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DATA DEPENDENCE OF FIXED POINTS IN GAUGE SPACES

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Abstract. Data dependence of fixed points for several classes of non-self generalized contractions in gauge spaces is studied.
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1. Introduction

Let $(X, \mathcal{P})$ be gauge space, $Y \subset X$ a nonempty subset of X and $f : Y \to X$ an operator. In what follow we shall use the following notations:

$F_f = \{x \in Y : f(x) = x\}$ – the fixed points set of f;
$I(f) = \{Z \subset Y : f(Z) \subset Z, Z \neq \emptyset\}$ – the set of invariant subsets of f;
$(MI)_f = \bigcup_{Z \in I(f)} Z$ – the maximal invariant subset of f;
$(BA)_f(x^*) = \{x \in Y : f^n(x)$ is defined for all $n \in \mathbb{N}$ and $f^n(x) \xrightarrow{\mathcal{P}} x^* \in F_f\}$.
– the attraction basin of $x^* \in F_f$ with respect to f;
$(BA)_f = \cup(BA)_f(x^*)$ – the attraction basin of f;
$(PH)_{\mathcal{P}} = ((PH)_{d_\alpha})_{\alpha \in \mathcal{A}}$, where
$(PH)_{d_\alpha}(A, B) := \max\left(\sup_{a \in A} \inf_{b \in B} d_\alpha(a, b), \sup_{b \in B} \inf_{a \in A} d_\alpha(a, b)\right)$

In [3], the authors, using the weakly Picard operator technique, give some data dependence results for the fixed points of nonself operators in a metric space. In this paper, the data dependence of fixed points for several classes of non-self generalized contractions in gauge spaces is studied.

2. Picard and weakly Picard non-self operators

We begin our considerations by some definitions. Let X be a nonempty set and let $\mathcal{P} = (d_\alpha)_{\alpha \in \mathcal{A}}$ be a separated gauge structure on X. Then the pair $(X, \mathcal{P})$ is said to be a gauge space (see [4], [10], [11]). Let Y be a nonempty subset of X.

Definition 2.1. *Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and let $f : Y \to X$ be an operator. We say that the operator f is a contraction with respect to $\mathcal{P}$ if for every $\alpha \in \mathcal{A}$ there exists $0 < a_\alpha < 1$ such that*

$$d_\alpha(f(x), f(y)) \le a_\alpha d_\alpha(x, y), \text{ for every } x, y \in Y \tag{2.1}$$

In this case we will say that f is an A-contraction, where $A = (a_\alpha)_{\alpha\in\mathcal{A}}$.

Definition 2.2. *An operator $f : Y \to X$ is said to be a Picard operator (PO) if:*
(i) $F_f = \{x_f^*\}$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.3. *An operator $f : Y \to X$ is said to be a weakly Picard operator (WPO) if:*
(i) $F_f \ne \emptyset$;
(ii) $(MI)_f = (BA)_f$.

Definition 2.4. *For each WPO $f : Y \to X$ we define the operator $f^\infty : (BA)_f \to (BA)_f$ by $f^\infty(x) = \lim\limits_{n\to\infty} f^n(x)$.*

Remark 2.1. *Notice that $f^\infty((BA)_f) = F_f$, thus f^∞ is a set retraction of $(BA)_f$ on F_f.*

Remark 2.2. *In terms of weakly Picard self operators theory, the above definitions take the following form:*

$$f : Y \to X \text{ is a WPO (PO) iff } f\,|_{(MI)_f} : (MI)_f \to (MI)_f \text{ is a WPO (PO).}$$

Let $\Psi = (\psi_\alpha)_{\alpha\in\mathcal{A}}$, where $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi_\alpha(0) = 0$, for every $\alpha \in \mathcal{A}$.

Definition 2.5. *An operator $f : Y \to X$ is said to be a Ψ-WPO (Ψ-PO) if f is a WPO (PO) and*

$$d_\alpha(x, f^\infty(x)) \le \psi_\alpha(d_\alpha(x, f(x)) \text{ for every } x \in (MI)_f \text{ and every } \alpha \in \mathcal{A}.$$

If $\psi_\alpha(t) := c_\alpha t$, $t \in \mathbb{R}_+$, for some $c_\alpha > 0$ for every $\alpha \in \mathcal{A}$, we say that f is a C-WPO (C-PO), where $C = (c_\alpha)_{\alpha\in\mathcal{A}}$.

Example 2.1. *Let Y be a nonempty subset of the gauge space $(X, \mathcal{P})$ and let $f : Y \to X$ be an A-contraction such that $F_f = \{x_f^*\}$. Then f is a C-PO with $C = \left(\frac{1}{1-a_\alpha}\right)_{\alpha\in\mathcal{A}}$.*

Indeed, for $x \in (MI)_f$ and $\alpha \in \mathcal{A}$ we have that $d_\alpha(f^n(x), x_f^*) \to 0$ as $n \to \infty$, i.e., f is a PO. On the other hand, for $x \in (MI)_f$ and $\alpha \in \mathcal{A}$ we have

$$\begin{aligned} d_\alpha(x, f^n(x)) &\le d_\alpha(x, f(x)) + d_\alpha(f(x), f^2(x)) + ... + d_\alpha(f^{n-1}(x), f^n(x)) \\ &\le (1 + a_\alpha + a_\alpha^2 + ... + a_\alpha^{n-1}) d_\alpha(x, f(x)) \\ &\le \frac{1}{1-a_\alpha} d_\alpha(x, f(x)). \end{aligned}$$

Hence $d_\alpha(x, x_f^*) \le \frac{1}{1-a_\alpha} d_\alpha(x, f(x))$, for every $\alpha \in \mathcal{A}$.

Example 2.2. *Let Y be a nonempty subset of the gauge space $(X, \mathcal{P})$ and let $f : Y \to X$ be a generalized contraction of Ćirić-Reich-Rus type, that is for every $\alpha \in \mathcal{A}$*

$$d_\alpha(f(x), f(y)) \leq a_\alpha d_\alpha(x, f(x)) + b_\alpha d_\alpha(y, f(y)) + c_\alpha d_\alpha(x, y) \text{ for all } x, y \in Y, \quad (2.2)$$

where $a_\alpha, b_\alpha, c_\alpha$ are non-negative numbers with $a_\alpha + b_\alpha + c_\alpha < 1$. We suppose that $F_f = \{x_f^*\}$. Then f is a C-PO, where $C = \left(\frac{1-b_\alpha}{1-a_\alpha-b_\alpha-c_\alpha}\right)_{\alpha\in\mathcal{A}}$.

Indeed, if we let in (2.2) $y = f(x)$, $x \in (BA)_f$, we obtain

$$d_\alpha(f(x), f^2(x)) \leq a_\alpha d_\alpha(x, f(x)) + b_\alpha d_\alpha(f(x), f^2(x)) + c_\alpha d_\alpha(x, f(x))$$

and thus

$$d_\alpha(f(x), f^2(x)) \leq \frac{a_\alpha + c_\alpha}{1 - b_\alpha} d_\alpha(x, f(x)), \text{ for all } x \in (BA)_f \text{ and all } \alpha \in \mathcal{A}. \quad (2.3)$$

Then, for every $n \in \mathbb{N}^*$ and every $\alpha \in \mathcal{A}$ we have

$$\begin{aligned} d_\alpha(x, f^n(x)) &\leq d_\alpha(x, f(x)) + d_\alpha(f(x), f^2(x)) + ... + d_\alpha(f^{n-1}(x), f^n(x)) \\ &\leq \frac{1 - b_\alpha}{1 - a_\alpha - b_\alpha - c_\alpha} d_\alpha(x, f(x)). \end{aligned}$$

Consequently, f is a C-PO.

Example 2.3. *Let Y be a nonempty subset of the gauge space $(X, \mathcal{P})$ and let $f : Y \to X$ be a generalized contraction of Ciric type, that is for every $\alpha \in \mathcal{A}$*

$$d_\alpha(f(x), f(y)) \leq q_\alpha \max\{d_\alpha(x, y), d_\alpha(x, f(x)), d_\alpha(y, f(y)), d_\alpha(x, f(y)), d_\alpha(y, f(x))\}$$

for all $x, y \in Y$ and some $q_\alpha \in [0, \frac{1}{2})$. We suppose that $F_f = \{x_f^*\}$. Then f is a C-PO, where $C = \left(\frac{1-q_\alpha}{1-2q_\alpha}\right)_{\alpha\in\mathcal{A}}$.

Example 2.4. *Let Y be a nonempty subset of the gauge space $(X, \mathcal{P})$ and let $f : Y \to X$ be a graphic contraction with closed graph, i.e., for every $\alpha \in \mathcal{A}$ there exists $a_\alpha \in (0, 1)$ such that, for all x for which $f^2(x)$ is defined we have:*

$$d_\alpha(f^2(x), f(x)) \leq a_\alpha d_\alpha(x, f(x)).$$

We suppose that $F_f \neq \emptyset$. Then f is a C-WPO, where $C = \left(\frac{1}{1-a_\alpha}\right)_{\alpha\in\mathcal{A}}$.

Indeed, the graphic contraction condition implies that for every $x \in (MI)_f$, the sequence $(f^n(x))$ is convergent. Since f has closed graph the limit of the sequence $(f^n(x))$ is a fixed point of f. Thus f is a WPO. In addition, if $x \in (BA)_f$, then for every $\alpha \in \mathcal{A}$, we have

$$\begin{aligned} d_\alpha(x, f^n(x)) &\leq d_\alpha(x, f(x)) + d_\alpha(f(x), f^2(x)) + ... + d_\alpha(f^{n-1}(x), f^n(x)) \\ &\leq (1 + a_\alpha + a_\alpha^2 + ... + a_\alpha^{n-1}) d_\alpha(x, f(x)) \\ &\leq \frac{1}{1 - a_\alpha} d_\alpha(x, f(x)). \end{aligned}$$

Letting $n \to \infty$, we obtain $d_\alpha(x, f^\infty(x)) \leq \frac{1}{1-a_\alpha} d_\alpha(x, f(x))$ for all $\alpha \in \mathcal{A}$.

Example 2.5. *Let $(X,\mathcal{P})$ be a gauge space, let $\Phi=(\varphi_\alpha)_{\alpha\in\mathcal{A}}$ be a family of functions such that for every $\alpha\in\mathcal{A}$, $\varphi_\alpha:\mathbb{R}_+\to\mathbb{R}_+$ is a strict comparison function (see* [8]*), i.e., for every $\alpha\in\mathcal{A}$:*

(a) *φ_α is increasing;*
(b) *$\varphi_\alpha^n(t)\to 0$ as $n\to\infty$, for all $t\in\mathbb{R}_+$;*
(c) *$t-\varphi_\alpha(t)\to+\infty$ as $t\to\infty$.*

Let $Y\subset X$ and $f:Y\to X$ be a strict Φ- contraction with respect to $\mathcal{P}$, i.e.,

$$d_\alpha(f(x),f(y))\le\varphi_\alpha(d_\alpha(x,y))\quad\text{for all } x,y\in Y \text{ and all } \alpha\in\mathcal{A}.$$

Suppose that $F_f\neq\emptyset$. Then f is a Ψ_Φ -PO, with $\Psi_\Phi=(\psi_{\varphi_\alpha})_{\alpha\in\mathcal{A}}$ where $\psi_{\varphi_\alpha}:\mathbb{R}_+\to\mathbb{R}_+$,

$$\psi_{\varphi_\alpha}(\eta)=\sup\{t\in\mathbb{R}_+:\ t-\varphi_\alpha(t)\le\eta\},$$

for every $\alpha\in\mathcal{A}$.

Notice that $F_f=\{x^*\}$. Then, for $x\in(BA)_f$ and $\alpha\in\mathcal{A}$ we have

$$\begin{aligned}d_\alpha(x,x^*)&\le d_\alpha(x,f(x))+d_\alpha(f(x),x^*)\\&\le d_\alpha(x,f(x))+\varphi_\alpha(d_\alpha(x,x^*)).\end{aligned}$$

Hence $d(_\alpha x,x^*)-\varphi_\alpha(d_\alpha(x,x^*))\le d_\alpha(x,f(x))$. Thus we get that $d_\alpha(x,x^*)\le\psi_{\varphi_\alpha}(d_\alpha(x,f(x)))$ for every $\alpha\in\mathcal{A}$. Therefore f is a Ψ_Φ- PO.

Remark 2.3. *It is obvious that if $f:X\to X$ is a WPO (PO), then $f|_Y:Y\to X$ is also a WPO (PO).*

3. Data dependence for Ψ-WPOs and Ψ-POs

Let $(X,\mathcal{P})$ be a gauge space, $Y\subset X$ be a nonempty subset of X and $f,g:Y\to X$ two operators.

Theorem 3.1. *Assume that the following conditions are satisfied:*

(i) *f and g are Ψ-WPOs;*
(ii) *$F_g\subset(BA)_f$ and $F_f\subset(BA)_g$;*
(iii) *for every $\alpha\in\mathcal{A}$ there exists $\eta_\alpha>0$ such that*

$$d_\alpha(f(x),g(x))\le\eta_\alpha\quad\text{for all } x\in Y.$$

Then

$$(PH)_{d_\alpha}(F_f,F_g)\le\psi_\alpha(\eta_\alpha),\ \text{for every } \alpha\in\mathcal{A}.$$

Proof. If $x\in F_g$, and $\alpha\in\mathcal{A}$ then

$$d_\alpha(x,f^\infty(x))\le\psi_\alpha(d_\alpha(x,f(x)))=\psi_\alpha(d_\alpha(g(x),f(x)))\le\psi_\alpha(\eta_\alpha).$$

If $y\in F_f$, and $\alpha\in\mathcal{A}$ then

$$d_\alpha(y,g^\infty(y))\le\psi_\alpha(d_\alpha(y,g(y)))=\psi_\alpha(d_\alpha(f(y),g(y)))\le\psi_\alpha(\eta_\alpha).$$

Now the conclusion follows from the next lemma from [8]. □

Lemma 3.1. *Let (X, d) be a metric space and A, $B \subset X$ be two nonempty sets. If $\tau > 0$ is such that:*
(1) *for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq \tau$;*
(2) *for each $b \in B$ there exists $a \in A$ such that $d(a, b) \leq \tau$,*
then $(PH)_d(A, B) \leq \tau$.

In the case of Picard operators, Theorem 3.1 takes the following form:

Theorem 3.2. *Assume that the following conditions are satisfied:*
(i) *f is a Ψ-PO ($F_f = \{x_f^*\}$);*
(ii) $\emptyset \neq F_g \subset (BA)_f$;
(iii) *for every $\alpha \in A$ there exists $\eta_\alpha > 0$ such that*

$$d_\alpha(f(x), g(x)) \leq \eta_\alpha \quad \textit{for all } x \in Y.$$

Then

$$d_\alpha(x_f^*, x_g^*) \leq \psi_\alpha(\eta_\alpha), \quad \textit{for all } x_g^* \in F_g \textit{ and all } \alpha \in \mathcal{A}.$$

For the case of strict Φ-contractions we have the following result:

Theorem 3.3. *Assume that the following conditions are satisfied:*
(i) *f is a strict Φ- contraction with respect to $\mathcal{P}$ with $F_f = \{x_f^*\}$;*
(ii) $F_g \neq \emptyset$;
(iii) *for every $\alpha \in \mathcal{A}$ there exists $\eta_\alpha > 0$ such that*

$$d_\alpha(f(x), g(x)) \leq \eta_\alpha, \quad \textit{for all } x \in Y.$$

Then

$$d_\alpha(x_g^*, x_f^*) \leq \psi_{\varphi_\alpha}(\eta_\alpha), \quad \textit{for all } x_g^* \in F_g \textit{ and all } \alpha \in \mathcal{A}.$$

Proof. Let $x_g^* \in F_g$ and $\alpha \in \mathcal{A}$. We have

$$\begin{aligned} d_\alpha(x_g^*, x_f^*) &\leq d_\alpha(x_g^*, f(x_g^*)) + d_\alpha(f(x_g^*), x_f^*) \\ &\leq \eta_\alpha + \varphi_\alpha(d_\alpha(x_g^*, x_f^*)). \end{aligned}$$

Hence $d_\alpha(x_g^*, x_f^*) - \varphi_\alpha(d_\alpha(x_g^*, x_f^*)) \leq \eta_\alpha$, i.e., we get that $d_\alpha(x_g^*, x_f^*) \leq \psi_{\varphi_\alpha}(\eta_\alpha)$ for every $\alpha \in \mathcal{A}$. □

We also have the following result:

Theorem 3.4. *Assume that the following conditions are satisfied:*
(i) *there exist $a_\alpha, b_\alpha \in \mathbb{R}_+$, $a_\alpha + 2b_\alpha < 1$ such that*

$$d_\alpha(f(x), f(y)) \leq a_\alpha d(x, y) + b_\alpha[d_\alpha(x, f(x)) + d_\alpha(y, f(y))]$$

for all $x, y \in X$ and each $\alpha \in A$. Suppose that $F_f = \{x_f^*\}$;
(ii) $F_g \neq \emptyset$;
(iii) *for every $\alpha \in \mathcal{A}$ there exists $\eta_\alpha > 0$ such that*

$$d_\alpha(f(x), g(x)) \leq \eta_\alpha, \quad \textit{for all } x \in Y.$$

Then

$$d_\alpha(x_g^*, x_f^*) \leq \frac{1 + b_\alpha}{1 - a_\alpha}\eta_\alpha, \textit{ for all } x_g^* \in F_g \textit{ and all } \alpha \in A. \tag{3.1}$$

Proof. Let $x_g^* \in F_g$ and $\alpha \in \mathcal{A}$. We have

$$\begin{aligned} d_\alpha(x_g^*, x_f^*) &\le d_\alpha(x_g^*, f(x_g^*)) + d_\alpha(f(x_g^*), x_f^*) \\ &\le \eta_\alpha + a_\alpha d_\alpha(x_g^*, x_f^*) + b_\alpha d_\alpha(x_g^*, f(x_g^*)) \\ &\le \eta_\alpha + a_\alpha d_\alpha(x_g^*, x_f^*) + b_\alpha \eta_\alpha. \end{aligned}$$

□

Remark 3.1. *In applications one use continuation principles in order to satisfies condition (i) in Theorem 3.1 and Theorem 3.2 (see* [1], [2], [5], [7]*).*

4. Data dependence for operators satisfying the Ψ-condition

4.1. **The Ψ- condition in the case $F_f = \{x_f^*\}$.** Let $\Psi = (\psi_\alpha)_{\alpha \in A}$, where $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi_\alpha(0) = 0$, for every $\alpha \in A$. Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f : Y \to X$ be an operator with $F_f = \{x_f^*\}$.

Definition 4.1. *We say that the operator f satisfies the Ψ-condition if for every $\alpha \in A$ one have*

$$d_\alpha(x, x_f^*) \le \psi_\alpha(d_\alpha(x, f(x))) \text{ for all } x \in Y.$$

Example 4.1. *If $f : Y \to X$ is an A-contraction then f satisfies the Ψ-condition with respect to* $\Psi = \left(\frac{t}{1-a_\alpha}\right)_{\alpha \in A}$.

Example 4.2. *If $f : Y \to X$ is a strict Φ-contraction with respect to $\mathcal{P}$, (with $\Phi := (\varphi_\alpha)_{\alpha \in \mathcal{A}}$ a family of strict comparison functions), then f satisfies the Ψ-condition, where $\Psi = (\psi_\alpha)_{\alpha \in \mathcal{A}}$, with $\psi_\alpha(r) = \sup\{t \in \mathbb{R}_+ : t - \varphi_\alpha(t) \le r\}$, for any $\alpha \in \mathcal{A}$.*

The above examples give rise to the following problems:

Problem 4.1. *Which generalized contractions on gauge space satisfy the Ψ-condition, where $\Psi = (\psi_\alpha)_{\alpha \in \mathcal{A}}$, for some functions $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$?*

Problem 4.2. *Let $Y = X$. For which generalized contractions we have that:*
(i) f satisfies the Ψ- condition with respect to $\Psi = (\psi_\alpha)_{\alpha \in \mathcal{A}}$;
(ii) f is not a Ψ-PO.

Theorem 4.1. *Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f : Y \to X$ be an operator with $F_f = \{x_f^*\}$. If f satisfies the Ψ-condition, then the fixed point problem is well posed for f.*

Proof. Let $(x_n)_{n \in \mathbb{N}} \subset Y$ be such that $d_\alpha(x_n, f(x_n)) \to 0$ as $n \to \infty$, for every $\alpha \in \mathcal{A}$. Then, from the Ψ-condition, for every $\alpha \in \mathcal{A}$, we have that $d_\alpha(x_n, x_f^*) \le \psi_\alpha(d_\alpha(x_n, f(x_n))) \to 0$ as $n \to \infty$. □

Theorem 4.2. *Let $(X, \mathcal{P})$ be a gauge space, $f : Y \to X$ be an operator with $F_f = \{x_f^*\}$. Assume that the following conditions are satisfied:*
(i) $Y = X$;
(ii) f satisfies the Ψ- condition with respect to $\Psi = (\psi_\alpha)_{\alpha \in \mathcal{A}}$;
(iii) f is asymptotically regular, i.e., for every $\alpha \in \mathcal{A}$

$$d_\alpha(f^n(x), f^{n+1}(x)) \to 0 \text{ as } n \to \infty \text{ for every } x \in X.$$

Then f is a Ψ-PO.

Proof. Let $x \in X$. For every $\alpha \in A$, we have that

$$d_\alpha(f^n(x), x_f^*) \le \psi_\alpha(d_\alpha(f^n(x), f^{n+1}(x))) \to 0 \text{ as } n \to \infty.$$

So, f is a PO. Now (ii) implies that f is a Ψ-PO. □

Now we state a data dependence result for operators satisfying the Ψ-condition.

Theorem 4.3. *Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f, g : Y \to X$ be two operators. We suppose that:*
(i) f satisfies the Ψ- condition;
(ii) for every $\alpha \in A$ there exists $\eta_\alpha > 0$ such that

$$d_\alpha(f(x), g(x)) \le \eta_\alpha, \qquad \text{for every } x \in Y.$$

Then

$$d_\alpha(x_f^*, x_g^*) \le \psi_\alpha(\eta_\alpha) \text{ for every } x_g^* \in F_g \text{ and every } \alpha \in A.$$

Proof. Let $x_g^* \in F_g$ and $\alpha \in A$. Then (i) and (ii) guarantee that:

$$d_\alpha(x_g^*, x_f^*) \le \psi_\alpha(d_\alpha(x_g^*, f(x_g^*))) = \psi_\alpha(d_\alpha(g(x_g^*), f(x_g^*))) \le \psi_\alpha(\eta_\alpha).$$

□

In the case of Φ-contractions (see Example 2.5) we have the following result:

Theorem 4.4. *Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f, g : Y \to X$ be two operators. We suppose that:*
(i) f is a Φ-contraction;
(ii) for every $\alpha \in A$ there exists $\eta_\alpha > 0$ such that

$$d_\alpha(f(x), g(x)) \le \eta_\alpha \qquad \text{for every } x \in Y.$$

Then

$$d_\alpha(x_f^*, x_g^*) \le \psi_\alpha(\eta_\alpha) \text{ for every } x_g^* \in F_g \text{ and every } \alpha \in A.$$

4.2. **The Ψ-condition in the case $F_f \neq \emptyset$.** Let $\Psi = (\psi_\alpha)_{\alpha \in A}$ where $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, continuous in 0 and $\psi_\alpha(0) = 0$, for every $\alpha \in A$. Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f : Y \to X$ be an operator with $F_f \neq \emptyset$.

Definition 4.2. *The operator f satisfies the Ψ- condition if there exists a set retraction $\chi_f : Y \to F_f$ such that*

$$d_\alpha(x, \chi_f(x)) \le \psi_\alpha(d_\alpha(x, f(x))) \text{ for every } x \in Y \text{ and } \alpha \in A.$$

Example 4.3. *Let $Y = X$ and let $f : X \to X$ be a Ψ-WPO. If we take $\chi_f = f^\infty$, then f satisfies the Ψ-condition.*

Example 4.4. *Let $(X, \mathcal{P})$ be a gauge space, $Y \subset X$ and $f : Y \to X$. We suppose that*
(i) $Y = \bigcup_{i \in I} Y_i$ *is a partition of Y such that $F_f \cap Y_i = \{x_i^*\}$, $i \in I$;*
(ii) $f|_{Y_i} : Y_i \to X$ *is an A-contraction, $i \in I$.*
Then f satisfies the Ψ- condition with respect to $\Psi = \left(\frac{t}{1-a_\alpha}\right)_{\alpha \in \mathcal{A}}$.

Problem 4.3. *Which generalized contractions on gauge space satisfy the* Ψ*-condition with respect to some family of functions* $\Psi = (\psi_\alpha)_{\alpha \in \mathcal{A}}$?

Problem 4.4. *In the case* $Y = X$, *for which generalized contractions on gauge space we have that:*
(i) f *satisfies the* Ψ*-condition;*
(ii) f *is not a* $\Psi - WPO$?

We have the following data dependence result.

Theorem 4.5. *Let* $(X, \mathcal{P})$ *be a gauge space,* $Y \subset X$ *and* $f, g : Y \to X$ *two operators. We suppose that:*
(i) f, g *satisfy the* Ψ*-condition and* $F_f \neq \emptyset$;
(ii) *for every* $\alpha \in A$ *there exists* $\eta_\alpha > 0$ *such that*

$$d_\alpha(f(x), g(x)) \leq \eta_\alpha, \quad \textit{for all } x \in Y;$$

(iii) $F_g \neq \emptyset$.
Then for every $\alpha \in A$ *we have that* $(PH)_{d_\alpha}(F_f, F_g) \leq \psi_\alpha(\eta_\alpha)$.

Proof. Let $x \in F_g$ and let $\alpha \in A$. Then

$$d_\alpha(x, \chi_f(x)) \leq \psi_\alpha(d_\alpha(x, f(x))) = \psi_\alpha(d_\alpha(g(x), f(x))) \leq \psi_\alpha(\eta_\alpha).$$

Similarly, if $y \in F_f$, then

$$d_\alpha(y, \chi_g(y)) \leq \psi_\alpha(d_\alpha(y, g(y))) = \psi_\alpha(d_\alpha(f(y), g(y))) \leq \psi_\alpha(\eta_\alpha).$$

Now from Lemma 3.1 we have that $(PH)_{d_\alpha}(F_f, F_g) \leq \psi(\eta_\alpha)$ for every $\alpha \in A$.

□

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