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A NONLINEAR INTEGRAL EQUATION VIA PICARD OPERATORS

CECILIA CRĂCIUN* AND MARCEL-ADRIAN ŞERBAN*

* Department of Applied Mathematics
Babeş-Bolyai University of Cluj-Napoca
M. Kogălniceanu Street No.1, 400048-Cluj-Napoca
Romania
E-mails: ceciliacraciun@yahoo.com mserban@math.ubbcluj.ro

Abstract. In this paper we study the following mixed type Volterra-Fredholm functional integral equation

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H(t,s,x(s))ds\right).$$

Using the Picard operator technique we establish existence, uniqueness, data dependence and Gronwall results for the solutions. Also, it is studied the Ulam-Hyers stability of this equation.

Key Words and Phrases: Picard operators, mixed type Volterra-Fredholm functional integral equation, data dependence, comparison theorem, Ulam-Hyers stability, Gronwall lemma, operatorial inequalities.
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1. Introduction

The theory of integral equation is an important chapter of nonlinear analysis and the most used tool for proving the existence of the solution is the fixed point technique (see [2], [3], [6], [7], [16], [18], etc.)

In this paper we consider the following mixed type Volterra-Fredholm functional nonlinear integral equation:

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H(t,s,x(s))ds\right), \quad (1.1)$$

where $[a_1; b_1] \times \ldots \times [a_m; b_m]$ be an interval in $\mathbb{R}^m$, $K, H : [a_1; b_1] \times \ldots \times [a_m; b_m] \times [a_1; b_1] \times \ldots \times [a_m; b_m] \times \mathbb{R} \to \mathbb{R}$ continuous functions and $F : [a_1; b_1] \times \ldots \times [a_m; b_m] \times \mathbb{R}^3 \to \mathbb{R}$. The mixed type Volterra-Fredholm integral equations have been studied by many authors (see [1], [4], [15], [17] [27], etc.).

We apply the Picard operators technique to prove the existence and the uniqueness, data dependence and comparison results for the solutions of (1.1). This technique was applied by many authors to study some functional nonlinear integral equation, see

[1], [4], [9], [12], [13], [14], [21], [23], [26], [27], [28]. We use the terminologies and notations from [19], [23] and [26]. For the convenience of the reader we recall some of them.

Let (X,d) be a metric space and $A : X \to X$ an operator. We denote by $A^0 := 1_X$, $A^1 := A$, $A^{n+1} := A^n \circ A$, $n \in \mathbb{N}$ the iterate operators of the operator A. We also have:

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\}$$
$$F_A := \{x \in X \mid A(x) = x\}$$
$$I(A) := \{Y \in P(X) \mid A(Y) \subseteq Y\}$$

Definition 1.1. *$A : X \to X$ is called a Picard operator (briefly PO) if:*
(i) $F_A = \{x^*\}$;
(ii) $A^n(x) \to x^*$ as $n \to \infty$, for all $x \in X$.

The operator A is Picard if and only if the discrete dynamical system generated by A has an equilibrium state which is globally asymptotically stable.

Definition 1.2. *Let (X,d) be a metric space and $c > 0$. By definition, the operator A is c-PO if A is PO and*

$$d(x,x^*) \leq c \cdot d(x,A(x)), \quad \forall x \in X.$$

Definition 1.3. *$A : X \to X$ is said to be a weakly Picard operator (briefly WPO) if the sequence $(A^n(x))_{n\in N}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of A.*

If $A : X \to X$ is a WPO, then we may define the operator $A^\infty : X \to X$ by

$$A^\infty(x) := \lim_{n\to\infty} A^n(x).$$

Obviously, $A^\infty(X) = F_A$. Moreover, if A is a PO and we denote by x^* its unique fixed point, then $A^\infty(x) = x^*$, for each $x \in X$.

Also, in this paper we study the following integral inequalities

$$x(t) \leq F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\right) \quad (1.2)$$

$$x(t) \geq F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\right) \quad (1.3)$$

using the Picard operators technique and Abstract Gronwall Lemma (I.A. Rus [22], [19]).

2. Existence and uniqueness

We prove the existence and uniqueness for the solution of integral equation (1.1) by standard techniques as in [1], [4], [5], [9], our integral equation (1.1) being more general than integral equations considered in above mentioned papers.

Theorem 2.1. *We assume that:*

(i) $K, H \in C([a_1,b_1] \times \cdots \times [a_m,b_m] \times [a_1,b_1] \times \cdots \times [a_m,b_m] \times \mathbb{R})$;

(ii) *$F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3)$;*
(iii) *there exist α, β, γ nonnegative constants such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m], u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$;
(iv) *there exist L_K and L_H nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u, v \in \mathbb{R}$;
(v) *$\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m) < 1$.*

Then, the equation (1.1) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$.

Proof. We consider the Banach space $X = C([a_1, b_1] \times \cdots \times [a_m, b_m], \| \cdot \|_C)$, where $\| \cdot \|_C$ is the Cebyshev's norm, and the operator

$$\begin{gathered} A : X \to X, \\ A(x)(t) = F\left(t, x(t), \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} K(t, s, x(s))ds, \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} H(t, s, x(s))ds\right) \end{gathered} \tag{2.1}$$

Conditions (iii) and (iv) imply that:

$$\begin{aligned} |A(u)(t) - A(v)(t)| &\leq \alpha|u(t) - v(t)| + \beta\left|\int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} (K(t, s, u(s)) - K(t, s, v(s)))ds\right| \\ &\quad + \gamma\left|\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} (K(t, s, u(s)) - H(t, s, v(s)))ds\right| \\ &\leq \alpha|u(t) - v(t)| + \beta\int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} L_K|u(s) - v(s)|ds \\ &\quad + \gamma\int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} L_H|u(s) - v(s)|ds \\ &\leq [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m)]\|u - v\|_C, \end{aligned}$$

therefore:

$$\|A(u) - A(v)\|_C \leq [\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m)]\|u - v\|_C.$$

From condition (v) we have that the operator A is a contraction and using the contraction principle we obtain that the operator A has a unique fixed point, $F_A = \{x^*\}$, i.e. the equation (1.1) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$. □

Remark 2.1. *In the conditions of the Theorem 2.1, the operator A, given by (2.1), is PO.*

Proof. From the proof of Theorem 2.1 we have that A is a contraction with $L_A = \alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m)$. □

Remark 2.2. *The conclusion of Theorem 2.1 remains true if instead of condition (v) we put the condition* (v') *there exists* $\tau > 0$ *such that*

$$\alpha + \frac{\beta L_K}{\tau^m} + \frac{\gamma L_H}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(b_i - a_i)} < 1.$$

Proof. We consider the Banach space $X = C([a_1, b_1] \times \cdots \times [a_m, b_m], \|\cdot\|_B)$, where $\|\cdot\|_B$ is the Bielecki's norm

$$\|x\|_B = \max_{[a_1,b_1]\times\cdots\times[a_m,b_m]} |x(t)| \cdot \prod_{i=1}^{m} e^{-\tau(t_i - a_i)}, \ \tau > 0$$

and A defined by (2.1). We have

$$\begin{aligned} |A(u)(t) - A(v)(t)| \quad &\leq \quad \alpha \|u-v\|_B \cdot \prod_{i=1}^{m} e^{\tau(t_i - a_i)} + \frac{\beta L_K}{\tau^m} \|u-v\|_B \cdot \prod_{i=1}^{m} e^{\tau(t_i - a_i)} + \\ &\quad + \frac{\gamma L_H}{\tau^m} \|u-v\|_B \cdot \prod_{i=1}^{m} e^{\tau(t_i - a_i + b_i - t_i)} \end{aligned}$$

therefore

$$\|A(u) - A(v)\|_B \leq \left[\alpha + \frac{\beta L_K}{\tau^m} + \frac{\gamma L_H}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(b_i - a_i)}\right] \|u-v\|_B \,,$$

thus A is L_A-contraction with $L_A = \alpha + \frac{\beta L_K}{\tau^m} + \frac{\gamma L_H}{\tau^m} \cdot \prod\limits_{i=1}^{m} e^{\tau(b_i - a_i)}$ and conclusion is obtained from contraction principle. □

Example 2.1. *Let consider the integral equation*

$$x(t) = F\left(t, x(t), \int_a^t K(t, s, x(s))ds, \int_a^b H(t, s, x(s))ds\right), \tag{2.2}$$

under the following hypothesis:

(i) $F \in C([a, b] \times [a, b] \times \mathbb{R}^3)$, $K,\ H \in C([a, b] \times [a, b] \times \mathbb{R})$*;*

(ii) *there exist* α, β, γ *nonnegative constants such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a, b], u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{R}$*;*

(iii) *there exist* L_K *and* L_H *nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$
$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|,$$

for all $t, s \in [a, b],\ u, v \in \mathbb{R}$*;*

(iv) $\alpha + (\beta L_K + \gamma L_H)(b - a) < 1$ *or there exists* $\tau > 0$ *such that*

$$\alpha + \frac{\beta L_K}{\tau} + \frac{\gamma L_H}{\tau} \cdot e^{\tau(b-a)} < 1.$$

Then, the equation (2.2) has a unique solution $x^* \in C([a, b])$*.*

Proof. We apply Theorem 2.1 in particular case of $m = 1$. □

The equation (2.2) is a general case of equations considered in [1], [4], [17], [27], their existence and uniqueness results are a consequence of Theorem 2.1.

Example 2.2. *Let consider the integral equation*

$$x(t) = f(t, x(t)) + \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s))ds + \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s))ds, \quad (2.3)$$

under the following hypothesis:

(i) $f \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$, K, $H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$;

(ii) *there exists* $\alpha > 0$ *such that:*

$$|f(t, u_1) - f(t, u_2)| \leq \alpha |u_1 - u_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u_1, u_2 \in \mathbb{R}$;

(iii) *there exist* L_K *and* L_H *nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K |u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \leq L_H |u - v|,$$

for all $t, s \in [a, b]$, $u, v \in \mathbb{R}$;

(iv) $\alpha + (L_K + L_H)(b_1 - a_1) \ldots (b_m - a_m) < 1$ *or there exists* $\tau > 0$ *such that* $\alpha + \frac{L_K}{\tau^m} + \frac{L_H}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(b_i - a_i)} < 1$

Then, the equation (2.3) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$.

Proof. This is the special case when F is linear with respect to the last two variables. We apply Theorem 2.1 for $F : [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3 \to \mathbb{R}$,

$$F(t, u, v, w) = f(t, u) + v + w.$$

In this case $\beta = \gamma = 1$. □

Example 2.3. *Let consider the Darboux problem*

$$\begin{cases} x_{t_1 t_2}(t_1, t_2) & = f(t_1, t_2, x(t_1, t_2)), & (t_1, t_2) \in [a_1; b_1] \times [a_2; b_2] \\ x(t_1, a_2) & = \varphi(t_1), & t_1 \in [a_1; b_1] \\ x(a_1, t_2) & = \psi(t_2), & t_2 \in [a_2; b_2], \ \varphi(a_1) = \psi(a_2) \end{cases} \quad (2.4)$$

under the following hypothesis:

(i) $f \in C([a_1; b_1] \times [a_2; b_2] \times \mathbb{R})$, $\varphi \in C([a_1; b_1])$, $\psi \in C([a_2; b_2])$;

(ii) *there exists* $L_f > 0$ *such that:*

$$|f(t_1, t_2, u_1) - f(t_1, t_2, u_2)| \leq L_f \cdot |u_1 - u_2|,$$

for all $(t_1, t_2) \in [a_1; b_1] \times [a_2; b_2]$, $u_1, u_2 \in \mathbb{R}$.

Then, the equation Darboux problem (2.4) has a unique solution $x^* \in C([a_1; b_1] \times [a_2; b_2])$.

Proof. $x \in C([a_1; b_1] \times [a_2; b_2])$ is a solution of *Darboux problem (2.4) iff it is a solution of the integral equation*

$$x(t_1, t_2) = \varphi(t_1) + \psi(t_2) - \varphi(a_1) + \int_{a_1}^{t_1} \int_{a_2}^{t_2} f(\xi_1, \xi_2, x(\xi_1, \xi_2))\, d\xi_1 d\xi_2. \tag{2.5}$$

So, we apply Theorem 2.1 in particular case of $m = 2$ and $F : [a_1, b_1] \times [a_2, b_2] \times \mathbb{R}^3 \to \mathbb{R}$,

$$F(t_1, t_2, u, v, w) = \varphi(t_1) + \psi(t_2) - \varphi(a_1) + v.$$

In this case we have $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $L_K = L_f$ and $L_H = 0$. Also, the condition (v') from Remark 2.2 is satisfied: there exists $\tau > 0$ such that $\frac{L_f}{\tau^2} < 1$, for example we can choose $\tau = L_f + 1$. $\square$

3. Data dependence: continuity

In this section we prove the continuous dependence of the solution for integral equation (1.1) using the following Abstract Data Dependence Lemma

Lemma 3.1. *(I.A. Rus* [19], [23], [26]*) (Abstract data dependence) Let (X, d) be a metric space and $A, B : X \to X$ two operators such that:*

(i) *A is c-PO with respect to the metric d, we denote by x_A^* the unique fixed point of operator A;*
(ii) *there exists $x_B^* \in F_B$;*
(iii) *there exists $\eta > 0$, such that:*

$$d(A(x), B(x)) \leq \eta, \quad \forall x \in X.$$

Then:

$$d(x_A^*, x_B^*) \leq c \cdot \eta.$$

We consider the following equations:

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K_1(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H_1(t, s, x(s))ds\right) \tag{3.1}$$

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K_2(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H_2(t, s, x(s))ds\right), \tag{3.2}$$

where $K_i, H_i \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$, $i = 1, 2$.

Theorem 3.1. *We assume that:*

(i) *F, K_1, H_1 satisfy the conditions from Theorem 2.1;*
(ii) *there exists a nonnegative constant η_1 such that:*

$$|K_1(t, s, u) - K_2(t, s, u)| \leq \eta_1, \ \forall\ t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m], \ \forall\ u \in \mathbb{R};$$

(iii) *there exists a nonnegative constant η_2 such that:*

$$|H_1(t, s, u) - H_2(t, s, u)| \leq \eta_2, \ \forall\ t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m], \ \forall\ u \in \mathbb{R}.$$

If x_2^* is a solution of the corresponding equation (3.2) then:

$$\|x_1^* - x_2^*\|_C \leq \frac{(\beta\eta_1 + \gamma\eta_2)(b_1 - a_1)\dots(b_m - a_m)}{1 - [\alpha + (\beta L_{K_1} + \gamma L_{H_1})(b_1 - a_1)\dots(b_m - a_m)]},$$

where x_1^* is the unique solution of the corresponding equation (3.1).

Proof. We consider the Banach space $X = C([a_1, b_1] \times \dots \times [a_m, b_m], \|\cdot\|_C)$ and operators

$$A_i : X \to X$$

$$A_i(x)(t) = F\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K_i(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H_i(t, s, x(s))ds\right) \tag{3.3}$$

$\forall\ t \in [a_1, b_1] \times \dots \times [a_m, b_m]$, $\forall\ i = 1, 2$.

From condition (i) we have that the operator A_1 is contraction with $L_{A_1} = \alpha + (\beta L_{K_1} + \gamma L_{H_1})(b_1 - a_1)\dots(b_m - a_m)$, (see Theorem 2.1). Hence, A_1 is c-PO with $c = \frac{1}{1 - L_{A_1}}$.

From (ii) and (iii) we get:

$$|A_1(x)(t) - A_2(x)(t)| \leq (\beta\eta_1 + \gamma\eta_2)(b_1 - a_1)\dots(b_m - a_m),$$

for $\forall\ t \in [a_1, b_1] \times \dots \times [a_m, b_m]$, which implies that

$$\|A_1(x) - A_2(x)\|_C \leq (\beta\eta_1 + \gamma\eta_2)(b_1 - a_1)\dots(b_m - a_m).$$

The conclusion is obtained from Lemma 3.1 for $\eta = (\beta\eta_1 + \gamma\eta_2)(b_1 - a_1)\dots(b_m - a_m)$ and $c = \frac{1}{1 - L_{A_1}}$. □

4. Data dependence: comparison results

In this section we prove a comparison result for the solution of integral equation (1.1) using the following Abstract Comparison Lemma

Lemma 4.1. *(I.A. Rus* [19], [23], [26]*) (Comparison lemma) Let $(X, d, \leq)$ be an ordered metric space and $A, B, C : X \to X$ operators such that:*

(i) *$A \leq B \leq C$;*
(ii) *A, B, C are WPOs;*
(iii) *the operator B is increasing.*

Then

$$x \leq y \leq z \Longrightarrow A^{\infty}(x) \leq B^{\infty}(y) \leq C^{\infty}(z).$$

We consider the nonlinear integral equations:

$$x(t) = F_i\left(t, x(t), \int_{a_1}^{t_1} \dots \int_{a_m}^{t_m} K_i(t, s, x(s))ds, \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} H_i(t, s, x(s))ds\right) \tag{4.1}$$

where $K_i, H_i \in C([a_1, b_1] \times \dots \times [a_m, b_m] \times [a_1, b_1] \times \dots \times [a_m, b_m] \times \mathbb{R})$, $i \in \{1, 2, 3\}$.

Theorem 4.1. *We assume that:*

(i) *F_i, K_i, H_i satisfy the conditions from Theorem 2.1 for $i = \{1, 2, 3\}$;*
(ii) *the functions $F_2(t, \cdot, \cdot, \cdot)$, $K_2(t, s, \cdot)$ and $H_2(t, s, \cdot)$ are increasing;*
(iii) *$F_1 \leq F_2 \leq F_3$, $K_1 \leq K_2 \leq K_3$ and $H_1 \leq H_2 \leq H_3$;*

If x_i^* is the solution of the equation (4.1) corresponding to F_i, K_i, H_i, $i \in \{1,2,3\}$, then:

$$x_1^* \leq x_2^* \leq x_3^*.$$

Proof. We consider the Banach space $X = C([a_1, b_1] \times \cdots \times [a_m, b_m], \| \cdot \|_C)$ and operators $A_i : X \to X$ defined by (2.1) corresponding to F_i, K_i, H_i, $i \in \{1,2,3\}$. From condition (i) we have that A_i are PO, $i \in \{1,2,3\}$, therefore $F_{A_i} = \{x_i^*\}$. Condition (ii) implies that A_2 is an increasing operator and condition (iii) implies that $A_1 \leq A_2 \leq A_3$.

Let $x \in X$ and we denote by $u_i = A_i(x)$, $i = \{1,2,3\}$. It is obvious that:

$$u_1 \leq u_2 \leq u_3$$

and

$$x_i^* = A_i^\infty(u_i), \ i = \{1,2,3\},$$

thus, from Lemma 4.1, we have:

$$u_1 \leq u_2 \leq u_3 \Longrightarrow A_1^\infty(u_1) \leq A_2^\infty(u_2) \leq A_3^\infty(u_3)$$

Hence, the conclusion follows. □

5. Gronwall lemmas

In this section we study the integral inequalities (1.2) and (1.3) using the Abstract Gronwall Lemma and Abstract Gronwall-comparison Lemma.

Lemma 5.1. *(I.A. Rus* [19], [23], [26]*) (Abstract Gronwall lemma) Let $(X, d, \leq)$ be an ordered metric space and $A : X \to X$ be an operator. We assume that:*

(i) *A is a PO;*
(ii) *A is increasing.*

If we denote by x_A^* the unique fixed point of A, then:

(a) $x \leq A(x) \Longrightarrow x \leq x_A^*$;
(b) $x \geq A(x) \Longrightarrow x \geq x_A^*$.

Lemma 5.2. *(I.A. Rus* [19], [23], [26]*) (Abstract Gronwall-comparison lemma) Let $(X, \to, \leq)$ be an ordered metric space and $A_1, A_2 : X \to X$ be two operators. We assume that:*

(i) *A_1 is increasing;*
(ii) *A_1 and A_2 are a POs.*
(iii) $A_1 \leq A_2$

If we denote by x_2^* the unique fixed point of A_2, then

$$x \leq A_1(x) \Longrightarrow x \leq x_2^*.$$

Theorem 5.1. *We consider the equation (1.1). We assume that:*

(i) *F, K, H satisfy the conditions from Theorem 2.1;*
(ii) *$K(t,s,\cdot), H(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ are increasing functions for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$;*
(iii) *$F(t,\cdot,\cdot,\cdot) : \mathbb{R}^3 \to \mathbb{R}$ is increasing, for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$.*

Then we have:

(a) *If x is a solution of (1.2) then $x \leq x^*$, where x^* is the unique solution of (1.1);*
(b) *If x is a solution of (1.3) then $x \geq x^*$,where x^* is the unique solution of (1.1).*

Proof. We consider the operator A defined by (2.1). From Theorem 2.1 we have that A is PO. Conditions (ii) and (iii) imply that A is increasing. In terms of the operator A the integral inequality (1.2) means

$$x \leq A(x),$$

and the integral inequality (1.3) means

$$x \geq A(x).$$

The conclusion is obtained from Abstract Gronwall Lemma, Lemma 5.1. □

Remark 5.1. *To have an effective Gronwall Lemma we need to "construct" x^*, which is usualy a very difficult problem.*

In this direction, if we use the Abstract Gronwall-comparison lemma, Lemma 5.2, we obtain the following result:

Theorem 5.2. *We consider the integral equation (1.2) corresponding to F_i, K_i, H_i for $i = \{1, 2\}$. We assume that:*
(i) *F_i, K_i, H_i satisfy the conditions from Theorem 2.1 for $i = \{1, 2\}$;*
(ii) *$F_1(t, \cdot, \cdot, \cdot)$, $K_1(t, s, \cdot)$ and $H_1(t, s, \cdot)$ are increasing functions for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$;*
(iii) *$F_1 \leq F_2$, $K_1 \leq K_2$ and $H_1 \leq H_2$.*

If x is a solution of (1.2) corresponding to F_1, K_1, H_1 then $x \leq x_2^*$, where x_2^* is the unique solution of (1.1) corresponding to F_2, K_2, H_2.

Proof. We consider the operator A_1, A_2 defined by (2.1), corresponding to F_1, K_1, H_1 and F_2, K_2, H_2. From Theorem 2.1 we have that A_1 and A_2 are POs, we denote by x_i^* the unique fixed point of operator A_i, $i = \{1, 2\}$. Condition (ii) implies that A_1 is increasing and condition (iii) implies that $A_1 \leq A_2$. If x is a solution of (1.2) corresponding to F_1, K_1, H_1 then

$$x \leq A_1(x).$$

The conclusion is obtained from Abstract Gronwall-comparison Lemma, Lemma 5.2. □

Example 5.1. *(Wendroff inequality* [11]*) If*

$$x(t_1, t_2) \leq a(t_1) + b(t_2) + \int_0^{t_1} \int_0^{t_2} v(\xi_1, \xi_2)\, x(\xi_1, \xi_2)\, d\xi_1 d\xi_2 \tag{5.1}$$

where $a(t_1)$, $b(t_2) > 0$, $a'(t_1)$, $b'(t_2) \geq 0$, $x(t_1, t_2)$, $v(t_1, t_2) \geq 0$, then:

$$x(t_1, t_2) \leq x^*(t_1, t_2) \leq \frac{[a(0) + b(t_2)]\,[a(t_1) + b(0)]}{a(0) + b(0)} \cdot e^{\int_0^{t_1} \int_0^{t_2} v(\xi_1, \xi_2) d\xi_1 d\xi_2}, \tag{5.2}$$

where $x^*(t_1,t_2)$ *is the solution of the Darboux problem*

$$\begin{cases} x_{t_1t_2}(t_1,t_2) & = x(t_1,t_2)\,v(t_1,t_2), \quad (t_1,t_2)\in[0;b_1]\times[0;b_2] \\ x(t_1,0) & = a(t_1)+b(0), \qquad t_1\in[0;b_1] \\ x(0,t_2) & = a(0)+b(t_2), \qquad t_2\in[0;b_2] \end{cases} \tag{5.3}$$

Proof. Let $b_1>0$ and $b_2>0$ and we consider the Banach space

$$X=\left(C\left([0;b_1]\times[0;b_2],\mathbb{R}_+\right),\|\cdot\|_B\right),$$

where

$$\|u\|_B=\max_{[0;b_1]\times[0;b_2]}|u(x,y)|\cdot e^{-\tau(x+y)},\ \tau>0.$$

We define the operators $A_1,A_2:X\to X$

$$A_1(x)(t_1,t_2)=a(t_1)+b(t_2)+\int_0^{t_1}\int_0^{t_2}v(\xi_1,\xi_2)\,x(\xi_1,\xi_2)\,d\xi_1 d\xi_2, \tag{5.4}$$

$$A_2(x)(t_1,t_2)=a(0)+b(t_2)+\int_0^{t_1}\left(\frac{a'(\xi_1)}{a(\xi_1)+b(0)}+\int_0^{t_2}v(\xi_1,\xi_2)\,d\xi_2\right)\cdot x(\xi_1,t_2)\,d\xi_1, \tag{5.5}$$

It is clear that any solution of the Darboux problem (5.3) is a fixed point of operator A_1. We have that $A_1:X\to X$ is PO (see Example 2.3), thus $F_{A_1}=\{x^*\}$. Also, A_1 is an increasing operator and from Abstract Gronwall Lemma we have

$$x\le A_1(x)\Longrightarrow x\le x^*,$$

which means that any x satisfying (5.1) will satisfy the inequality $x\le x^*$.

The function

$$w^*(t_1,t_2)=\frac{[a(0)+b(t_2)]\,[a(t_1)+b(0)]}{a(0)+b(0)}\cdot e^{\int_0^{t_1}\int_0^{t_2}v(\xi_1,\xi_2)d\xi_1 d\xi_2} \tag{5.6}$$

is not the solution of (5.3), w^* is the solution of the problem

$$\begin{cases} w_{t_1}(t_1,t_2) & =\left(\frac{a'(t_1)}{a(t_1)+b(0)}+\int_0^{t_2}v(t_1,\xi_2)\,d\xi_2\right)\cdot w(t_1,t_2) \\ w(0,t_2) & = a(0)+b(t_2) \end{cases} \tag{5.7}$$

or

$$\begin{cases} \frac{\partial}{\partial t_2}\left(\frac{w_{t_1}(t_1,t_2)}{w(t_1,t_2)}\right) & = v(t_1,t_2) \\ w(t_1,0) & = a(t_1)+b(0) \\ w(0,t_2) & = a(0)+b(t_2) \end{cases} \tag{5.8}$$

In order to prove (5.2) we will apply the Abstract Gronwall-comparison lemma, Lemma 5.2.

We consider the set

$$Y=\{x\in X: x_{t_1}\ge 0, x_{t_2}\ge 0, x(t_1,0)=a(t_1)+b(0)\}.$$

It is clear that $Y\subseteq X$ is a closed subset, so it is a complete metric space. Moreover, Y is an invariant set of A_1 and $A_1:Y\to Y$ is a contraction, so $x^*\in Y$ and $A_1:Y\to Y$

is PO. It is easy to check that also $A_2 : Y \to Y$ is a contraction, so it is PO and $F_{A_2} = \{w^*\}$. Now we prove that $A_1(x) \le A_2(x)$ for all $x \in Y$.

We have:

$$\begin{aligned} A_1(x)(t_1,t_2) &= a(0) + b(t_2) + \int_0^{t_1} \left(a'(\xi_1) + \int_0^{t_2} v(\xi_1,\xi_2)\, x(\xi_1,\xi_2)\, d\xi_2 \right) d\xi_1 \\ &\le a(0) + b(t_2) + \int_0^{t_1} \left(\frac{a'(\xi_1)}{a(\xi_1)+b(0)} + \int_0^{t_2} v(\xi_1,\xi_2)\, d\xi_2 \right) x(\xi_1,t_2)\, d\xi_1 \\ &= A_2(x)(t_1,t_2), \end{aligned}$$

since

$$a(\xi_1) + b(0) = x(\xi_1, 0) \le x(\xi_1,\xi_2), \ \forall (\xi_1,\xi_2) \in [0;b_1] \times [0;b_2]$$

and

$$x(\xi_1,\xi_2) \le x(\xi_1,t_2), \ \forall \xi_1 \in [0;b_1], \ 0 \le \xi_2 \le t_2 \le b_2.$$

All the conditions of the Abstract Gronwall-comparison lemma are satisfied, therefore

$$x \le A_1(x) \le A_2(x) \Longrightarrow x \le x^* \le w^*$$

and the proof is complete. □

If we consider the case of $a(t_1) + b(t_2) \equiv c$, $c \in \mathbb{R}_+$, we obtain th results from C. Crăciun, N. Lungu [5], N. Lungu [8]. In some particular cases for $v(t_1,t_2)$ we can find the expresion of x^* from (5.2), for example, if $a(t_1) + b(t_2) \equiv c$ and $v(t_1,t_2) \equiv \alpha^2$ then $x^*(t_1,t_2) = cJ_0\left(2\alpha\sqrt{t_1t_2}\right)$, where $J_0\left(2\alpha\sqrt{t_1t_2}\right)$ is the Bessel function (see C. Crăciun, N. Lungu [5]).

For other applications of Abstract Gronwall lemma and Abstract Gronwall-comparison lemma see N. Lungu [9], N. Lungu, I.A. Rus [10], I.A. Rus [19], [22].

6. Ulam-Hyers stability

Definition 6.1. *(I.A. Rus [24]) Let (X,d) be a metric space and $A : X \to X$ be an operator. By definition, the fixed point equation*

$$x = A(x) \tag{6.1}$$

is Ulam-Hyers stable if there exists a real number $c_A > 0$ such that for each $\varepsilon > 0$ and each solution y^* of the inequation

$$d(y, A(y)) \le \varepsilon$$

there exists a solution x^* of (6.1) such that

$$d(y^*, x^*) \le c_A \varepsilon.$$

Theorem 6.1. *In the conditions of the Theorem 2.1, the integral equation (1.1) is Ulam-Hyers stable, in more precise manner, let $\varepsilon > 0$, if $y^* \in C([a_1,b_1] \times \cdots \times [a_m,b_m])$ is a solution of the inequation*

$$\left| y(t) - F\left(t, y(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,y(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,y(s))ds \right) \right| \le \varepsilon,$$

for every $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, *then there exists a solution* $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$ *of the equation (1.1) such that*

$$|y^*(t) - x^*(t)| \leq \frac{1}{1 - L_A}\varepsilon,$$

for every $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, *where* $L_A = \alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m)$.

Proof. We consider the operator A, given by (2.1). In the conditions of the Theorem 2.1, the operator A is contraction, therefore A is c-PO with the constant $c = (1 - L_A)^{-1}$ and the conclusion is an application of the Remark 2.1 from I.A. Rus [24]. □

7. Integral equations in Banach space

Let $(\mathbb{B}, |\cdot|)$ a Banach space. The Theorem 2.1 remains also true if we consider the mixed type Volterra-Fredholm functional nonlinear integral equation (1.1) in the Banach space $\mathbb{B}$ instead of Banach space $\mathbb{R}$.

Theorem 7.1. *We assume that:*

(i) $K, H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{B}, \mathbb{B})$;
(ii) $F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{B}^3, \mathbb{B})$;
(iii) *there exist* α, β, γ *nonnegative constants such that:*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \leq \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m], u_1, u_2, v_1, v_2, w_1, w_2 \in \mathbb{B}$;
(iv) *there exist* L_K *and* L_H *nonnegative constants such that:*

$$|K(t, s, u) - K(t, s, v)| \leq L_K|u - v|,$$

$$|H(t, s, u) - H(t, s, v)| \leq L_H|u - v|,$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$, $u, v \in \mathbb{B}$;
(v) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \ldots (b_m - a_m) < 1$.

Then, the equation (1.1) has a unique solution $x^* \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$.

Remark 7.1. *The conclusion of Theorem 7.1 remains true if instead of condition* (v) *we put the condition* (v') *from Remark 2.2.*

Example 7.1. *We consider the following infinite system of integral equation*

$$x_n(t) = f_n(t) + \int_{a_1}^{t_1} \ldots \int_{a_m}^{t_m} k(t, s)x_{n+1}(s)ds + \int_{a_1}^{b_1} \ldots \int_{a_m}^{b_m} h(t, s)x_{n+2}(s)ds, \ n \in \mathbb{N}, \tag{7.1}$$

under the following hypothesis:

(i) $f_n \in C([a_1, b_1] \times \cdots \times [a_m, b_m])$, $n \in \mathbb{N}$, $f_n(t) \to 0$, $n \to +\infty$, *for every* $t \in [a_1, b_1] \times \cdots \times [a_m, b_m]$;
(ii) $k, h \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m])$;

(iii) $(m_k + m_h)(b_1 - a_1)\dots(b_m - a_m) < 1$ *or there exists* $\tau > 0$ *such that* $\frac{m_k}{\tau^m} + \frac{m_h}{\tau^m} \cdot \prod_{i=1}^{m} e^{\tau(b_i - a_i)} < 1$, *where*

$$\begin{aligned} m_k &= \max_{t,s\in[a_1,b_1]\times\cdots\times[a_m,b_m]} |k(t,s)|, \\ m_h &= \max_{t,s\in[a_1,b_1]\times\cdots\times[a_m,b_m]} |h(t,s)|. \end{aligned}$$

Then, the equation (7.1) has a unique solution.

Proof. Let $(\mathbb{B}, \|\cdot\|)$ the Banach space, where

$$\mathbb{B} = c_0 = \{\mathbf{u} = (u_0, u_1, \dots, u_n, \dots) \in s(\mathbb{R}) : u_n \to 0\}$$

and

$$\|\mathbf{u}\| = \max_{n\in\mathbb{N}} |u_n|.$$

Let $\mathbf{u} = (u_0, u_1, \dots, u_n, \dots) \in \mathbb{B}$. We denote by

$$\begin{aligned} \mathbf{f} &= (f_0, f_1, \dots, f_n, \dots), \\ \mathbf{K} &= (K_0, K_1, \dots, K_n, \dots), \\ \mathbf{H} &= (H_0, H_1, \dots, H_n, \dots), \end{aligned}$$

where

$$\begin{aligned} K_n(t,s,\mathbf{u}) &= k(t,s)u_{n+1}, \\ H_n(t,s,\mathbf{u}) &= h(t,s)u_{n+2}, \end{aligned}$$

$t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$. From (i) and (ii) we have that $\mathbf{f} \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$ and $\mathbf{K}, \mathbf{H} \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$. Also,

$$\begin{aligned} \|\mathbf{K}(t,s,\mathbf{u}) - \mathbf{K}(t,s,\mathbf{v})\| &\le m_k \|\mathbf{u} - \mathbf{v}\|, \\ \|\mathbf{H}(t,s,\mathbf{u}) - \mathbf{H}(t,s,\mathbf{v})\| &\le m_h \|\mathbf{u} - \mathbf{v}\|, \end{aligned}$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$ and $\mathbf{u}, \mathbf{v} \in \mathbb{B}$. All the conditions of Theorem 7.1 are satisfied, therefore we get that the equation (7.1) has a unique solution $\mathbf{x}^* = (x_0^*, x_1^*, \dots, x_n^*, \dots) \in C([a_1, b_1] \times \cdots \times [a_m, b_m], \mathbb{B})$. □

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