-T)^{-1}$ is continuous as Lemma 2.1 shows. Also, $\liminf_{s\to\infty}(s-\psi(s)) > 0$ is necessary for obtaining boundedness of solutions in many integral equations with a nonlinear contraction term (see Section 3 and 4). We may readily verify that $\psi(r) = r^2/(r+1)$ for $r \ge 0$ satisfies (2.5) even if $\lim_{r\to\infty}\psi(r)/r = 1$.

Proof of Lemma 2.1. We first show that $I-T$ is one to one. Let $x_1, x_2 \in S$ with $\|x_1 - x_2\| > 0$ and suppose that $(I-T)x_1 = (I-T)x_2$. Then $x_1 - x_2 = Tx_1 - Tx_2$. Since $\psi(r) < r$ for $r > 0$, we have

$$\|x_1 - x_2\| = \|Tx_1 - Tx_2\| \le \psi(\|x_1 - x_2\|) < \|x_1 - x_2\|,$$

a contradiction. Thus, $(I-T)x_1 \ne (I-T)x_2$ and hence $(I-T)^{-1}$ exists. We now show that $(I-T)^{-1}$ is continuous on $(I-T)S$. Let $\{y_n\}$ be a sequence in $(I-T)S$ with $y_n \to y^* \in (I-T)S$. We need to show that $(I-T)^{-1}y_n \to (I-T)^{-1}y^*$ as $n\to\infty$. Set $x_n = (I-T)^{-1}y_n$ and $x^* = (I-T)^{-1}y^*$. Then $(I-T)x_n = y_n$ and $(I-T)x^* = y^*$. Suppose that $x_n \not\to x^*$. Then there exists an $\varepsilon_0 > 0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x^*\| \ge \varepsilon_0$ for all $k = 1, 2, \cdots$. Observe that

$$\begin{aligned}\|y_{n_k} - y^*\| &= \|(I-T)x_{n_k} - (I-T)x^*\|\\ &\ge \|x_{n_k} - x^*\| - \|Tx_{n_k} - Tx^*\|\\ &\ge \|x_{n_k} - x^*\| - \psi(\|x_{n_k} - x^*\|).\end{aligned} \tag{2.6}$$

Let $r_k = \|x_{n_k} - x^*\|$. Then $r_k \ge \varepsilon_0$. There must be a subsequence $\{r_{k_j}\}$ of $\{r_k\}$ such that $r_{k_j} \to r_0$ $(\varepsilon_0 \le r_0 \le \infty)$ as $j\to\infty$. By (2.5) and (2.6), we obtain

$$\lim_{j\to\infty}\|y_{n_{k_j}} - y^*\| \ \ge\ \liminf_{j\to\infty}\,[r_{k_j} - \psi(r_{k_j})] \ge \liminf_{s\to r_0}\,[s-\psi(s)] > 0.$$

This contradicts the fact that $\lim_{j\to\infty}\|y_{n_{k_j}} - y^*\| = 0$. Thus, we must have $x_n \to x^*$ as $n\to\infty$, and hence $(I-T)^{-1}$ is continuous on $(I-T)S$. This completes the proof.

Concerning the terminology of a compact mapping, Krasnoselskii used the convention of Smart [27, p. 25] to mean the following: Let A map a set M into a topological space X. If AM is contained in a compact subset of X, we say that A is compact. In particular, M need not be bounded.

Theorem 2.2. Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ are mappings such that A is continuous and AM is contained in a compact subset of S, B is a nonlinear contraction with ψ satisfying (2.5), and $x^* = Bx^* \in M/\partial M$. Then either

(i) $x = Bx + Ax$ has a solution in M, or
(ii) there is a point $\tilde{x} \in \partial M$ and $\lambda \in (0,1)$ with $\tilde{x} = B\tilde{x} + \lambda A\tilde{x}$.

Proof. For each $y \in M$ and $\lambda \in [0,1]$, define $P : S \to S$ by $P(z) = Bz + \lambda Ay$. Then P is a nonlinear contraction satisfying (2.5), and so, by Theorem 2.1, there exists a unique $z \in S$ such that $z = P(z) = Bz + \lambda Ay$. This implies that $\lambda AM \subset (I-B)S$.

Now $(I-B)^{-1}$ exists and is continuous on $(I-B)S$ by Lemma 2.1. Since A is continuous and compact on M, so is $(I-B)^{-1}(\lambda A)$ for each $\lambda \in [0,1]$ (The proof given by Kreyszig [18, p. 412, 656] is valid for general metric spaces).
Define $U : [0,1] \times M \to S$ by $U(\lambda,\phi) = (I-B)^{-1}(\lambda A\phi)$. Then $U_\lambda(\phi) = U(\lambda,\phi)$ is a continuous mapping of $[0,1]\times M$ into a compact subset of S. Indeed, set $\Gamma = \{\lambda A\phi : \lambda \in [0,1],\ \phi \in M\}$ and let $\{(\lambda_n,\phi_n)\}$ be a sequence in $[0,1]\times M$. We may assume that $\lambda_n \to \lambda_0 \in [0,1]$ as $n \to \infty$. Since AM is contained in a compact subset of S, there exists a convergent subsequence $\{A\phi_{n_k}\}$ of $\{A\phi_n\}$. Now $\{\lambda_{n_k} A\phi_{n_k}\}$ converges in S. This implies that Γ is pre-compact, and so is $(I-B)^{-1}\Gamma$.
Observe that $U_0(\phi) = (I-B)^{-1}(0) = x^*$, for all $\phi \in M$, where $x^* \in M/\partial M$ is the unique fixed point of B. Notice that if there exists a fixed point of $B+\lambda A$ on ∂M for $\lambda = 1$, then (i) holds. Thus, if (ii) fails, we may assume that $B+\lambda A$ is fixed point free on ∂M for $0<\lambda\le 1$. Then the same is true for U_λ. By Theorem 1.7, U_1 must have a fixed point in $M/\partial M$. This again implies that (i) holds, and the proof is complete.

As indicated early, to prove the operator $B+A$ has a fixed point in M, we must impose conditions that prevent occurrence of (ii) by showing $B+\lambda A$ is fixed point free on ∂M for $0<\lambda<1$. We can achieve this by establishing the existence of an *a priori* bound for all possible fixed points of $B+\lambda A$. Since there is no λ involved in the term Bx, our "deformation" homotopy takes a form simpler than those in Theorem 1.8 and Theorem 1.9. This is especially helpful when we try to derive an *a priori* bound for all possible fixed points of $B+\lambda A$.

The following theorems are corollaries of Theorem 2.2, but have their own right in applications.

Theorem 2.3. Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ are mappings such that

(i) A is continuous and AM is contained in a compact subset of S,
(ii) B is a nonlinear contraction with ψ satisfying (2.5),
(iii) for $0 \le \lambda < 1$, $x = Bx + \lambda Ax \Rightarrow x \in M/\partial M$.

Then there exists y in M such that $By + Ay = y$.

Proof. Observe that by (iii) with $\lambda = 0$, the unique fixed point x^* of B is in $M/\partial M$ and $B+\lambda A$ is fixed point free on ∂M for $\lambda \in (0,1)$.

The following theorem may be viewed as a Krasnoselskii theorem of continuation type with a nonlinear contraction, and its conditions can easily be verified in application (see Section 3 and 4). Observe also that we don't need the continuity of ψ in (2.1) from the left.

Theorem 2.4. Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that $A : M \to S$ and $B : S \to S$ are mappings such that

(i) A is continuous and AM is contained in a compact subset of S,
(ii) B is a nonlinear contraction and ψ in (2.1) is nondecreasing, right-continuous with
$\liminf_{r\to\infty}(r - \psi(r)) > 0$.
(iii) for $0 \le \lambda < 1$, $x = Bx + \lambda Ax \Rightarrow x \in M/\partial M$.

Then there exists y in M such that $By + Ay = y$.

Proof. We first show that $\limsup_{s\to r}\psi(s) < r$ for each $r > 0$. Since ψ is nondecreasing and $\psi(r) < r$, we see that $\limsup_{s\to r^-}\psi(s) \le \psi(r) < r$. By the continuity of ψ from the right of r, we have $\lim_{s\to r^+}\psi(s) = \psi(r)$, and hence, $\limsup_{s\to r^+}\psi(s) = \psi(r) < r$. According to Remark 2.2, this implies that (2.5) holds since $\liminf_{r\to\infty}(r-\psi(r)) > 0$. Now the result follows from Theorem 2.3.

To conclude this section, we present a generalized version of Schaefer's Theorem with a nonlinear contraction. Theorems of this kind are especially useful in proving the existence of periodic solutions for differential and integral equations. We again point out that in the theorem below there is no λ involved in the term Bx, contrary to some existing results mentioned in the introduction.

Theorem 2.5. Let $(V, \|\cdot\|)$ be a Banach space, $A,\ B : V \to V$, B a nonlinear contraction with ψ satisfying (2.5), and A continuous with A mapping bounded sets into compact sets. Then either

(i) $x = Bx + \lambda Ax$ has a solution in V for $\lambda = 1$, or

(ii) the set of all such solutions, $0 < \lambda < 1$, is unbounded.

Proof. For each positive integer n, define $M_n = \{x \in V : \|x\| \le n\}$. We choose n sufficiently large so that the unique fixed point x^* of B is in $M_n/\partial M_n$. By Theorem 2.2, either $x = Bx + Ax$ has a solution in M_n or there exists $x_n \in \partial M_n$ such that $x_n = Bx_n + \lambda Ax_n$ for some $\lambda \in (0, 1)$. In the later case, we have $\|x_n\| = n$. Thus, if (i) does not hold, then $\|x_n\| \to \infty$ as $n \to \infty$. This completes the proof.

3. Periodicity in an Integral Equation

Let $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\phi : R \to R$ with the supremum norm.

Consider the integral equation

$$x(t) = h(t, x(t)) - \int_{-\infty}^{t} D(t, s)g(s, x(s))ds \tag{3.1}$$

where $h : R \times R \to R,\ D : R \times R \to R,\ g : R \times R \to R$ are continuous.

We shall show the existence of a periodic solution of (3.1) by applying Theorem 2.4. The technique used here has its roots in Burton [8]. We assume that

(H_1) there exists a constant $T > 0$ such that $D(t+T, s+T) = D(t, s),\ h(t+T, x) = h(t, x),\ g(t + T, x) = g(t, x)$ for all $t \in R$ and all $x \in R$,

(H_2) $D_s(t, s) \ge 0,\ D_{st}(t, s) \le 0$ for $-\infty < s \le t < \infty$ with $D_s(t, s)$ and $D_{st}(t, s)$ continuous,

(H_3) there exists $U_1 > 0$ such that $xg(t, x) \ge 0$ for all $|x| \ge U_1$ and $t \in R$,

(H_4) $|h(t, x) - h(t, y)| \le \psi(|x - y|)$ for all $t, x, y \in R$, where ψ is nondecreasing, right-continuous with $\psi(r) < r$ for all $r > 0$ and

$$\lim_{r\to\infty}(r - \psi(r)) = \infty \tag{3.2}$$

(H_5) $\displaystyle\int_{-\infty}^{t}[|D(t, s)| + D_s(t, s)(t - s)^2 + |D_{st}(t, s)|(t - s)^2]ds$ is continuous in t, with $\displaystyle\lim_{s\to-\infty}(t - s)D(t, s) = 0$ for fixed t,

(H_6) there exists a constant $Q > 0$ such that

$$\int_{-\infty}^{t_1} |D(t_1, s) - D(t_2, s)|ds \leq Q|t_2 - t_1| \text{ for } 0 \leq t_1 \leq t_2 \leq T.$$

Theorem 3.1. If (H_1)-(H_6) hold, then (3.1) has a T-periodic solution.

Proof. For $x \in P_T$, define

$$(Bx)(t) = h(t, x(t)) \text{ and } (Ax)(t) = -\int_{-\infty}^{t} D(t, s)g(s, x(s))ds.$$

A change of variable shows that if $\phi \in P_T$, then $(A\phi)(t + T) = (A\phi)(t)$. Thus, $A, B : P_T \to P_T$ are well defined. We will apply Theorem 2.4 with $S = P_T$ to show that $B + A$ has a fixed point which is a T-periodic solution of (3.1). By (H_4), B is a nonlinear contraction on P_T satisfying (ii) of Theorem 2.4. To establish that $A : M \to P_T$ is compact for some closed convex subset $M \subset P_T$, we need several steps to follow. Let us first claim that there exists a constant $K > 0$ such that $\|x\| < K$ whenever $x \in P_T$ and $x = Bx + \lambda Ax$ for $\lambda \in (0, 1]$. Suppose that $x \in P_T$ satisfying

$$x(t) = h(t, x(t)) - \lambda \int_{-\infty}^{t} D(t, s)g(s, x(s))ds \tag{3.3}$$

and define

$$V(t, x(\cdot)) = \lambda^2 \int_{-\infty}^{t} D_s(t, s) \left(\int_s^t g(v, x(v))dv \right)^2 ds.$$

Then $V(t, x(\cdot))$ is T-periodic and

$$\begin{aligned} V'(t, x(\cdot)) &= \lambda^2 \int_{-\infty}^{t} D_{st}(t, s) \left(\int_s^t g(v, x(v))dv \right)^2 ds \\ &\quad +2\lambda^2 g(t, x(t)) \int_{-\infty}^{t} D_s(t, s) \int_s^t g(v, x(v))dvds. \end{aligned}$$

If we integrate the last term by parts, we have

$$2\lambda^2 g(t, x(t)) \left[D(t, s) \int_s^t g(v, x(v))dv \Big|_{s=-\infty}^{s=t} + \int_{-\infty}^{t} D(t, s)g(s, x(s))ds \right].$$

The first term vanishes at both limits by (H_5); the first term of V' is not positive since $D_{st}(t, s) \leq 0$, and if we use (3.3) on the last term, then we obtain

$$\begin{aligned} V'(t, x(\cdot)) &\leq 2\lambda^2 g(t, x(t)) \int_{-\infty}^{t} D(t, s)g(s, x(s))ds \\ &= 2\lambda g(t, x(t))[-x(t) + h(t, x(t))]. \end{aligned}$$

If $|x(t)| \geq U_1$, then

$$\begin{aligned} V'(t, x(\cdot)) &\leq -2\lambda|g(t, x(t))||x(t)| + 2\lambda|g(t, x(t))||h(t, x(t))| \\ &\leq -2\lambda|g(t, x(t))||x(t)| + 2\lambda|g(t, x(t))|[\psi(|x(t)|) + |h(t, 0)|] \\ &\leq -2\lambda|g(t, x(t))|[|x(t)| - \psi(|x(t)|) - h^*] \end{aligned}$$

where $h^* = \sup\{|h(t,0)| : 0 \le t \le T\}$. By (3.2) in ($\mathrm{H}_4$), we find $U_1^* > U_1$ such that $r \ge U_1^*$ implies $r - \psi(r) - h^* \ge 1$. Thus, if $|x(t)| \ge U_1^*$, then $V'(t,x(\cdot)) \le -\lambda|g(t,x(t))|$. It is clear that $V'(t,x(\cdot))$ is bounded above for $0 \le |x(t)| \le U_1^*$, and hence there exists a constant L depending on U_1^* such that

$$V'(t,x(\cdot)) \quad \le \quad -\lambda|g(t,x(t))| + \lambda L. \tag{3.4}$$

Since (H_5) holds, we have

$$\begin{aligned} \int_{-\infty}^{t} D_s(t,s)ds &= \lim_{b\to-\infty}\int_{b}^{t} D_s(t,s)ds \\ &= \lim_{b\to-\infty}[D(t,t) - D(t,b)] = D(t,t) \end{aligned}$$

and so

$$\sup_{0\le t\le T}\int_{-\infty}^{t} D_s(t,s)ds = \sup_{0\le t\le T} D(t,t) =: J.$$

By the Schwarz inequality, we have

$$\begin{aligned} &\lambda^2\left(\int_{-\infty}^{t} D_s(t,s)\int_{s}^{t} g(v,x(v))dvds\right)^2 \\ \le\ &\lambda^2\int_{-\infty}^{t} D_s(t,s)ds\int_{-\infty}^{t} D_s(t,s)\left(\int_{s}^{t} g(v,x(v))dv\right)^2 ds \\ \le\ &JV(t,x(\cdot)). \end{aligned}$$

We have just integrated the left side by parts, obtaining

$$\left(\int_{-\infty}^{t} D(t,s)g(s,x(s))ds\right)^2 = \left(\int_{-\infty}^{t} D_s(t,s)\int_{s}^{t} g(v,x(v))dvds\right)^2$$

so that by (3.3) we now have

$$(x(t) - h(t,x(t)))^2 \le JV(t,x(\cdot)). \tag{3.5}$$

Since V is T-periodic, there exists a sequence $\{t_n\} \uparrow \infty$ with $V(t_n,x(\cdot)) \ge V(s,x(\cdot))$ for $s \le t_n$. Thus,

$$0 \le \quad V(t_n,x(\cdot)) - V(s,x(\cdot)) \le -\lambda\textstyle\int_s^{t_n} |g(v,x(v))|dv + \lambda L(t_n - s).$$

and so $\lambda\int_s^{t_n}|g(v,x(v))|dv \le \lambda L(t_n - s)$. Thus,

$$\begin{aligned} V(t_n,x(\cdot)) &= \lambda^2\int_{-\infty}^{t_n} D_s(t_n,s)\left(\int_{s}^{t_n} g(v,x(v))dv\right)^2 ds \\ &\le \lambda^2\int_{-\infty}^{t_n} D_s(t_n,s)L^2(t_n - s)^2 ds \le \gamma L^2 \end{aligned}$$

where

$$\gamma = \sup_{0\le t\le T}\int_{-\infty}^{t} D_s(t,s)(t-s)^2 ds.$$

This implies that $V(t,x(\cdot)) \leq \gamma L^2$ for all $t \in R$, and therefore by (3.5) we obtain

$$(x(t) - h(t,x(t)))^2 \leq JV(t,x(\cdot)) \leq \gamma JL^2. \tag{3.6}$$

Observe that

$$\begin{aligned} |x(t) - h(t,x(t))| &\geq |x(t)| - |h(t,x(t)) - h(t,0)| - |h(t,0)| \\ &\geq |x(t)| - \psi(|x(t)|) - h^*. \end{aligned}$$

By (3.2) in (H_4), there exists a constant $K > U_1^*$ such that $r \geq K$ implies

$$r - \psi(r) - h^* > \sqrt{\gamma JL^2}. \tag{3.7}$$

We now claim that $|x(t)| < K$ for all $t \in [0,T]$. If for some $t^* \in [0,T]$ such that $|x(t^*)| \geq K$, then by (3.6) and (3.7), we have

$$\begin{aligned} \gamma JL^2 &< (|x(t^*)| - \psi(|x(t^*)|) - h^*)^2 \\ &\leq |x(t^*) - h(t^*,x(t^*))|^2 \leq JV(t^*,x(t^*)) \leq \gamma JL^2, \end{aligned}$$

a contradiction, and thus, $\|x\| < K$ whenever x is a solution of (3.3). For $\lambda = 0$, (3.3) becomes $x = Bx$ which has a unique solution x^* by Theorem 2.1. We may now assume that $K > \|x^*\|$ and define $M = \{x \in P_T : \|x\| \leq K\}$. It is clear that M is a closed convex subset of P_T. By the argument above, if $x = Bx + \lambda Ax$ for $0 \leq \lambda < 1$, then $\|x\| < K$. This implies $x \in M/\partial M$. Thus, (iii) of Theorem 2.4 holds.

Next, we show that $A : M \to P_T$ is continuous and AM is contained in a compact subset of P_T, and hence, (i) of Theorem 2.4 holds. Let $\phi_1, \phi_2 \in M$. Then for all $t \in [0,T]$, we have

$$|A\phi_1(t) - A\phi_2(t)| \leq \int_{-\infty}^{t} |D(t,s)||g(s,\phi_1(s)) - g(s,\phi_2(s))|ds. \tag{3.8}$$

Since g is uniformly continuous on $\{(t,x) : 0 \leq t \leq T, |x| \leq K\}$, then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\phi_1 - \phi_2\| < \delta$ implies $|g(s,\phi_1(s)) - g(s,\phi_2(s))| < \varepsilon$ for all $s \in [0,T]$. It then follows from (3.8) that $\|A\phi_1 - A\phi_2\| \leq J^*\varepsilon$, where $J^* = \sup_{0\leq t\leq T} \int_{-\infty}^{t} |D(t,s)|ds$. Thus, A is continuous on M. Now let $\phi \in M$ and define

$$g^* = \sup\{|g(t,x)| : 0 \leq t \leq T, |x| \leq K\}.$$

If $0 \leq t_1 \leq t_2 \leq T$, then

$$\begin{aligned} & |A\phi(t_1) - A\phi(t_2)| \\ = & \left| \int_{-\infty}^{t_1} D(t_1,s)g(s,\phi(s))ds - \int_{-\infty}^{t_2} D(t_2,s)g(s,\phi(s))ds \right| \\ \leq & \int_{-\infty}^{t_1} |D(t_1,s) - D(t_2,s)||g(s,\phi(s))|ds + \left| \int_{t_1}^{t_2} D(t_2,s)g(s,\phi(s))ds \right| \\ \leq & Qg^*|t_2 - t_1| + D^*g^*|t_2 - t_1| \end{aligned}$$

where $D^* = \sup\{|D(t,s)| : 0 \leq s \leq t \leq T\}$. Here we have used ($H_6$) in the last inequality. This implies that AM is equi-continuous. The uniform boundedness of

AM follows from the following inequality

$$|A\phi(t)| \leq \int_{-\infty}^{t} |D(t,s)||g(s,\phi(s))|ds \leq g^* J^*$$

for all $\phi \in M$. So, by Ascoli-Arzela Theorem, AM lies in a compact subset of P_T. By Theorem 2.4, there exists $y \in M$ such that $y = By + Ay$. Thus, y is a T-periodic solution of (3.1), and the proof is complete.

4. Boundedness and Attractivity in Integral Control Equations

Consider the scalar equation

$$x(t) = h(t, x(t)) - \int_0^t D(t,s)g(s, x(s))ds + u(t) \tag{4.1}$$

where $h : R^+ \times R \to R$, $D : R^+ \times R^+ \to R$, $g : R^+ \times R \to R$ are all continuous, $g(t,x)$ is bounded for x bounded, and $u : R^+ \to R$, the control, belongs to a class $\mathcal{G}$ of functions.

We denote by $C(X,Y)$ the set of all continuous functions $\phi : X \to Y$ for normed spaces X and Y. We also denote by $BC(R^+, R)$ the Banach space of bounded continuous functions $\phi : R^+ \to R$ with the supremum norm $\|\cdot\|$. For a given $u \in \mathcal{G}$, we say that $x : R^+ \to R$ is a solution $x = x(t, x_0)$ of (4.1) on R^+ with an initial value x_0 if x is continuous and satisfies (4.1) on R^+ with $x(0) = x_0$. It is to be understood that $x(0) = h(0, x(0)) + u(0)$. If $x(t)$ is specified to be a certain initial function on an initial interval, say

$$x(t) = \phi(t) \text{ for } 0 \leq t \leq t_0,$$

we are then looking for a solution of

$$x(t) = h(t, x(t)) - \int_0^{t_0} D(t,s)g(s,\phi(s))ds - \int_{t_0}^{t} D(t,s)g(s,x(s))ds + u(t),\ t \geq t_0. \tag{4.2}$$

However, a change of variable $y(t) = x(t + t_0)$ will reduce the problem back to one of form (4.1). Thus, the initial function on $[0, t_0]$ is absorbed into the forcing function, and hence, it suffices to consider (4.1) with the simple initial condition $x(0) = h(0, x(0)) + u(0)$.

The project is to characterize $\mathcal{G}$ so that the attractivity ($x(t) \to 0$ as $t \to \infty$) of solutions of (4.1) is independent of special choice of $u \in \mathcal{G}$. Such a property may be referred to as absolute stability (or asymptotic stability) of a control system (cf. Lefschetz [20], Banaś and Rzepka [3], Burton and Zhang [11]). We define the set $\mathcal{G}$ with respect to (4.1) as

$$\mathcal{G} = \{\, u \in C(R^+, R) \ : \ \sup_{t \geq 0} |h(t,0) + u(t)| < \infty\}. \tag{4.3}$$

As indicated in Section 1, equation (4.1) may be viewed as an integral form of a neutral differential equation, say (1.1). Note again that the initial function for the differential equation is absorbed into the term $h(t,x)$.

To consider the existence of a solution of (4.1) by fixed point theory, one must overcome difficulties presented by non-compactness associated with unbounded intervals. A simple solution was to construct a mapping set which degenerated to a curve as $t \to \infty$. This meant that an equicontinuous subset was, in fact, contained in a compact set. The following compactness result from Burton and Furumochi [9] is needed. It is an Ascoli-Arzela type theorem on R^+.

Lemma 4.1 (Burton and Furumochi) Let $q : R^+ \to R^+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\phi_k\}$ is an equicontinuous sequence of R^n-valued functions on R^+ with $|\phi(t)| \le q(t)$ for all $t \in R^+$, then there exists a subsequence of $\{\phi_k\}$ that converges uniformly on R^+ to a continuous function ϕ with $|\phi(t)| \le q(t)$ for all $t \in R^+$.

We shall first show the existence of a bounded solution of (4.1) by applying again Theorem 2.4. Note that other theorems in Section 2 can also be applied. The following conditions are parallel to those of Theorem 3.1 without periodicity requirement.

($\widetilde{\mathrm{H}}_1$) For each $\mu > 0$, there exist $g_\mu \in BC(R^+, R)$ and $L_\mu > 0$ such that if $|x| \le \mu$, $|y| \le \mu$, then $|g(t,x)| \le g_\mu(t)$ and $|g(t,x) - g(t,y)| \le L_\mu |x-y|$ for all $t \ge 0$.

($\widetilde{\mathrm{H}}_2$) $D(t,0) \ge 0$, $D_s(t,s) \ge 0$, $D_t(t,0) \le 0$, and $D_{st}(t,s) \le 0$ with $D_s(t,s)$ and $D_{st}(t,s)$ continuous for all $t \ge s \ge 0$.

($\widetilde{\mathrm{H}}_3$) There exists $U_1 > 0$ such that $xg(t,x) \ge 0$ for all $|x| \ge U_1$ and $t \ge 0$.

($\widetilde{\mathrm{H}}_4$) $|h(t,x) - h(t,y)| \le \psi(|x-y|)$ for all $t \ge 0$ and $x, y \in R$, where ψ is nondecreasing, right-continuous with $\psi(r) < r$ for all $r > 0$ and

$$\lim_{r\to\infty} (r - \psi(r)) = \infty \tag{4.4}$$

($\widetilde{\mathrm{H}}_5$) $\sup_{t\ge 0} \left[|D(t,t)| + |D(t,0)|t^2 + \int_0^t D_s(t,s)(t-s)^2 ds \right] < \infty$ with

$\int_0^t D(t,s) g_\mu(s) ds \to 0,$ as $t \to \infty$ for each $\mu > 0,$ where g_μ is defined in ($\widetilde{\mathrm{H}}_1$).

($\widetilde{\mathrm{H}}_6$) For each $\mu > 0$, there exists a constant $Q_\mu > 0$ such that

$$\int_0^{t_1} |D(t_1,s) - D(t_2,s)| g_\mu(s) ds \le Q_\mu \, |t_2 - t_1|, \text{ for all } 0 \le t_1 \le t_2 < \infty.$$

The proof of the following theorem is similar to that of Theorem 3.1 with P_T replaced by $S = BC(R^+, R)$. We may omit some of the calculations.

Theorem 4.1. Suppose that ($\widetilde{\mathrm{H}}_1$) - ($\widetilde{\mathrm{H}}_6$) hold. Then (4.1) has a bounded solution for each $u \in \mathcal{G}$.

Proof. We first define some constants to simply notations. Integrating by parts in the first integral of ($\widetilde{\mathrm{H}}_5$), we obtain

$$\begin{aligned} \int_0^t D_s(t,s)(t-s)^2 ds &= D(t,s)(t-s)^2 \Big|_0^t + 2\int_0^t D(t,s)(t-s)ds \\ &= -D(t,0)t^2 + 2\int_0^t D(t,s)(t-s)ds. \end{aligned}$$

This implies by $(\widetilde{\mathrm{H}}_5)$ that

$$\sup_{t\geq 0}\int_0^t D(t,s)(t-s)ds = J_1 < \infty. \tag{4.5}$$

We also observe that

$$\begin{aligned}\int_0^t D(t,s)ds &= \int_0^t D(t,t-u)du = \int_0^1 D(t,t-u)du + \int_1^t D(t,t-u)du \\ &\leq D(t,t) + \int_1^t D(t,t-u)u\,du \\ &\leq D(t,t) + \int_0^t D(t,s)(t-s)ds.\end{aligned}$$

Here we have used the condition $D_s(t,s) \geq 0$ which implies that $D(t,t) \geq D(t,t-u)$ for all $0 \leq u \leq t$. Thus, by $(\widetilde{\mathrm{H}}_5)$ and (4.5), we have

$$\int_0^t D(t,s)ds = J_0 < \infty. \tag{4.6}$$

We also set

$$\sup_{t\geq 0}\int_0^t D_s(t,s)(t-s)^2 ds = J_2. \tag{4.7}$$

Next define

$$\sup_{t\geq 0} D(t,t) = D_1 \text{ and } \sup_{t\geq 0} D(t,0)t^2 = D_2. \tag{4.8}$$

Notice that

$$\int_0^t D_s(t,s)ds = D(t,t) - D(t,0) \leq D_1. \tag{4.9}$$

We now follow the proof of Theorem 3.1. Let $x \in S = BC(R^+, R)$ and define $A, B : S \to S$ by

$$(Bx)(t) = h(t,x(t)) + u(t) \text{ and } (Ax)(t) = -\int_0^t D(t,s)g(s,x(s))ds.$$

Applying $(\widetilde{\mathrm{H}}_4)$, we see

$$\begin{aligned}|(Bx)(t)| &= |h(t,x(t)) + u(t)| \leq |h(t,x(t)) - h(t,0)| + |h(t,0) + u(t)| \\ &\leq \psi(|x(t)|) + M^* \leq \psi(\|x\|) + M^*\end{aligned} \tag{4.10}$$

where $M^* = \sup\{|h(t,0) + u(t)| \ : \ t \in R^+\}$. Thus, $Bx \in S$. Also, by $(\widetilde{\mathrm{H}}_1)$, for each $\mu > 0$, there exists $g_\mu \in BC(R^+, R)$ such that if $|x| \leq \mu$, then $|g(t,x)| \leq g_\mu(t)$ for all $t \in R^+$. Now, for $x \in S$ with $\|x\| \leq \mu$, we have

$$|(Ax)(t)| = \left|\int_0^t D(t,s)g(s,x(s))ds\right| \leq \int_0^t D(t,s)g_\mu(s)ds \leq J_0\|g_\mu\|.$$

Thus, $A, B : S \to S$ are well defined, and by $(\widetilde{\mathrm{H}}_4)$, B is a nonlinear contraction satisfying (ii) of Theorem 2.4.

We will apply Theorem 2.4 to show that $B+A$ has a fixed point which is a bounded solution of (4.1). To this end, we shall prove that $A : M \to S$ is compact for some closed convex subset $M \subset S$. Let us first claim that there exists a constant $\mu > 0$ such that $\|x\| < \mu$ whenever $x \in S$ and

$$x = Bx + \lambda Ax \ \lambda \in [0, 1]. \tag{4.11}$$

Now suppose that $x \in S$ satisfying (4.11); that is,

$$x(t) = h(t, x(t)) - \lambda \int_0^t D(t, s)g(s, x(s))ds + u(t) \tag{4.12}$$

and define the Liapunov functional

$$V(t, x(\cdot)) = \lambda^2 \int_0^t D_s(t, s) \left(\int_s^t g(v, x(v))dv \right)^2 ds + \lambda^2 D(t, 0) \left(\int_0^t g(v, x(v))dv \right)^2$$

for $\lambda \in (0, 1]$. We now differentiate $V(t, x(\cdot))$ with respect to t to obtain

$$\begin{aligned} V'(t, x(\cdot)) &= \lambda^2 \int_0^t D_{st}(t, s) \left(\int_s^t g(v, x(v))dv \right)^2 ds \\ &+ 2\lambda^2 g(t, x(t)) \int_0^t D_s(t, s) \int_s^t g(v, x(v))dvds \\ &+ \lambda^2 D_t(t, 0) \left(\int_0^t g(v, x(v))dv \right)^2 \\ &+ 2\lambda^2 D(t, 0)g(t, x(t)) \int_0^t g(v, x(v))dv. \end{aligned}$$

Integrate the third to last term by parts to obtain

$$\begin{aligned} & 2\lambda^2 g(t, x(t)) \left[D(t, s) \int_s^t g(v, x(v))dv \Big|_{s=0}^{s=t} + \int_0^t D(t, s)g(s, x(s))ds \right] \\ = \ & 2\lambda^2 g(t, x(t)) \left[-D(t, 0) \int_0^t g(s, x(s))ds + \int_0^t D(t, s)g(s, x(s))ds \right]. \end{aligned}$$

Cancel terms, use the sign conditions, and use (4.1) in the last step of the process to unite the Liapunov functional and the equation obtaining

$$\begin{aligned} V'(t, x(\cdot))) = \lambda^2 \int_0^t D_{st}(t, s) \left(\int_s^t g(v, x(v))dv \right)^2 ds + \lambda^2 D_t(t, 0) \left(\int_0^t g(v, x(v))dv \right)^2 \\ + 2\lambda^2 g(t, x(t)) \int_0^t D(t, s)g(s, x(s))ds \le 2\lambda^2 g(t, x(t)) \int_0^t D(t, s)g(s, x(s))ds \\ = 2\lambda g(t, x(t))[-x(t) + h(t, x(t)) + u(t)]. \end{aligned} \tag{4.13}$$

By ($\widetilde{\mathrm{H}}_3$) and ($\widetilde{\mathrm{H}}_4$), we see that if $|x(t)| \ge U_1$, then

$$V'(t, x(\cdot)) \le -2\lambda |g(t, x(t))| [|x(t)| - \psi(|x(t)|) - M^*]$$

where M^* is defined in (4.10). Taking into account condition (4.4) and applying the argument similar to that above (3.4), we arrive at

$$V'(t,x(\cdot)) \le -\lambda|g(t,x(t))| + \lambda L \tag{4.14}$$

where L depends on U_1 and M^*. By the Schwarz inequality, we have

$$\begin{aligned}
&\lambda^2\left(\int_0^t D_s(t,s)\int_s^t g(v,x(v))dvds\right)^2\\
\le\ &\lambda^2\int_0^t D_s(t,s)ds\int_0^t D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds \le D_1V(t,x(\cdot))
\end{aligned} \tag{4.15}$$

where D_1 is defined in (4.8). We have just integrated the left-hand side by parts, obtaining

$$\begin{aligned}
&\left(\int_0^t D_s(t,s)\int_s^t g(v,x(v))dvds\right)^2\\
=\ &\left(-D(t,0)\int_0^t g(s,x(s))ds + \int_0^t D(t,s)g(s,x(s))ds\right)^2
\end{aligned}$$

so that by (4.15) and (4.12) we now have

$$\begin{aligned}
D_1V(t,x(\cdot)) &\ge \lambda^2\left(\int_0^t D_s(t,s)\int_s^t g(v,x(v))dvds\right)^2\\
&= \lambda^2\left(-D(t,0)\int_0^t g(s,x(s))ds + \int_0^t D(t,s)g(s,x(s))ds\right)^2\\
&= \left(x(t)-h(t,x(t))-u(t)+\lambda D(t,0)\int_0^t g(s,x(s))ds\right)^2\\
&\ge \frac{1}{2}[x(t)-h(t,x(t))-u(t)]^2 - \lambda^2\left(D(t,0)\int_0^t g(s,x(s))ds\right)^2\\
&\ge \frac{1}{2}[x(t)-h(t,x(t))-u(t)]^2 - D_1V(t,x(\cdot)).
\end{aligned}$$

Here we have used the inequality $2(a^2+b^2) \ge (a+b)^2$. It is now clear that

$$[x(t)-h(t,x(t))-u(t)]^2 \le 4D_1V(t,x(\cdot)). \tag{4.16}$$

We now show that $V(t,x(\cdot))$ is bounded. If $V(t,x(\cdot))$ is not bounded, then there exists a sequence $\{t_n\} \uparrow \infty$ with $V(t_n,x(\cdot)) \to \infty$ as $n \to \infty$ and

$$V(t_n,x(\cdot)) \ge V(s,x(\cdot)) \text{ for } 0 \le s \le t_n.$$

It then follows from (4.14) that

$$\begin{aligned} 0 &\leq V(t_n, x(\cdot)) - V(s, x(\cdot)) \\ &\leq -\lambda \int_s^{t_n} |g(v, x(v))| dv + \lambda L(t_n - s). \end{aligned}$$

This implies

$$\int_s^{t_n} |g(v, x(v))| dv \leq L(t_n - s). \tag{4.17}$$

Substitute (4.17) into $V(t_n, x(\cdot))$ to obtain

$$\begin{aligned} V(t_n, x(\cdot)) &= \lambda^2 \int_0^{t_n} D_s(t_n, s) \left(\int_s^{t_n} g(v, x(v)) dv \right)^2 ds \\ &\quad + \lambda^2 D(t_n, 0) \left(\int_0^{t_n} g(v, x(v)) dv \right)^2 \\ &\leq \lambda^2 \int_0^{t_n} D_s(t_n, s) \left[L^2 (t_n - s)^2 \right] ds + \lambda^2 D(t_n, 0) L^2 (t_n)^2 \leq (J_2 + D_2) L^2, \end{aligned}$$

a contradiction. This implies that $V(t, x(\cdot)) \leq (J_2 + D_2)L^2$ for all $t \geq 0$, and therefore, by (4.16) we have

$$[x(t) - h(t, x(t)) - u(t)]^2 \leq 4D_1 V(t, x(\cdot)) \leq 4D_1(J_2 + D_2)L^2. \tag{4.18}$$

Again, observe that

$$|x(t) - h(t, x(t)) - u(t)| \geq |x(t)| - \psi(|x(t)|) - M^*.$$

By (4.4) in ($\widetilde{\mathrm{H}}_4$), there exists a constant $\mu > 0$ such that $r \geq \mu$ implies

$$r - \psi(r) - M^* > \sqrt{4D_1(J_2 + D_2)}\, L. \tag{4.19}$$

We now claim that $|x(t)| < \mu$ for all $t \geq 0$. Suppose there exists $t^* \geq 0$ with $|x(t^*)| \geq \mu$. Then by (3.18) and (3.19), we have

$$\begin{aligned} 4D_1(J_2 + D_2)L^2 &< [|x(t^*)| - \psi(|x(t^*)|) - M^*]^2 \\ &\leq |x(t^*) - h(t^*, x(t^*)) - u(t^*)|^2 \\ &\leq 4D_1(J_2 + D_2)L^2 \end{aligned}$$

a contradiction, and thus $\|x\| < \mu$ whenever x is a solution of (4.11) with $\lambda > 0$. For $\lambda = 0$, (4.11) becomes $x = Bx$ which has a unique solution x^* by Theorem 2.1. We may now assume that $\mu > \|x^*\|$ and define

$$M = \{x \in S : \|x\| \leq \mu\} \tag{4.20}$$

which is a closed convex subset of S. By the argument above, if $\lambda \in [0, 1]$ and $x = Bx + \lambda Ax$, then $\|x\| < \mu$. This yields $x \in M/\partial M$.

Next we show that $A : M \to S$ is continuous and AM is contained in a compact subset of S. Let $\phi_1, \phi_2 \in M$. By ($\widetilde{\mathrm{H}}_1$), there exists a constant L_μ such that

$$|g(s, \phi_1(s)) - g(s, \phi_2(s))| \leq L_\mu |\phi_1(s) - \phi_2(s)| \leq L_\mu \|\phi_1 - \phi_2\|.$$

Thus, for $t \geq 0$, we have

$$
\begin{aligned}
|(A\phi_1)(t) - (A\phi_2)(t)| &\leq \int_0^t D(t,s)|g(s,\phi_1(s)) - g(s,\phi_2(s))|ds \\
&\leq J_0 L_\mu \|\phi_1 - \phi_2\|.
\end{aligned}
$$

This implies that A is continuous on M. If $0 \leq t_1 \leq t_2$ and $\phi \in M$, then

$$
\begin{aligned}
& |A\phi(t_1) - A\phi(t_2)| \\
= & \left| \int_0^{t_1} D(t_1,s)g(s,\phi(s))ds - \int_0^{t_2} D(t_2,s)g(s,\phi(s))ds \right| \\
\leq & \int_0^{t_1} |D(t_1,s) - D(t_2,s)||g(s,\phi(s))|ds + \left| \int_{t_1}^{t_2} D(t_2,s)g(s,\phi(s))ds \right| \\
\leq & \int_0^{t_1} |D(t_1,s) - D(t_2,s)|g_\mu(s)ds + \int_{t_1}^{t_2} D(t_2,t_2)g_\mu(s)ds \\
\leq & Q_\mu|t_2 - t_1| + D_1\|g_\mu\||t_2 - t_1|
\end{aligned}
$$

where Q_μ is defined in $(\widetilde{\mathrm{H}}_6)$. This implies that AM is equi-continuous. For each $\phi \in M$, we also have

$$
|(A\phi)(t)| \leq \int_0^t D(t,s)|g(s,x(s))|ds \leq \int_0^t D(t,s)g_\mu(s)ds := q(t).
$$

Then $q(t) \to 0$ as $t \to \infty$ by $(\widetilde{\mathrm{H}}_5)$, and so by Lemma 4.1, AM lies in a compact subset of S. By Theorem 2.4, there exists $y \in M$ satisfying $y = By + Ay$. Thus, y is a bounded solution of (4.1) on R^+. This completes the proof.

Remark 4.1. In Theorem 4.1, a unique bounded solution may be obtained by assuming that $h(t,x)$ and $g(t,x)$ satisfy additional Lipschitz conditions and applying Gronwall's inequality. We omit this part. Notice also that by the proof of Theorem 4.1, any solution $x = x(t)$ of (4.1) defined on R^+ is bounded.

For attractivity of solutions in (4.1), we shall define

$$
\mathcal{G}_0 = \{\, u \in \mathcal{G} \;:\; h(t,0) + u(t) \to 0 \text{ as } t \to \infty\} \tag{4.21}
$$

where $\mathcal{G}$ is given in (4.3) and make the following assumptions:

(P_1) $xg(t,x) \geq 0$ for all $t \geq 0$ and $x \in R$,
(P_2) $D(t,0)t^2 \to 0$ as $t \to \infty$,
(P_3) $\displaystyle\int_0^p D_s(t,s)(t-s)^2 ds \to 0$ as $t \to \infty$ for each fixed p,
(P_4) For each $\mu > 0$ and $\alpha > 0$, there exists $\beta > 0$ such that $|x| \leq \mu$ implies $|g(t,x)| \leq \alpha + \beta(|x| - \psi(|x|))$ for all $t \geq 0$, where ψ is defined in $(\widetilde{\mathrm{H}}_4)$.

Note that (P_3) is a fading memory condition of the integral $\int_0^t D_s(t,s)(t-s)^2 ds$. If $D(t,s) = D(t-s)$ is of convolution type, then $(\widetilde{\mathrm{H}}_5)$ implies (P_3). If we choose $g(t,x) = x^3$ and ψ as defined in (2.4), then (P_4) is satisfied. In fact, suppose that

$\mu > 0$ and $\alpha > 0$ are given with $|x| \le \mu$. Then for $x \ne 0$ we have

$$\begin{aligned} |g(t,x)| &= |x^3| \le |x|^2\mu \le \alpha + \left(\frac{\mu^2}{\alpha}\right)|x|^4 \\ &= \alpha + \left(\frac{\mu^2}{\alpha}\right)\frac{|x|^4}{|x|-\psi(|x|)}(|x|-\psi(|x|)) \le \alpha + \beta(|x|-\psi(|x|)) \end{aligned}$$

where $\beta = (\mu^2/\alpha)\max\{64\mu, 64\mu^3/3\}$. Here we have used the inequality $2ab \le \delta a^2 + b^2/\delta$ with $\delta > 0$.

Theorem 4.2. Suppose that $(\widetilde{\mathrm{H}}_1)$ - $(\widetilde{\mathrm{H}}_6)$ and (P_1) - (P_4) hold. Then every solution of (4.1) defined on R^+ tends to zero as $t \to \infty$ for each $u \in \mathcal{G}_0$.

Proof. Let $x = x(t)$ be a fixed solution of (4.1) on R^+. Since $(\widetilde{\mathrm{H}}_1)$ - $(\widetilde{\mathrm{H}}_6)$ hold, by Theorem 4.1, all solutions of (4.1) on R^+ are bounded (see Remark 4.1). Let $\|x\| \le \mu$ for some $\mu > 0$ and define

$$V(t,x(\cdot)) = \int_0^t D_s(t,s)\left(\int_s^t g(v,x(v))dv\right)^2 ds + D(t,0)\left(\int_0^t g(v,x(v))dv\right)^2.$$

Then by (4.13) with $\lambda = 1$, we have

$$\begin{aligned} V'(t,x(\cdot))) &\le 2g(t,x(t))[-x(t)+h(t,x(t))+u(t)] \\ &= 2g(t,x(t))[-x(t)+h(t,x(t))-h(t,0)+h(t,0)+u(t)] \\ &\le -2|g(t,x(t))|\,[|x(t)|-\psi(|x(t)|)] + 2\|g_\mu\|\,[|h(t,0)+u(t)|] \\ &\le -2|g(t,x(t))|[|x(t)|-\psi(|x(t)|)] + M_\mu(t) \end{aligned} \tag{4.22}$$

where $M_\mu(t) = \sup\{2\|g_\mu\||h(s,0)+u(s)| \;:\; s \ge t\}$. Note that $M_\mu(t)$ is decreasing and converges to zero as $t \to \infty$ for each fixed $u \in \mathcal{G}_0$. We first claim that $V(t,x(\cdot)) \to 0$ as $t \to \infty$. Observe that by $(\widetilde{\mathrm{H}}_5)$, $V(t,x(\cdot))$ is bounded for $\|x\| \le \mu$. Now suppose that

$$\limsup_{t\to\infty} V(t,x(\cdot)) = P \ge 0.$$

Then for any $\varepsilon > 0$, there exists a positive constant $K > 0$ and a sequence $\{t_n\} \uparrow \infty$ with

$$V(t_n,x(\cdot)) \ge V(s,x(\cdot)) - \varepsilon \text{ for } K \le s \le t_n. \tag{4.23}$$

In fact, by the definition of $\limsup_{t\to\infty} V(t,x(\cdot))$, for any $\varepsilon > 0$, there exists $K > 0$ such that $t \ge K$ implies

$$-\frac{\varepsilon}{2} < \sup_{s\ge t} V(s,x(\cdot)) - P < \frac{\varepsilon}{2}.$$

Thus, there exists a sequence $\{t_n\} \uparrow \infty$ with $t_1 \ge K$ such that

$$-\frac{\varepsilon}{2} < V(t_n,x(\cdot)) - P < \frac{\varepsilon}{2}.$$

and therefore

$$V(t_n, x(\cdot)) > P - \frac{\varepsilon}{2} = \left(P + \frac{\varepsilon}{2}\right) - \varepsilon > V(s, x(\cdot)) - \varepsilon$$

for all $K \le s \le t_n$ and for $n = 1, 2, \cdots$. By (4.22) and (4.23), we now see that

$$\begin{aligned} -\varepsilon &\le V(t_n, x(\cdot)) - V(s, x(\cdot)) \\ &\le -\int_s^{t_n} |g(s, x(v))|[|x(v)| - \psi(|x(v)|)]dv + M_\mu(K)(t_n - s) \end{aligned}$$

or

$$\int_s^{t_n} |g(s, x(v))|[|x(v)| - \psi(|x(v)|)]dv \le \varepsilon + M_\mu(K)(t_n - s) \tag{4.24}$$

for all $K \le s \le t_n$. Apply (P_1), (P_4), and (4.24) in the following argument to obtain

$$\begin{aligned} V(t_n, x(\cdot)) &= \int_0^K D_s(t_n, s)\left(\int_s^{t_n} g(v, x(v))dv\right)^2 ds \\ &+ \int_K^{t_n} D_s(t_n, s)\left(\int_s^{t_n} g(v, x(v))dv\right)^2 ds + D(t_n, 0)\left(\int_0^{t_n} g(v, x(v))dv\right)^2 \\ &\le \|g_\mu\|^2 \int_0^K D_s(t_n, s)(t_n - s)^2 ds + \|g_\mu\|^2 D(t_n, 0)|t_n^2 \\ &+ \int_K^{t_n} D_s(t_n, s)\left[(t_n - s)\int_s^{t_n} |g(v, x(v))|^2 dv\right] ds \\ &\le \|g_\mu\|^2 \int_0^K D_s(t_n, s)(t_n - s)^2 ds + \|g_\mu\|^2 D(t_n, 0)|t_n^2 \\ &+ \int_K^{t_n} D_s(t_n, s)\left\{(t_n - s)\int_s^{t_n} |g(v, x(v))|\left[\alpha + \beta(|x(v)| - \psi(|x(v)|)\right] dv\right\} ds \\ &\le \|g_\mu\|^2\left\{\int_0^K D_s(t_n, s)(t_n - s)^2 ds + |D(t_n, 0)|t_n^2\right\} + \alpha\|g_\mu\|\int_0^{t_n} D_s(t_n, s)(t_n - s)^2 ds \\ &+ \beta\int_K^{t_n} D_s(t_n, s)(t_n - s)\left\{\int_s^{t_n} |g(v, x(v))|\left[|x(v)| - \psi(|x(v)|)\right] dv\right\} ds \\ &\le \|g_\mu\|^2\left\{\int_0^K D_s(t_n, s)(t_n - s)^2 ds + |D(t_n, 0)|t_n^2\right\} + \alpha\|g_\mu\| J_2 \\ &+ \beta\int_K^{t_n} D_s(t_n, s)(t_n - s)[\varepsilon + M_\mu(K)(t_n - s)]ds \\ &\le \|g_\mu\|^2\left\{\int_0^K D_s(t_n, s)(t_n - s)^2 ds + |D(t_n, 0)|t_n^2\right\} + \alpha\|g_\mu\| J_2 \\ &+ \varepsilon\beta(J_2 + D_1) + M_\mu(K)\beta J_2 \end{aligned} \tag{4.25}$$

where J_2 and D_1 are defined in (4.7) and (4.9), respectively. Now, for a given $\delta > 0$, choose $K > 0$ so large, $\varepsilon > 0$ and $\alpha > 0$ so small that $\alpha J_2\|g_\mu\| + \varepsilon\beta(J_2 + D_1) < \delta$, and $M_\mu(K)\beta J_2 < \delta$. Since $\int_0^K D_s(t_n, s)(t_n - s)^2 ds + |D(t_n, 0)|t_n^2 \to 0$ as $n \to \infty$ by

(P_2) and (P_3), so that as $\delta \to 0$, we see that $V(t_n, x(\cdot)) \to 0$ as $t \to \infty$. This implies that $P = 0$, and therefore, $V(t, x(\cdot)) \to 0$ as $t \to \infty$.

We now show that $x(t) \to 0$ as $t \to \infty$. It follows from (4.16) with $\lambda = 1$ that

$$4D_1V(t, x(\cdot)) \geq [x(t) - h(t, x(t)) - u(t)]^2$$

and thus,

$$\begin{aligned} 4D_1V(t, x(\cdot)) &\geq [x(t) - (h(t, x(t)) - h(t, 0)) - (h(t, 0) + u(t))]^2 \\ &\geq \frac{1}{2}[x(t) - (h(t, x(t)) - h(t, 0))]^2 - [h(t, 0) + u(t)]^2 \\ &\geq \frac{1}{2}[|x(t)| - \psi(|x(t)|)]^2 - [h(t, 0) + u(t)]^2 \end{aligned}$$

This implies that

$$[|x(t)| - \psi(|x(t)|)]^2 \leq 8D_1V(t, x(\cdot)) + 2[h(t, 0) + u(t)]^2 \to 0 \tag{4.26}$$

as $t \to \infty$, and therefore, by ($\widetilde{\mathrm{H}}_4$), we have $x(t) \to 0$ as $t \to \infty$. This completes the proof.

Concluding Remark. The fixed point theory presented here combined with Liapunov's direct method seems to provide a systematic way to solve very different problems. All of these applications and examples are important classical problems and are not merely contrived to make our point here, and they can easily be generalized to systems. We refer the reader to, for example, the work of Burton ([6]-[8]), Chukwu [12], Gripenberg, Londen, and Staffans [15], Kolmanovskii and Myshkis [16] for further reference in neutral differential equations, integral equations and control systems.

References

[1] R.P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge University Press, 2001.

[2] K. Balachandran, R. Sakthivel, *Controllability of functional semilinear integrodifferential systems in Banach spaces*, J. Math. Anal. Appl., **255**(2001), 447-457.

[3] J. Banaś, R. Rzepka, *An application of a measure of noncompactness in the study of asymptotic stability*, Appl. Math. Lett., **16**(2003), 1-6.

[4] D.W. Boyd, J.S.W. Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., **20**(1969), 458-464.

[5] F.E. Browder, *Problèmes non-linéaires*, Lecture Notes, University of Montreal, 1966.

[6] T.A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, Mineola, New York, 2006.

[7] T.A. Burton, *Integral equations, implicit functions, and fixed points*, Proc. Amer. Math. Soc., **124**(1996), 2383-2389.

[8] T.A. Burton, *Boundedness and periodicity in integral and integro-differential equations*, Diff. Eq. Dynam. Syst., **1**(1993), 161-172.

[9] T.A. Burton, T. Furumochi, *A note on stability by Schauder's theorem*, Funkcialaj Ekvacioj, **44**(2001), 73-82.

[10] T A. Burton, C. Kirk, *A fixed point theorem of Krasnoselskii-Schaefer type*, Math. Nachr., **189**(1998), 23-31.

[11] T.A. Burton, B. Zhang, *Fixed points and stability of an integral equation: nonuniqueness*, Appl. Math. Lett., **17**(2004), 839-486.

[12] E.N. Chukwu, *Stability and Time-optimal Control of Hereditary Systems*, Academic Press, 1992.

[13] H. Gao, B. Zhang, *Fixed points and controllability in delay systems*, Fixed Point Theory and Applications, **2006**(2006), 1-14.
[14] A. Granas, J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.
[15] G. Gripenberg, S.O. Londen, O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
[16] V. Komanovskii, A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*, Kluwer Academic Publishers, 1999.
[17] M.A. Krasnoselskii, *Some problems of nonlinear analysis*, Amer. Math. Soc. Transl., **10**(1958), 345-409.
[18] E. Kreyszig, *Introductory Functional Analysis with Applications*, Wiley, New York, 1978.
[19] T. Küpper, Y. Li, B. Zhang, *Periodic solutions for dissipative-repulsive systems*, Tohoku Math. J., **52**(2000), 321-329.
[20] S. Lefschitz, *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.
[21] X. Liao, *Absolute Stability of Nonlinear Control Systems*, Springer, New York, 1993.
[22] Y. Liu, Z. Liu, *Schaefer type theorem and periodic solutions of evolution equations*, J. Math. Anal. Appl., **316**(2006), 237-255.
[23] D. O'Regan, *Fixed-point theory for the sum of two operators*, Appl. Math. Lett., **9**(1996), 1-8.
[24] A.J.B. Potter, *An elementary version of the Leray-Schauder theorem*, J. London Math. Soc., **5**(1972), 414-416.
[25] H. Schaefer, *Über Die Methode Der a Priori Schranken*, Math. Ann., **129**(1955), 415-416.
[26] J. Schauder, *Über den Zusammenhang zwischen der Eindeutigkeit and Lösbarkeit partieller Differentialgleichungen zweiter Ordung von Elliptischen Typus*, Math. Ann., **106**(1932), 661-721.
[27] D.R. Smart, *Fixed Point Theorems*, Cambridge University Press, Cambridge, 1980.
[28] J. Wu, H. Xia, B. Zhang, *Topological transversality and periodic solutions of neutral functional differential equations*, Proc. Royal Soc. Edinburgh, **129**(1999), 199-220.
[29] E. Zeidler, *Nonlinear Functional Analysis and Its Applications I: Fixed Point Theorems*, Springer-Verlag, New York, 1986.
[30] B. Zhang, *Periodic solutions of nonlinear abstract differential equations with infinite delay*, Funkcialaj Ekvacioj, **36**(1993), 433-378.
[31] B. Zhang, *Fixed points and stability in differential equations with variable delays*, Nonlinear Anal., **63**(2005), 233-242.

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