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FIXED POINT METHODS FOR THE STABILITY OF GENERAL QUADRATIC FUNCTIONAL EQUATION

M. ESHAGHI GORDJI*, H. KHODAEI** AND J. M. RASSIAS***

*Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
E-mail: madjid.eshaghi@gmail.com

**Department of Mathematics, Semnan University,
P. O. Box 35195-363, Semnan, Iran
E-mail: khodaei.hamid.math@gmail.com

***Section of Mathematics and Informatics, Pedagogical
Department National and Capodistrian University of Athens, 4, Agamemnonos
St., Aghia Paraskevi, Athens 15342, Greece
E-mail: jrassias@primedu.uoa.gr

Abstract. In this paper we obtain the general solution and prove the stability in Banach spaces and also the stability using the alternative fixed point of quadratic functional equation:

$$f(ax+by+2cz)+f(ax+by-2cz)+f(ax-by+2cz)+f(ax-by-2cz)$$

$$=4a^2f(x)+4b^2f(y)+16c^2f(z)$$

for any fixed integers a,b,c with $a,b,c\neq 0,\pm 1$ and $a\pm b\neq 0$.

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1. Introduction

Traditionally when we investigate stability of functional equations the problem posed by S.M. Ulam in 1940 [27] and the well-known theorem of Hyers [13] which came within a year are taken as a starting point. Following Ulam and D.H. Hyers a great number of papers on the subject have been published, generalizing Ulam's problem in various directions. One of these possible generalizations is to allow the Cauchy difference to be unbounded, to be controlled by a function, not necessarily by a constant. Perhaps Tosio Aoki in 1950 was the first author treating this problem [2]. He proved that if a mapping $f:X\to Y$ between two Banach spaces satisfies

$$\|f(x+y)-f(x)-f(y)\|\leq\varphi(x,y)$$

for all $x, y \in X$, where $\varphi(x,y) = K(\| x \|^p + \| y \|^p)$ with $(K \geq 0, 0 \leq p < 1)$, then there exists a unique additive function $A : X \to Y$ such that

$$\| f(x) - A(x) \| \leq \frac{K}{1 - 2^{p-1}} \| x \|^p$$

for all $x \in X$. In (D.G. Bourgin [4], 1951) as well as in (Th.M. Rassias [24], 1978), (G.L. Forti [10, 11]), (J.M. Rassias, [19]–[23]) and (Gavruta [12], 1994) the stability problem with unbounded Cauchy differences is considered.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is related to symmetric bi–additive function and is called a quadratic functional equation and every solution of the quadratic equation (1.1) is said to be a quadratic function. It is well known that a function f between two real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x,x)$ for all x where

$$B(x,y) = \frac{1}{4}(f(x+y) - f(x-y))$$

(see [1, 14]). Skof proved Hyers–Ulam stability problem for quadratic functional equation (1.1) for a class of functions $f : A \longrightarrow B$, where A is normed space and B is a Banach space (see [26]). In 1992, Czerwik [9] proved the Hyers–Ulam–Rassias stability of the equation (1.1). Recently, J.H. Bae and W.G. Park [3] have investigated the stability of the following n-dimensional quadratic functional equation:

$$f(\sum_{i=1}^{n} x_i) + \sum_{1 \leq i < j \leq n} f(x_i - x_j) = n \sum_{i=1}^{n} f(x_i)$$

in Banach modules over a Banach algebra and a C*-algebra.

In this paper, we investigate the stability in Banach spaces and the stability using the alternative fixed point of general quadratic type functional equation:

$$\begin{aligned} f(ax+by+2cz) + f(ax+by-2cz) + f(ax-by+2cz) + f(ax-by-2cz) \\ = 4a^2 f(x) + 4b^2 f(y) + 16c^2 f(z) \end{aligned} \tag{1.2}$$

for any fixed integers $a, b, c \neq 0, \pm 1$ and $a \pm b \neq 0$.

2. Quadratic functional equation

Let X and Y be real vector spaces. We here present the general solution of (1.2).

Theorem 2.1. *A function $f : X \to Y$ satisfies the functional equation (1.2) if and only if f is quadratic.*

Proof. Suppose that f satisfies the functional equation (1.2) for any fixed integers $a, b, c \neq 0, \pm 1$ and $a \pm b \neq 0$. Putting $x = y = z = 0$ in (1.2), we get

$$4(a^2 + b^2 + 4c^2 - 1) f(0) = 0$$

but since $a, b, c \neq 0$, therefore $f(0) = 0$. Letting $z := -z$ in (1.2), we have

$$f(ax+by-2cz) + f(ax+by+2cz) + f(ax-by-2cz) + f(ax-by+2cz)$$

$$= 4a^2 f(x) + 4b^2 f(y) + 16c^2 f(-z) \tag{2.1}$$

for all $x, y, z \in X$. If we compare (1.2) with (2.1), we obtain

$$16c^2 f(-z) = 16c^2 f(z)$$

for all $z \in X$. But since $c \neq 0$ so $f(-z) = f(z)$, which implies that f is even. Setting $y = z = 0$ in (1.2), gives $f(ax) = a^2 f(x)$ for all $x \in X$. Letting $x = z = 0$ in (1.2) and using the evenness of f, gives $f(by) = b^2 f(y)$ for all $y \in X$. So

$$f(abx) = a^2 b^2 f(x)$$

for all $x \in X$. Replacing x, y and z by bx, ay and 0 in (1.2), respectively, we have

$$f(abx - aby) + f(abx + aby) = 2a^2 f(bx) + 2b^2 f(ay) \tag{2.2}$$

for all $x, y \in X$. Now, since $a, b \neq 0$ by using $f(ax) = a^2 f(x)$, $f(bx) = b^2 f(x)$ and $f(abx) = a^2 b^2 f(x)$, it follows from (2.2) that

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

for all $x, y \in X$. Therefore f is quadratic.

Conversely, suppose that f is quadratic thus f satisfies the functional equation (1.1). Putting $x = y = 0$ in (1.1), we get $f(0) = 0$. Setting $x = 0$ in (1.1) to get $f(-y) = f(y)$. Letting $y = x$ and $y = 2x$ in (1.1), respectively, we obtain that $f(2x) = 4f(x)$ and $f(3x) = 9f(x)$ for all $x \in X$. By induction, we lead to $f(kx) = k^2 f(x)$ for all positive integer k.

Replacing x and y by $2x + y$ and $2x - y$ in (1.1), respectively, then by using the identity $f(kx) = k^2 f(x)$, we get

$$f(2x + y) + f(2x - y) = 8f(x) + 2f(y) \tag{2.3}$$

for all $x, y \in X$. It follows from (1.1) and (2.3) that

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x) \tag{2.4}$$

for all $x, y \in X$. Now, replacing x and y by $3x + y$ and $3x - y$ in (1.1), respectively, then by using (1.1) and the identity $f(kx) = k^2 f(x)$, we have

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x) \tag{2.5}$$

for all $x, y \in X$. Thus by using the above method, by induction, we infer that

$$f(cx + y) + f(cx - y) = f(x + y) + f(x - y) + 2(c^2 - 1)f(x) \tag{2.6}$$

for all $x, y \in X$ and each positive integer $c > 1$ and for a negative integer $c < -1$, replacing c by $-c$ one can easily prove the validity of (2.6). Therefore (1.1) implies (2.6) for any integer $c \neq 0, \pm 1$.

Putting $x = x + y$ and $y = x - y$ in (2.4) and then by using the identity $f(2x) = 4f(x)$, we obtain

$$f[(c+1)x+(c-1)y]+f[(c-1)x+(c+1)y] = 4(f(x)+f(y))+2(c^2-1)f(x+y) \tag{2.7}$$

for all $x, y \in X$. Replacing x and y by $x + cz$ and $y + cz$ in (2.7), respectively, one gets that

$$f[(c + 1)x + (c - 1)y + 2c^2 z] + f[(c - 1)x + (c + 1)y + 2c^2 z]$$

$$= 4(f(x+cz)+f(y+cz))+2(c^2-1)f(x+y+2cz) \tag{2.8}$$

for all $x, z, y \in X$. Also, replacing x and y by $x-cz$ and $y-cz$ in (2.7), respectively, one gets that

$$\begin{aligned} &f[(c+1)x+(c-1)y-2c^2z]+f[(c-1)x+(c+1)y-2c^2z] \\ &\quad = 4(f(x-cz)+f(y-cz))+2(c^2-1)f(x+y-2cz) \end{aligned} \tag{2.9}$$

for all $x, z, y \in X$. Now, by adding (2.8) and (2.9), we arrive at

$$\begin{aligned} &f[(c+1)x+(c-1)y+2c^2z]+f[(c+1)x+(c-1)y-2c^2z] \\ &+f[(c-1)x+(c+1)y+2c^2z]+f[(c-1)x+(c+1)y-2c^2z] \\ &\quad = 4(f(x+cz)+f(x-cz)+f(y+cz)+f(y-cz)) \\ &\qquad +2(c^2-1)(f(x+y+2cz)+f(x+y-2cz)) \end{aligned} \tag{2.10}$$

for all $x, z, y \in X$. On the other hand, we substitute $x = x+cz$ and $y = y-cz$ in (2.7), we obtain

$$\begin{aligned} &f[(c+1)x+(c-1)y+2cz]+f[(c-1)x+(c+1)y-2cz] \\ &\quad = 4(f(x+cz)+f(y-cz))+2(c^2-1)f(x+y) \end{aligned} \tag{2.11}$$

for all $x, z, y \in X$. And putting $x = x-cz$ and $y = y+cz$ in (2.7), we get

$$\begin{aligned} &f[(c+1)x+(c-1)y-2cz]+f[(c-1)x+(c+1)y+2cz] \\ &\quad = 4(f(x-cz)+f(y+cz))+2(c^2-1)f(x+y) \end{aligned} \tag{2.12}$$

for all $x, z, y \in X$. Adding (2.11) to (2.12), we lead to

$$\begin{aligned} &f[(c+1)x+(c-1)y+2cz]+f[(c+1)x+(c-1)y-2cz] \\ &+f[(c-1)x+(c+1)y+2cz]+f[(c-1)x+(c+1)y-2cz] \\ &\quad = 4(f(x+cz)+f(x-cz)+f(y+cz)+f(y-cz)) \\ &\qquad +4(c^2-1)f(x+y) \end{aligned} \tag{2.13}$$

for all $x, z, y \in X$. Now, replacing z by cz in (2.13), gives

$$\begin{aligned} &f[(c+1)x+(c-1)y+2c^2z]+f[(c+1)x+(c-1)y-2c^2z] \\ &+f[(c-1)x+(c+1)y+2c^2z]+f[(c-1)x+(c+1)y-2c^2z] \\ &\quad = 4(f(x+c^2z)+f(x-c^2z)+f(y+c^2z)+f(y-c^2z)) \\ &\qquad +4(c^2-1)f(x+y) \end{aligned} \tag{2.14}$$

for all $x, z, y \in X$. If we compare (2.10) with (2.14), we conclude that

$$\begin{aligned} &4(f(x+cz)+f(x-cz)+f(y+cz)+f(y-cz)) \\ &\quad +2(c^2-1)(f(x+y+2cz)+f(x+y-2cz)) \\ &= 4(f(x+c^2z)+f(x-c^2z)+f(y+c^2z)+f(y-c^2z)) \\ &\qquad +4(c^2-1)f(x+y) \end{aligned} \tag{2.15}$$

for all $x, z, y \in X$. Also, if substituting y by cy into (2.6) and then using the identity $f(cx) = c^2f(x)$, we get

$$f(x+cy)+f(x-cy) = c^2f(x+y)+c^2f(x-y)+2(1-c^2)f(x) \tag{2.16}$$

for all $x, y \in X$ and any integer $c \neq 0, \pm 1$. It follows from (2.16) that

$$\begin{aligned} &f(x+cz)+f(x-cz)+f(y+cz)+f(y-cz) \\ &= c^2(f(x+z)+f(x-z)+f(y+z)+f(y-z))+2(1-c^2)(f(x)+f(y)) \quad (2.17) \end{aligned}$$

and

$$\begin{aligned} &f(x+c^2z)+f(x-c^2z)+f(y+c^2z)+f(y-c^2z) \\ &= c^4(f(x+z)+f(x-z)+f(y+z)+f(y-z))+2(1-c^4)(f(x)+f(y)) \quad (2.18) \end{aligned}$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. Hence, according to (2.15) and (2.17) and (2.18), we obtain that

$$\begin{aligned} &f(x+y+2cz)+f(x+y-2cz) \\ &= 2f(x+y)+2c^2(f(x+z)+f(x-z)+f(y+z)+f(y-z))-4c^2(f(x)+f(y)) \quad (2.19) \end{aligned}$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. It follows from (1.1) and (2.19) that

$$f(x+y+2cz)+f(x+y-2cz) = 2f(x+y)+8c^2f(z) \quad (2.20)$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. Replacing y by $-y$ in (2.20), we get

$$f(x-y+2cz)+f(x-y-2cz) = 2f(x-y)+8c^2f(z) \quad (2.21)$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. Adding (2.20) to (2.21), we obtain that

$$\begin{aligned} &f(x+y+2cz)+f(x+y-2cz)+f(x-y+2cz)+f(x-y-2cz) \\ &= 2(f(x+y)+f(x-y))+16c^2f(z) \quad (2.22) \end{aligned}$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. We substitute $x = ax$ and $y = by$ in (2.22), we lead to

$$\begin{aligned} &f(ax+by+2cz)+f(ax+by-2cz)+f(ax-by+2cz)+f(ax-by-2cz) \\ &= 2(f(ax+by)+f(ax-by))+16c^2f(z) \quad (2.23) \end{aligned}$$

for all $x, y, z \in X$ and any integer $c \neq 0, \pm 1$. On the other hand, it is noted that Eq.(2.6) implies the following equation

$$f(ax+y)+f(ax-y) = f(x+y)+f(x-y)+2(a^2-1)f(x) \quad (2.24)$$

for all $x, y \in X$ and any integer $a \neq 0, \pm 1$. Also, it is noted that Eq.(2.16) implies the following equation

$$f(x+by)+f(x-by) = f(x+y)+f(x-y)+2(1-b^2)f(x) \quad (2.25)$$

for all $x, y \in X$ and any integer $b \neq 0, \pm 1$. Replacing y by by in (2.24), we observe that

$$f(ax+by)+f(ax-by) = f(x+by)+f(x-by)+2(a^2-1)f(bx) \quad (2.26)$$

for all $x, y \in X$ and any integers $a, b \neq 0, \pm 1$. Hence, according to (2.25) and (2.26), we get

$$f(ax+by)+f(ax-by) = b^2f(x+y)+b^2f(x-y)+2(a^2-b^2)f(x) \quad (2.27)$$

for all $x, y \in X$ and any integers $a, b \neq 0, \pm 1$ and $a \pm b \neq 0$. Now, by using (1.1) and (2.27), we have

$$f(ax+by)+f(ax-by) = 2a^2f(x)+2b^2f(y) \quad (2.28)$$

for all $x, y \in X$ and any integers $a, b \neq 0, \pm 1$ and $a \pm b \neq 0$. Finally, it follows from (2.23) and (2.28) that

$$f(ax+by+2cz)+f(ax+by-2cz)+f(ax-by+2cz)+f(ax-by-2cz) \\ = 4a^2f(x)+4b^2f(y)+16c^2f(z)$$

for all $x, y, z \in X$ and any integers $a, b, c \neq 0, \pm 1$ and $a \pm b \neq 0$. $\square$

3. Stability in Banach spaces

From this point on, let X be a real vector space and let Y be a Banach space. Before taking up the main subject, for the given function $f : X \to Y$ we define the difference operator $\Delta_f : X \times X \times X \to Y$ by

$$\Delta_f(x,y,z) := f(ax+by+2cz)+f(ax+by-2cz)+f(ax-by+2cz)+f(ax-by-2cz) \\ -4a^2f(x)-4b^2f(y)-16c^2f(z)$$

for all $x, y, z \in X$ and any integers $a, b, c \neq 0, \pm 1$ and $a \pm b \neq 0$.

Theorem 3.1. *Let $j \in \{-1, 1\}$ is fixed, and let $\varphi : X \times X \times X \to [0, \infty)$ be a function such that*

$$\widetilde{\varphi}(x) := \sum_{i=\frac{1-j}{2}}^{\infty} \frac{1}{a^{2ij}} \varphi(a^{ij}x, 0, 0) < \infty \tag{3.1}$$

$$\lim_{n\to\infty} \frac{1}{a^{2nj}} \varphi(a^{nj}x, a^{nj}y, a^{nj}z) = 0 \tag{3.2}$$

for all $x, y, z \in X$. Suppose that $f : X \to Y$ is a function that satisfies

$$\|\Delta_f(x,y,z)\| \leq \varphi(x,y,z) \tag{3.3}$$

for all $x, y, z \in X$. Furthermore, assume that $f(0) = 0$ in (3.3) for the case $j = 1$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{4a^{1+j}} \widetilde{\varphi}(\frac{x}{a^{\frac{1-j}{2}}}), \tag{3.4}$$

for all $x \in X$.

Proof. For $j = 1$, putting $y = z = 0$ in (3.3), we have

$$\|4f(ax) - 4a^2f(x)\| \leq \varphi(x, 0, 0) \tag{3.5}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{a^2}f(ax)\| \leq \frac{1}{4a^2}\varphi(x, 0, 0) \tag{3.6}$$

for all $x \in X$. Replacing x by ax in (3.6) and dividing by a^2 and summing the resulting inequality with (3.6), we get

$$\|f(x) - \frac{1}{a^4}f(a^2x)\| \leq \frac{1}{4a^2}(\varphi(x,0,0) + \frac{\varphi(ax,0,0)}{a^2}) \tag{3.7}$$

for all $x \in X$. Hence

$$\|\frac{1}{a^{2k}}f(a^kx) - \frac{1}{a^{2m}}f(a^mx)\| \leq \frac{1}{4a^2}\sum_{i=k}^{m-1}\frac{1}{a^{2i}}\varphi(a^ix, 0, 0) \tag{3.8}$$

for all nonnegative integers m and k with $m > k$ and for all $x \in X$. It follows from (3.1) and (3.8) that the sequence $\{\frac{1}{a^{2n}} f(a^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{a^{2n}} f(a^n x)\}$ converges. Therefore, one can define the function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{a^{2n}} f(a^n x)$$

for all $x \in X$. By (3.2) for $j = 1$ and (3.3),

$$\|\Delta_Q(x, y, z)\| = \lim_{n\to\infty} \frac{1}{a^{2n}} \|\Delta_f(a^n x, a^n y, a^n z)\| \le \lim_{n\to\infty} \frac{1}{a^{2n}} \varphi(a^n x, a^n y, a^n z) = 0$$

for all $x, y, z \in X$. So $\Delta_Q(x, y, z) = 0$. By Theorem 2.1, the function $Q : X \to Y$ is quadratic. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.8), we get the inequality (3.4) for $j = 1$.

Now, let $Q' : X \to Y$ be another quadratic function satisfying (1.3) and (3.4). So

$$\|Q(x) - Q'(x)\| = \frac{1}{a^{2n}} \|Q(a^n x) - Q'(a^n x)\|$$

$$\le \frac{1}{a^{2n}} (\|Q(a^n x) - f(a^n x)\| + \|Q'(a^n x) - f(a^n x)\|)$$

$$\le \frac{1}{2a^2 a^{2n}} \widetilde{\varphi}(a^n x)$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = Q'(x)$ for all $x \in X$. This proves the uniqueness of Q.

Also, for $j = -1$, it follows from (3.5) that

$$\|f(x) - a^2 f(\frac{x}{a})\| \le \frac{1}{4} \varphi(\frac{x}{a}, 0, 0) \tag{3.9}$$

for all $x \in X$. Hence

$$\|a^{2k} f(\frac{x}{a^k}) - a^{2m} f(\frac{x}{a^m})\| \le \frac{1}{4} \sum_{i=k}^{m-1} a^{2i} \varphi(\frac{x}{a^{i+1}}, 0, 0) \tag{3.10}$$

for all nonnegative integers m and k with $m > k$ and for all $x \in X$. It follows from (3.10) that the sequence $\{a^{2n} f(\frac{x}{a^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{a^{2n} f(\frac{x}{a^n})\}$ converges. So one can define the function $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} a^{2n} f(\frac{x}{a^n})$$

for all $x \in X$. By (3.2) for $j = -1$ and (3.3),

$$\|\Delta_Q(x, y, z)\| = \lim_{n\to\infty} a^{2n} \|\Delta_f(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n})\| \le \lim_{n\to\infty} a^{2n} \varphi(\frac{x}{a^n}, \frac{y}{a^n}, \frac{z}{a^n}) = 0,$$

for all $x, y, z \in X$. So $\Delta_Q(x, y, z) = 0$. By Theorem 2.1, the function $Q : X \to Y$ is quadratic. Moreover, letting $k = 0$ and passing the limit $m \to \infty$ in (3.10), we get the inequality (3.4) for $j = -1$. The rest of the proof is similar to the proof of previous section. □

4. Stability using alternative fixed point

In this section, we will investigate the stability of quadratic functional equation (1.2) using alternative fixed point (see [5]–[8], [17, 18]). Before proceeding to the proof, we will state the following theorem.

Theorem 4.1. *(the alternative of fixed point* [16, 25]*). Suppose that we are given a complete generalized metric space (Ω, d) and a strictly contractive mapping $T : \Omega \to \Omega$ with Lipschitz constant L. Then for each given $x \in \Omega$, either*

$d(T^n x, T^{n+1}x) = \infty$ *for all* $n \geq 0$,

or other exists a natural number n_0 such that

⋆ $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \geq n_0$;

⋆ *the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;*

⋆ *y^* is the unique fixed point of T in the set $\Lambda = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;*

⋆ $d(y, y^*) \leq \frac{1}{1-L} d(y, Ty)$ *for all* $y \in \Lambda$.

Theorem 4.2. *Suppose that $j \in \{-1, 1\}$ is fixed, and Let $f : X \to Y$ a function with $f(0) = 0$ for which there exists be a function $\varphi : X \times X \times X \to [0, \infty)$ such that*

$$\lim_{n\to\infty} \frac{1}{a^{2nj}} \varphi(a^{nj}x, a^{nj}y, a^{nj}z) = 0 \tag{4.1}$$

$$\|\Delta_f(x, y, z)\| \leq \varphi(x, y, z) \tag{4.2}$$

for all $x, y, z \in X$. If there exists $L < 1$ such that the function φ has the property

$$\varphi(\frac{x}{a}, 0, 0) \leq L.\ a^2.\ \varphi(\frac{x}{a^2}, 0, 0) \tag{4.3}$$

for all $x \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that, we have the inequality

$$\|f(x) - Q(x)\| \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)}\ \varphi(\frac{x}{a}, 0, 0) \tag{4.4}$$

for all $x \in X$.

Proof. Consider the set $\Omega = \{g |\ g : X \to Y,\ g(0) = 0\}$ and introduce the generalized metric

$$d(g, h) = d_\varphi(g, h) = \inf\{K \in (0, \infty) :\ \|g(x) - h(x)\| \leq K\varphi(\frac{x}{a}, 0, 0),\ x \in X\}$$

on Ω. It is easy to see that (Ω, d) is complete.

Now we define a function $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{a^{2j}}\ g(a^j x)$ for all $x \in X$. Note that for all $g, h \in \Omega$,

$$\begin{aligned}
d(g, h) < K\ &\Rightarrow\ \|g(x) - h(x)\| \leq K\varphi(\frac{x}{a}, 0, 0),\ \text{for all } x \in X,\\
&\Rightarrow\ \|\frac{1}{a^{2j}} g(a^j x) - \frac{1}{a^{2j}} h(a^j x)\| \leq \frac{1}{a^{2j}}\ K\ \varphi(a^{j-1}x, 0, 0),\ \text{for all } x \in X,\\
&\Rightarrow\ \|\frac{1}{a^{2j}} g(a^j x) - \frac{1}{a^{2j}} h(a^j x)\| \leq L\ K\ \varphi(\frac{x}{a}, 0, 0)),\ \text{for all } x \in X,\\
&\Rightarrow\ d(Tg, Th) \leq L\ K.
\end{aligned}$$

Hence we see that

$$d(Tg, Th) \leq L\ d(g, h)$$

for all $g, h \in \Omega$, that is, T is a strictly self-mapping of Ω with the Lipschitz constant L.

Putting $y = z = 0$ in (4.2), we have

$$\|4f(ax) - 4a^2 f(x)\| \leq \varphi(x, 0, 0) \tag{4.5}$$

for all $x \in X$. Now, by using (4.3) and (4.5), we obtain that

$$\|f(x) - \frac{1}{a^2} f(ax)\| \leq \frac{1}{4a^2}\ \varphi(x, 0, 0) \leq \frac{L}{4}\ \varphi(\frac{x}{a}, 0, 0)$$

for all $x \in X$, that is, $d(f, Tf) \leq \frac{L}{4} < \infty$.

If we substitute $x = \frac{x}{a}$ in (4.5), we see that

$$\|f(x) - a^2 f(\frac{x}{a})\| \leq \frac{1}{4}\ \varphi(\frac{x}{a}, 0, 0)$$

for all $x \in X$, that is, $d(f, Tf) \leq \frac{1}{4} < \infty$.

Now, from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{n\to\infty} \frac{1}{a^{2nj}} f(a^{nj} x) \tag{4.6}$$

for all $x \in X$, since $\lim_{n\to\infty} d(T^n f, Q) = 0$.

Also, if we replace x, y and z by $a^{nj}\ x$, $a^{nj}\ y$ and $a^{nj}\ z$ in (2.30), respectively, and divide by a^{2nj}. Then it follows from (4.1) and (4.6) that

$$\|\Delta_Q(x, y, z)\| = \lim_{n\to\infty} \frac{1}{a^{2nj}} \|\Delta_f(a^{nj}x, a^{nj}y, a^{nj}z)\|$$

$$\leq \lim_{n\to\infty} \frac{1}{a^{2nj}} \varphi(a^{nj}x, a^{nj}y, a^{nj}z) = 0$$

for all $x, y, z \in X$, so $\Delta_Q(x, y, z) = 0$. By Theorem 2.1, the function Q is quadratic.

According to the fixed point alterative, since Q is the unique fixed point of T in the set $\Lambda = \{g \in \Omega : d(f, g) < \infty\}$, Q is the unique function such that

$$\|f(x) - Q(x)\| \leq K\ \varphi(\frac{x}{a}, 0, 0)$$

for all $x \in X$ and $K > 0$. Again using the fixed point alterative, gives

$$d(f, Q) \leq \frac{1}{1-L} d(f, Tf) \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)}$$

so we conclude that

$$\|f(x) - Q(x)\| \leq \frac{L^{\frac{j+1}{2}}}{4(1-L)}\ \varphi(\frac{x}{a}, 0, 0)$$

for all $x \in X$. This completes the proof. □

Corollary 4.3. *Let $\varepsilon, p, q, r \geq 0$ be real numbers such that $p, q, r < 2$ or $p, q, r > 2$. Suppose that a function $f : X \to Y$ with $f(0) = 0$ satisfies*

$$\|\Delta_f(x, y, z)\| \leq \varepsilon\ (\|x\|^p + \|y\|^q + \|z\|^r)$$

for all $x, y, z \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\varepsilon}{4|a^2 - a^p|}\|x\|^p \tag{4.7}$$

for all $x \in X$.

Proof. In Theorem 4.2, put $\varphi(x, y, z) := \varepsilon\ (\|x\|^p + \|y\|^q + \|z\|^r)$ for all $x, y, z \in X$. Then the relation (4.1) is true for $p, q, r < 2$ or $p, q, r > 2$ and also the inequality (4.3) holds with $L = a^{(p-2)j}$. So from (4.4), yields (4.7). □

Corollary 4.4. *Assume that $\theta \geq 0$ is fixed. Let $f : X \to Y$ be a function such that*

$$\|\Delta_f(x, y, z)\| \leq \theta$$

for all $x, y, z \in X$, then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{\theta}{12(a^2 - 1)}$$

holds for all $x \in X$.

Remark 4.5. *Corollary* 4.4 *that we obtained in this paper is similar to but more precise than Corollary* 4.5 *obtained by Y.S. Lee and S.Y. Chung* [15].

Theorem 4.6. *Suppose that $j \in \{-1, 1\}$ be fixed, and let $f : X \to Y$ be an even function with $f(0) = 0$ for which there exists a function $\varphi : X \times X \times X \to [0, \infty)$ satisfies* (4.2) *for all $x, y, z \in X$. If there exists a constant $0 < L < 1$ such that*

$$\varphi(c^j x, c^j y, c^j z) \leq c^{2j} L\ \varphi(x, y, z) \tag{4.8}$$

for all $x, y, z \in X$. Then there exists a unique quadratic function $Q : X \to Y$ such that

$$\|f(z) - Q(z)\| \leq \frac{L^{\frac{1-j}{2}}}{4c^2(1 - L)}\varphi(0, 0, z) \tag{4.9}$$

for all $z \in X$.

Proof. It follows from (4.8) that

$$\lim_{n\to\infty} \frac{1}{c^{2nj}}\varphi(c^{nj}x, c^{nj}y, c^{nj}z) = 0$$

for all $x, y, z \in X$. Putting $x = y = 0$ in (4.2), we obtain by evenness of f and $f(0) = 0$ that

$$\|4f(cz) - 4c^2 f(z)\| \leq \varphi(0, 0, z) \tag{4.10}$$

for all $z \in X$. It follows from (4.8) and (4.10) that

$$\|\frac{1}{c^{2j}} f(c^j z) - f(z)\| \leq \frac{L^{\frac{1-j}{2}}}{4c^2}\varphi(0, 0, z) \tag{4.11}$$

for all $z \in X$. Let E be the set of all even functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{\, K \in [0, \infty] : \|g(z) - h(z)\| \leq K\varphi(0, 0, z) \quad \text{for all } z \in X \,\}.$$

It is easy to show that (E,d) is a generalized complete metric space. Now we consider the function $\Lambda : E \to E$ defined by

$$(\Lambda g)(z) = \frac{1}{c^{2j}}\ g(c^j z), \quad \text{for all } g \in E \text{ and } z \in X.$$

Let $g, h \in E$ and let $K \in [0,\infty]$ be an arbitrary constant with $d(g,h) \le K$. From the definition of d, we have

$$\|g(z) - h(z)\| \le K\varphi(0,0,z)$$

for all $z \in X$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(z) - (\Lambda h)(z)\| = \frac{1}{c^{2j}}\|g(c^j z) - h(c^j z)\| \le \frac{1}{c^{2j}}K\varphi(0,0,c^j z) \le KL\varphi(x,0,0,...,0)$$

for all $z \in X$. So

$$d(\Lambda g, \Lambda h) \le Ld(g,h)$$

for any $g, h \in E$. It follows from (4.11) that $d(\Lambda f, f) \le \frac{L^{\frac{1-j}{2}}}{4c^2} < \infty$. The rest of the proof is similar to the proof of Theorem 4.2 and we omit the details. □

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