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CONVERGENCE CRITERIA OF GENERALIZED HYBRID PROXIMAL POINT ALGORITHMS

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Abstract. In this paper, we introduce and analyze some generalized hybrid proximal point algorithms for finding a common element of the set of zeros of a maximal monotone operator and the set of fixed points of a nonexpansive mapping in a Hilbert space. These algorithms include the previously known proximal point algorithms as special cases. Weak and strong convergence of the proposed proximal point algorithms are proved under some mild conditions.
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1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let T be an operator with domain $D(T)$ and range $R(T)$ in H. A multivalued operator T is monotone if its graph $G(T) := \{(x,y) \in H \times H : x \in D(T),\ y \in Tx\}$ is a monotone set in $H \times H$. That is, T is monotone if and only if

$$(x_1, y_1), (x_2, y_2) \in G(T) \ \Rightarrow\ \langle x_1 - x_2, y_1 - y_2\rangle \geq 0. \tag{1.1}$$

A monotone operator T is maximal monotone if the graph $G(T)$ is not properly contained in the graph of any other monotone operator on H.

Let T be a maximal monotone operator on H. Then a point $z \in D(T)$ is called a zero of T if $0 \in Tz$. We denote by Ω the set of all zeros of T, i.e., $\Omega = T^{-1}(0)$. It is known that Ω is closed and convex.

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One of the major problems in the theory of maximal monotone operators is to find a point in Ω, assuming that Ω is nonempty. A variety of problems, for example, convex programming and variational inequalities, can be formulated as finding a zero of maximal monotone operators. The proximal point algorithm (for short, PPA) is recognized as a powerful and successful algorithm in finding a zero of maximal monotone operators. Starting from any initial guess $x_0 \in H$, the PPA generates a sequence $\{x_n\}$ via the following inclusion:

$$x_n \in x_{n+1} + c_n T x_{n+1}, \tag{1.2}$$

where $c_n > 0$ is a regularization parameter.

Based on the fact that solving the inclusion (1.2) may probably be as difficult as solving the original problem of finding a zero of Ω, Rockafellar [13] proposed the inexact proximal point algorithm (for short, IPPA) which is a more practical algorithm. Starting from any initial guess $x_0 \in H$, the IPPA generates a sequence $\{x_n\}$ via the following relation:

$$x_n + e_n \in x_{n+1} + c_n T x_{n+1}, \tag{1.3}$$

where $\{e_n\}$ is a sequence of errors.

In recent years, some iterative algorithms including the PPA and IPPA have played a powerful and successful role in solving variational inequality problems, optimization problems, the zero point problem of maximal monotone operators and many others, see [1,2,4,6-15,19,27-34] and the references therein. It is worth pointing out that the accuracy criteria for the errors $\{e_n\}$ in the IPPA (1.3) have been extensively studied so that the convergence of (1.3) is guaranteed. It is well-known that two criteria were introduced in [13]; these are

$$\|e_n\| \le \varepsilon_n, \quad \sum_{n=0}^{\infty} \varepsilon_n < \infty, \tag{1.4}$$

$$\|e_n\| \le \delta_n \|x_{n+1} - x_n\|, \quad \sum_{n=0}^{\infty} \delta_n < \infty, \tag{1.5}$$

Utilizing criterion (1.4), Rockafellar [13] proved the weak convergence of (1.3) provided the regularization sequence $\{c_n\}$ remains bounded away from zero. He [11] also obtained the rate of convergence of (1.3) by virtue of the criterion (1.5). He [11] also proposed another criterion as follows:

$$\|e_n\| \le \eta_n \|x_{n+1} - x_n\|, \quad \sum_{n=0}^{\infty} \eta_n^2 < \infty. \tag{1.6}$$

It is shown in [10] that if H is a finite dimensional Hilbert space, then the sequence $\{x_n\}$ generated by (1.3) converges to a point in Ω provided the criterion (1.6) holds and the regularization sequence $\{c_n\}$ remains bounded away from zero.

In [6], Eckstein and Bertsekas considered the following generalized proximal point algorithm (for short, GPPA):

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n w_n, \quad \forall n \ge 0, \tag{1.7}$$

where $\alpha_n \in (0,2)$ $(\forall n \geq 0)$ and

$$\|w_n - (I + c_n T)^{-1} x_n\| \leq \varepsilon_n, \quad \forall n \geq 0.$$

Weak convergence of (1.7) was proved under the conditions

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty, \ \inf_{n\geq 0} c_n > 0,$$

and there is some $\bar{\alpha} \in (0,2)$ with the property

$$\bar{\alpha} \leq \alpha_n \leq 2 - \bar{\alpha}, \quad \forall n \geq 0. \tag{1.8}$$

Since the PPA (1.2) does not, in general, have strong convergence (see [8]), an interesting topic is how to modify the PPA (1.2) so that strong convergence is guaranteed. Some effort has been made recently (see, e.g., [1,14,18,27]). In 2002, Xu [18] introduced a contraction proximal point algorithm as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(I + c_n T)^{-1} x_n + e_n, \quad \forall n \geq 0, \tag{1.9}$$

and proved strong convergence of (1.9) by virtue of the condition that the regularization sequence $\{c_n\}$ tends to the infinity. Later on, Marino and Xu [12] further considered algorithm (1.9) and obtained some strong convergence results under certain appropriate assumptions. Very recently, Yao and Noor [19] introduced and analyzed some generalized proximal point algorithms, which include as special cases the algorithms in Rockafellar [13], Han and He [10], Eckstein and Bertsekas [6], Marino and Xu [12], and Xu [18]. In [19], several weak and strong convergence results for Yao and Noor's GPPAs were established under some mild conditions.

On the other hand, let H be a real Hilbert space. Yamada ([22], see also [23]) recently introduced a hybrid steepest-descent method for the variational inequality problem (for short, VI(F,C)), that is, find a point $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in C,$$

where $F : H \to H$ is a nonlinear operator and C is the fixed point set of a nonexpansive mapping $A : H \to H$, i.e., $C = \{x \in H : Ax = x\}$. Recall that A is nonexpansive if

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in H,$$

and let

$$\mathrm{Fix}(A) = \{x \in H : Ax = x\}$$

denote the fixed point set of A. Yamada's idea is stated now. Let $F : H \to H$ be an operator such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\} \subset [0,1]$. Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} := Ax_n - \lambda_{n+1}\mu F(Ax_n), \quad \forall n \geq 0. \tag{1.10}$$

Then, Yamada [22] proved that under appropriate conditions, $\{x_n\}$ converges strongly to the unique solution of the VI(F,C). Subsequently, Xu and Kim [20] and Ceng, Xu and Yao [24] improved and extended Yamada's result [22].

Inspired and motivated by the above research work, we introduce and analyze some generalized hybrid proximal point algorithms for finding a common element of the set of zeros of a maximal monotone operator and the set of fixed points of a nonexpansive mapping in a Hilbert space. These algorithms include the Eckstein-Bertsekas type generalized proximal point algorithm and Marino-Xu type contraction proximal point algorithm as special cases; see, e.g., [6,9,12,18,19]. Moreover, weak and strong convergence results for these algorithms are proved under some mild conditions. Our proofs are different from many others. Results presented in this paper can be viewed as a significant improvement, refinement and development of the corresponding ones in Eckstein and Bertsekas [6], Han and He [10], Marino and Xu [12], Yao and Noor [19] and many others.

Throughout this paper, we use the following notations:

- $\rightharpoonup$ stands for weak convergence and $\to$ for strong convergence.
- $\omega_w(\{x_n\}) = \{x : \exists x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let T be a maximal monotone operator on H and Ω be the set of zeros of T. Let $A : H \to H$ be a nonexpansive mapping such that $\mathrm{Fix}(A) \cap \Omega \neq \emptyset$. For $r > 0$, we use J_r and A_r to denote the resolvent and Yosida approximation of T, respectively; that is,

$$J_r = (I + rT)^{-1} \quad \text{and} \quad A_r = \frac{1}{r}(I - J_r).$$

It is well-known that $A_r x \in T(J_r x)$ for all $x \in H$; for more details see [3,12].

Let C be a nonempty closed convex subset of H. Recall that a mapping $f : C \to C$ is called nonexpansive if

$$\|f(x) - f(y)\| \leq \|x - y\|$$

for all $x, y \in C$. It is known that the resolvent J_r is nonexpansive for $r > 0$. We use $\mathrm{Fix}(f) = \{x \in C : f(x) = x\}$ to denote the fixed point set of f. Note that $\mathrm{Fix}(f)$ is closed and convex in H. Note also that $\Omega = \mathrm{Fix}(J_r)$ for each $r > 0$. We use P_C to denote the projection from H onto C; that is, for each $x \in H$

$$P_C x = \arg\min_{y \in C} \|x - y\|.$$

It is known [25] that P_C is characterized by: for given $x \in H$ and $u \in C$, $u = P_C x$ if and only if

$$\langle x - u, u - y \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

Before starting the main results of this paper, we include some lemmas as follows

Lemma 2.1. (Xu [18]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq a_n + \sigma_n, \quad \forall n \geq 0,$$

where $\{\sigma_n\}$ is a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \sigma_n < \infty$. Then $\lim_{n\to\infty} a_n$ exists.

The following two lemmas are well-known.

Lemma 2.2. (see [25, Demiclosedness principle]). *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \to C$ be a nonexpansive mapping such* *that* $\mathrm{Fix}(f) \neq \emptyset$. *Assume that* $\{x_n\}$ *is a sequence in* C *which converges weakly to* $x \in C$ *and that* $\{(I-f)x_n\}$ *converges strongly to* $y \in H$. *Then* $(I-f)x = y$.

Lemma 2.3. (see [12]). *The resolvent identity. For* $\lambda, \mu > 0$, *there holds the identity*

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1-\frac{\mu}{\lambda}\right)J_\lambda x\right), \quad \forall x \in H.$$

Lemma 2.4. (Marino and Xu [12]). *Assume that* $0 < c_1 \le c_2$. *Then* $\|J_{c_1}x - x\| \le \|J_{c_2}x - x\|$ *for all* $x \in H$.

The following lemma is very important for proving our main results, one can find it in [5,16,17].

Lemma 2.5. *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a real Banach space* X *and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *with* $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$. *Suppose that* $x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n$ *for all integers* $n \ge 0$ *and* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$. *Then,* $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Lemma 2.6. (see [18,20]). *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \le (1-s_n)a_n + s_n t_n + \delta_n, \quad \forall n \ge 0,$$

where $\{s_n\} \subset [0,1]$ *and* $\{t_n\}$ *are such that*

(i) $\sum_{n=0}^{\infty} s_n = \infty$;

(ii) either $\limsup_{n\to\infty} t_n \le 0$ *or* $\sum_{n=0}^{\infty}|s_n t_n| < \infty$;

(iii) $\sum_{n=0}^{\infty}\delta_n < \infty$.

Then $\{a_n\}$ *converges to zero.*

Let $F: H \to H$ be an operator such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. That is, F satisfies the conditions

$$\|Fx - Fy\| \le \kappa\|x-y\|, \quad \forall x, y \in H,$$

and

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x-y\|^2, \quad \forall x, y \in H.$$

Let λ be a number in $[0,1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $A: H \to H$, we define the mapping $A^\lambda : H \to H$ by

$$A^\lambda x := Ax - \lambda\mu F(Ax), \quad \forall x \in H.$$

Lemma 2.7. (see [22]). *If* $0 \le \lambda \le 1$ *and* $0 < \mu < 2\eta/\kappa^2$, *then there holds for* $A^\lambda : H \to H$,

$$\|A^\lambda x - A^\lambda y\| \le (1-\lambda\tau)\|x-y\|, \quad \forall x, y \in H,$$

where $\tau = 1 - \sqrt{1-\mu(2\eta - \mu\kappa^2)} \in (0,1)$.

Lemma 2.8. *Let* H *be a Hilbert space. Then there hold the following statements:*

(i) (see [25]) for each $x, y \in H$ *and each* $\lambda \in [0,1]$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2;$$

(ii) for each $x, y \in H$

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle.$$

Recall now that a Banach space X satisfies Opial's property [21] provided, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X,\ y \neq x.$$

It is known [21] that the Hilbert space H and space l^p $(1 \le p < \infty)$ enjoy this property, while L^p does not unless $p = 2$. It is known [26] that any separable Banach space can be equivalently renormed so that it satisfies Opial's property.

3. Main results

We recall the following generalized proximal point algorithm proposed by Eckstein and Bertsekas in [6]:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_{c_n}(x_n) + e_n, \quad \forall n \ge 0, \tag{3.1}$$

where e_n is an error. Algorithm (3.1) generalizes Gol'shtein and Tre'yakov's algorithm [7] defined as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_c(x_n), \quad \forall n \ge 0, \tag{3.2}$$

which was considered in a finite dimensional Hilbert space setting where the parameters c does not vary with the iteration steps.

Very recently, Yao and Noor [19] consider Gol'shtein and Tre'yakov's algorithm (3.2) with errors; namely, the algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_c(x_n) + e_n, \quad \forall n \ge 0, \tag{3.3}$$

Now we introduce the following algorithm in the sense of Gol'shtein and Tre'yakov

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_c(y_n) + e_n, \\ y_n = (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)), \quad \forall n \ge 0. \end{cases} \tag{3.4}$$

Let $w_\omega(x_n)$ denote the weak ω-limit set of $\{x_n\}$. That is, $w_\omega(x_n)$ consists of the points which are the weak limits of subsequences of $\{x_n\}$. Now we give the following result.

Theorem 3.1. *Let $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $A : H \to H$ be nonexpansive such that* $\mathrm{Fix}(A) \cap \Omega \neq \emptyset$. *Let $\mu \in (0, 2\eta/\kappa^2)$, let $x_0 \in H$, $\{e_n\} \subset H$ and $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfying the conditions:*

(i) $\lim_{n\to\infty} \lambda_n = 0$;

(ii) $\lim_{n\to\infty} \|e_n\| = 0$;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;

(iv) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ generated by (3.4) converges weakly to a point in $\mathrm{Fix}(A) \cap \Omega$ *provided $\{x_n\}$ is bounded.*

Proof. First, let us show that $\{y_n\}$ is bounded. Indeed, pick $p \in \mathrm{Fix}(A) \cap \Omega$, then utilizing Lemma 2.7 we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)) - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|(Ax_n - \lambda_n \mu F(Ax_n)) - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|A^{\lambda_n}x_n - A^{\lambda_n}p\| + \|A^{\lambda_n}p - p\|] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1-\lambda_n\tau)\|x_n - p\| + \lambda_n\mu\|F(p)\|] \\ &\leq \|x_n - p\| + \lambda_n\mu\|F(p)\|. \end{aligned}$$

Hence from the boundedness of $\{x_n\}$ it follows that $\{y_n\}$ is bounded.

Set $x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n$. Note that $\{z_n\}$ is also bounded. Then, we have

$$\begin{aligned} z_{n+1} - z_n &= \tfrac{x_{n+2} - (1-\alpha_{n+1}x_{n+1})}{\alpha_{n+1}} - \tfrac{x_{n+1} - (1-\alpha_n)x_n}{\alpha_n} \\ &= J_c(y_{n+1}) - J_c(y_n) + \tfrac{e_{n+1}}{\alpha_{n+1}} - \tfrac{e_n}{\alpha_n}. \end{aligned} \tag{3.5}$$

From the nonexpansivity of J_c and (3.5), we have

$$\|z_{n+1} - z_n\| - \|y_{n+1} - y_n\| \leq \frac{1}{\alpha_{n+1}}\|e_{n+1}\| + \frac{1}{\alpha_n}\|e_n\|,$$

which implies that (noting that $\lim_{n\to\infty}\|e_n\| = 0$)

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|y_{n+1} - y_n\|) \leq 0. \tag{3.6}$$

Also, observe that

$$\begin{aligned} \|A^{\lambda_{n+1}}x_{n+1} - A^{\lambda_n}x_n\| &\leq \|A^{\lambda_{n+1}}x_{n+1} - A^{\lambda_{n+1}}x_n\| + \|A^{\lambda_{n+1}}x_n - A^{\lambda_n}x_n\| \\ &\leq (1-\lambda_{n+1}\tau)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\mu\|F(Ax_n)\|, \end{aligned}$$

and hence

$$\begin{gathered} \|y_{n+1} - y_n\| \leq |\beta_n - \beta_{n+1}|\|x_{n+1}\| + (1-\beta_n)\|x_{n+1} - x_n\| \\ +|\beta_{n+1} - \beta_n|\|A^{\lambda_{n+1}}x_{n+1}\| + \beta_n\|A^{\lambda_{n+1}}x_{n+1} - A^{\lambda_n}x_n\| \\ \leq |\beta_n - \beta_{n+1}|\|x_{n+1}\| + (1-\beta_n)\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|A^{\lambda_{n+1}}x_{n+1}\| \\ +\beta_n[(1-\lambda_{n+1}\tau)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\mu\|F(Ax_n)\|] \\ \leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\mu\|F(Ax_n)\| \\ +|\beta_n - \beta_{n+1}|(\|x_{n+1}\| + \|A^{\lambda_{n+1}}x_{n+1}\|). \end{gathered}$$

Since A and F are Lipschitzian and $\{x_n\}$ is bounded, we know that both $\{F(Ax_n)\}$ and $\{A^{\lambda_n}x_n\}$ are bounded. Hence from $\lambda_n \to 0$ and $|\beta_{n+1} - \beta_n| \to 0$ it follows that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

This together with (3.6) implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.7}$$

It follows from (3.7) and Lemma 2.5 that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

So $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Consequently, we have

$$\lim_{n\to\infty}\|J_c(y_n) - x_n\| = 0. \tag{3.8}$$

Furthermore, observe that for $p \in \mathrm{Fix}(A) \cap \Omega$

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)) - p\|^2 \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p) - \beta_n\lambda_n\mu F(Ax_n)\|^2 \\ &\le \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p)\|^2 + 2\|(1-\beta_n)(x_n - p) \\ &\quad +\beta_n(Ax_n - p)\|\|\beta_n\lambda_n\mu F(Ax_n)\| + \|\beta_n\lambda_n\mu F(Ax_n)\|^2 \\ &\le \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p)\|^2 + \lambda_n M\mu\|F(Ax_n)\| \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|Ax_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2 \\ &\quad +\lambda_n M\mu\|F(Ax_n)\| \\ &\le \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2 + \lambda_n M\mu\|F(Ax_n)\|,\end{aligned}$$

where M is a constant such that $M > 2\|x_n - p\| + \mu\|F(Ax_n)\|$. In the meantime, observe also that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n J_c(y_n) + e_n - p\|^2 \\ &\le \|(1-\alpha_n)(x_n - p) + \alpha_n(J_c(y_n) - p)\|^2 + 2\langle e_n, x_{n+1} - p\rangle \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|y_n - p\|^2 + 2\|e_n\|\|x_{n+1} - p\| \\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2 \\ &\quad +\lambda_n M\mu\|F(Ax_n)\|] + 2\|e_n\|\|x_{n+1} - p\| \\ &\le \|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Ax_n\|^2 + \lambda_n M\mu\|F(Ax_n)\|] \\ &\quad +2\|e_n\|\|x_{n+1} - p\| \\ &\le \|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Ax_n\|^2 + M(\lambda_n M + \|e_n\|),\end{aligned}$$

Consequently, we obtain

$$\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Ax_n\|^2 + M(\lambda_n M + \|e_n\|),$$

and hence

$$\begin{aligned}&\alpha_n\beta_n(1-\beta_n)\|x_n - Ax_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + M(\lambda_n M + \|e_n\|) \\ &\le |\|x_n - p\| - \|x_{n+1} - p\||(\|x_n - p\| + \|x_{n+1} - p\|) + M(\lambda_n M + \|e_n\|) \\ &\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + M(\lambda_n M + \|e_n\|).\end{aligned}$$

Since

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1,$$

$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$

$\lambda_n \to 0$ and $\|e_n\| \to 0$, we deduce from $\|x_n - x_{n+1}\| \to 0$ that

$$\lim_{n\to\infty} \|x_n - Ax_n\| = 0.$$

This together with (3.4) implies that

$$\|y_n - x_n\| \le \beta_n\|x_n - Ax_n\| + \beta_n\lambda_n\mu\|F(Ax_n)\| \le \|x_n - Ax_n\| + \lambda_n M \to 0.$$

That is,

$$\lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Thus, we get from (3.8)

$$\|J_c(x_n) - x_n\| \le \|J_c(x_n) - J_c(y_n)\| + \|J_c(y_n) - x_n\| \le \|x_n - y_n\| + \|J_c(y_n) - x_n\| \to 0.$$

That is,

$$\lim_{n\to\infty} \|J_c(x_n) - x_n\| = 0.$$

Now let us show that $\omega_w(x_n) \subset \mathrm{Fix}(A) \cap \Omega$. Indeed, since $\{x_n\}$ is bounded and H is reflexive, we know that $\omega_w(x_n) \neq \emptyset$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x} \in \omega_w(x_n)$. Since $\|x_n - Ax_n\| \to 0$ and $\|J_c(x_n) - x_n\| \to 0$, and both A and J_c are nonexpansive mappings, in terms of Lemma 2.2 we conclude that $\hat{x} \in \mathrm{Fix}(A) \cap \mathrm{Fix}(J_c) = \mathrm{Fix}(A) \cap \Omega$. This shows that $\omega_w(x_n) \subset \mathrm{Fix}(A) \cap \Omega$. Next let us show that $\omega_w(x_n)$ is a singleton. Indeed, let $\{x_{m_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{m_j} \rightharpoonup \bar{x} \in \omega_w(x_n)$. If $\hat{x} \neq \bar{x}$, utilizing Opial's property of H we reach the following contradiction:

$$\begin{aligned}\lim_{n\to\infty} \|x_n - \hat{x}\| &= \lim_{i\to\infty} \|x_{n_i} - \hat{x}\| \\ &< \lim_{i\to\infty} \|x_{n_i} - \bar{x}\| = \lim_{j\to\infty} \|x_{m_j} - \bar{x}\| \\ &< \lim_{j\to\infty} \|x_{m_j} - \hat{x}\| \\ &= \lim_{n\to\infty} \|x_n - \hat{x}\|.\end{aligned}$$

This shows that $\omega_w(x_n)$ is a singleton. The proof is therefore complete. □

Corollary 3.1. (see [19, Theorem 3.1]). *Let $\{x_n\}$ be generated by the algorithm (3.3). Assume that* $\lim_{n\to\infty} \|e_n\| = 0$ *and* $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$. *Then $\{x_n\}$ converges weakly to a point in Ω provided $\{x_n\}$ is bounded.*

Proof. In Theorem 3.1, put $A = I$ the identity mapping of H, and $\lambda_n = 0$ for all $n \ge 0$. Then we have

$$\begin{aligned}y_n &= (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)) \\ &= (1-\beta_n)x_n + \beta_n x_n = x_n.\end{aligned}$$

Hence algorithm (3.4) reduces to (3.3). Therefore, from Theorem 3.1 we immediately obtain the desired result. □

We remind the reader of the following fact. Although in [19, Theorem 3.1] there is no requirement of the boundedness of $\{x_n\}$, in the proof of [19, Theorem 3.1] one can not see that the combination of Lemma 2.1 with the condition $\lim_{n\to\infty} \|e_n\| = 0$ implies the existence of the limit $\lim_{n\to\infty} \|x_n - p\|$ for $p \in \Omega$. Thus, in [19, Theorem 3.1] the boundedness of $\{x_n\}$ must be required.

Next we let the parameter c_n to vary with the iteration steps and introduce the following generalized hybrid proximal point algorithm in the sense of Eckstein and Bertsekas:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n J_{c_n}(y_n) + e_n, \\ y_n = (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)), \quad \forall n \ge 0. \end{cases} \tag{3.9}$$

The convergence result for algorithm (3.9) is given as follows:

Theorem 3.2. *Let $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $A : H \to H$ be nonexpansive such that* $\mathrm{Fix}(A) \cap \Omega \neq \emptyset$. *Let $\mu \in (0, 2\eta/\kappa^2)$, let $x_0 \in H$, $\{c_n\} \subset (0,\infty)$, $\{e_n\} \subset H$ and* $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *satisfying the conditions:*

(i) $\sum_{n=0}^{\infty} \lambda_n < \infty$;

(ii) $\sum_{n=0}^{\infty} \|e_n\| < \infty$;

(iii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$;
(iv) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(v) $c_n \ge c$, *where* c *is some positive constant;*
(vi) $\lim_{n\to\infty} |c_{n+1} - c_n| = 0$.

Then the sequence $\{x_n\}$ *generated by (3.9) converges weakly to a point in* $\mathrm{Fix}(A)\cap\Omega$.

Proof. Pick $p \in \mathrm{Fix}(A)\cap\Omega$ and note that $J_r p = p$ for all $r > 0$. Then utilizing Lemma 2.7 we have from (3.9)

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n\mu F(Ax_n)) - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n\|(Ax_n - \lambda_n\mu F(Ax_n)) - p\| \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n[\|A^{\lambda_n}x_n - A^{\lambda_n}p\| + \|A^{\lambda_n}p - p\|] \\ &\le (1-\beta_n)\|x_n - p\| + \beta_n[(1-\lambda_n\tau)\|x_n - p\| + \lambda_n\mu\|F(p)\|] \\ &\le \|x_n - p\| + \lambda_n\mu\|F(p)\|, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)x_n + \alpha_n J_{c_n}(y_n) + e_n - p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|J_{c_n}(y_n) - p\| + \|e_n\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| + \|e_n\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n[\|x_n - p\| + \lambda_n\mu\|F(p)\|] + \|e_n\| \\ &\le \|x_n - p\| + \lambda_n\mu\|F(p)\| + \|e_n\|. \end{aligned} \tag{3.10}$$

From Lemma 2.1, (i), (ii) and (3.10), we conclude that $\lim_{n\to\infty} \|x_n - p\|$ exists. This implies that $\{x_n\}$ is bounded.

Put $x_{n+1} = (1-\alpha_n)x_n + \alpha_n z_n$. Then we have

$$\begin{aligned} z_{n+1} - z_n &= \tfrac{x_{n+2} - (1-\alpha_{n+1}x_{n+1})}{\alpha_{n+1}} - \tfrac{x_{n+1} - (1-\alpha_n)x_n}{\alpha_n} \\ &= J_{c_{n+1}}(y_{n+1}) - J_{c_n}(y_n) + \tfrac{e_{n+1}}{\alpha_{n+1}} - \tfrac{e_n}{\alpha_n}. \end{aligned}$$

If $c_n \le c_{n+1}$, from Lemma 2.3, utilizing the resolvent identity

$$J_{c_{n+1}}(y_{n+1}) = J_{c_n}\Big(\frac{c_n}{c_{n+1}}y_{n+1} + \Big(1 - \frac{c_n}{c_{n+1}}\Big)J_{c_{n+1}}(y_{n+1})\Big),$$

we obtain

$$\begin{aligned} \|J_{c_{n+1}}(y_{n+1}) - J_{c_n}(y_n)\| &\le \tfrac{c_n}{c_{n+1}}\|y_{n+1} - y_n\| + \big(1 - \tfrac{c_n}{c_{n+1}}\big)\|J_{c_{n+1}}(y_{n+1}) - y_n\| \\ &\le \|y_{n+1} - y_n\| + \tfrac{1}{c}|c_{n+1} - c_n|\|J_{c_{n+1}}(y_{n+1}) - y_n\|. \end{aligned}$$

If $c_n > c_{n+1}$, again by Lemma 2.3,

$$\begin{aligned} \|J_{c_n}(y_n) - J_{c_{n+1}}(y_{n+1})\| &= \|J_{c_{n+1}}\big(\tfrac{c_{n+1}}{c_n}y_n + \big(1 - \tfrac{c_{n+1}}{c_n}\big)J_{c_n}(y_n)\big) - J_{c_{n+1}}(y_{n+1})\| \\ &\le \tfrac{c_{n+1}}{c_n}\|y_n - y_{n+1}\| + \big(1 - \tfrac{c_{n+1}}{c_n}\big)\|J_{c_n}(y_n) - y_{n+1}\| \\ &\le \|y_{n+1} - y_n\| + \tfrac{1}{c}|c_{n+1} - c_n|\|J_{c_n}(y_n) - y_{n+1}\|. \end{aligned}$$

Hence, from the above estimates, we have

$$\|J_{c_{n+1}}(y_{n+1}) - J_{c_n}(y_n)\| \le \|y_{n+1} - y_n\| + \frac{K}{c}|c_{n+1} - c_n|, \tag{3.11}$$

where K is a constant such that $\sup\{\|J_{c_{n+1}}(y_{n+1}) - y_n\|, \|J_{c_n}(y_n) - y_{n+1}\|, n \ge 0\} \le K$. Therefore, we have

$$\begin{aligned} \|z_{n+1} - z_n\| &\le \|J_{c_{n+1}}(y_{n+1}) - J_{c_n}(y_n)\| + \tfrac{\|e_{n+1}\|}{\alpha_{n+1}} + \tfrac{\|e_n\|}{\alpha_n} \\ &\le \|y_{n+1} - y_n\| + \tfrac{K}{c}|c_{n+1} - c_n| + \tfrac{\|e_{n+1}\|}{\alpha_{n+1}} + \tfrac{\|e_n\|}{\alpha_n}, \end{aligned}$$

which implies

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|y_{n+1}-y_n\|)\le 0. \tag{3.12}$$

Also, observe that

$$\begin{aligned}\|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_n}x_n\| &\le \|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_{n+1}}x_n\|+\|A^{\lambda_{n+1}}x_n-A^{\lambda_n}x_n\|\\ &\le (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\mu\|F(Ax_n)\|,\end{aligned}$$

and hence

$$\|y_{n+1}-y_n\|\le|\beta_n-\beta_{n+1}|\|x_{n+1}\|+(1-\beta_n)\|x_{n+1}-x_n\|$$

$$+|\beta_{n+1}-\beta_n|\|A^{\lambda_{n+1}}x_{n+1}\|+\beta_n\|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_n}x_n\|$$

$$\le|\beta_n-\beta_{n+1}|\|x_{n+1}\|+(1-\beta_n)\|x_{n+1}-x_n\|+|\beta_{n+1}-\beta_n|\|A^{\lambda_{n+1}}x_{n+1}\|$$

$$+\beta_n[(1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\mu\|F(Ax_n)\|]$$

$$\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\mu\|F(Ax_n)\|$$

$$+|\beta_n-\beta_{n+1}|(\|x_{n+1}\|+\|A^{\lambda_{n+1}}x_{n+1}\|).$$

Since A and F are Lipschitzian and $\{x_n\}$ is bounded, we know that both $\{F(Ax_n)\}$ and $\{A^{\lambda_n}x_n\}$ are bounded. Hence from $\lambda_n\to 0$ and $|\beta_{n+1}-\beta_n|\to 0$ it follows that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\le 0.$$

This together with (3.12) implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.13}$$

It follows from (3.13) and Lemma 2.5 that

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

So $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Consequently, we have

$$\lim_{n\to\infty}\|J_{c_n}(y_n)-x_n\|=0. \tag{3.14}$$

Furthermore, observe that for $p\in \mathrm{Fix}(A)\cap\Omega$

$$\begin{aligned}\|y_n-p\|^2 &=\|(1-\beta_n)x_n+\beta_n(Ax_n-\lambda_n\mu F(Ax_n))-p\|^2\\ &=\|(1-\beta_n)(x_n-p)+\beta_n(Ax_n-p)-\beta_n\lambda_n\mu F(Ax_n)\|^2\\ &\le\|(1-\beta_n)(x_n-p)+\beta_n(Ax_n-p)\|^2+2\|(1-\beta_n)(x_n-p)\\ &\quad+\beta_n(Ax_n-p)\|\|\beta_n\lambda_n\mu F(Ax_n)\|+\|\beta_n\lambda_n\mu F(Ax_n)\|^2\\ &\le\|(1-\beta_n)(x_n-p)+\beta_n(Ax_n-p)\|^2+\lambda_nM\mu\|F(Ax_n)\|\\ &=(1-\beta_n)\|x_n-p\|^2+\beta_n\|Ax_n-p\|^2-\beta_n(1-\beta_n)\|x_n-Ax_n\|^2\\ &\quad+\lambda_nM\mu\|F(Ax_n)\|\\ &\le\|x_n-p\|^2-\beta_n(1-\beta_n)\|x_n-Ax_n\|^2+\lambda_nM\mu\|F(Ax_n)\|,\end{aligned}$$

where M is a constant such that $M > 2\|x_n - p\| + \mu\|F(Ax_n)\|$. In the meantime, observe also that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n)x_n + \alpha_n J_{c_n}(y_n) + e_n - p\|^2 \\ &\le \|(1-\alpha_n)(x_n-p) + \alpha_n(J_{c_n}(y_n)-p)\|^2 + 2\langle e_n, x_{n+1}-p\rangle \\ &\le (1-\alpha_n)\|x_n-p\|^2 + \alpha_n\|y_n-p\|^2 + 2\|e_n\|\|x_{n+1}-p\| \\ &\le (1-\alpha_n)\|x_n-p\|^2 + \alpha_n[\|x_n-p\|^2 - \beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\ &\quad +\lambda_n M\mu\|F(Ax_n)\|] + 2\|e_n\|\|x_{n+1}-p\| \\ &\le \|x_n-p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 + \lambda_n M\mu\|F(Ax_n)\|] \\ &\quad +2\|e_n\|\|x_{n+1}-p\| \\ &\le \|x_n-p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 + M(\lambda_n M + \|e_n\|),\end{aligned}$$

Consequently, we obtain

$$\|x_{n+1}-p\|^2 \le \|x_n-p\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 + M(\lambda_n M + \|e_n\|),$$

and hence

$$\alpha_n\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + M(\lambda_n M + \|e_n\|).$$

Since

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1,$$
$$0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1,$$

$\lambda_n \to 0$ and $\|e_n\| \to 0$, we deduce from the existence of $\lim_{n\to\infty}\|x_n - p\|$ that

$$\lim_{n\to\infty}\|x_n - Ax_n\| = 0.$$

This together with (3.9) implies that

$$\|y_n - x_n\| \le \beta_n\|x_n - Ax_n\| + \beta_n\lambda_n\mu\|F(Ax_n)\| \le \|x_n - Ax_n\| + \lambda_n M \to 0.$$

That is,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Thus, we get from (3.14)

$$\begin{aligned}\|J_{c_n}(x_n) - x_n\| &\le \|J_{c_n}(x_n) - J_{c_n}(y_n)\| + \|J_{c_n}(y_n) - x_n\| \\ &\le \|x_n - y_n\| + \|J_{c_n}(y_n) - x_n\| \to 0.\end{aligned}$$

That is,

$$\lim_{n\to\infty}\|J_{c_n}(x_n) - x_n\| = 0. \tag{3.15}$$

It follows from Lemma 2.4 and (3.15) that

$$\lim_{n\to\infty}\|J_c(x_n) - x_n\| = 0.$$

By the same argument as in the proof of Theorem 3.1 we obtain $\{x_n\}$ converges weakly to a point in $\mathrm{Fix}(A)\cap\Omega$. The proof is therefore complete. □

Corollary 3.2. ([19, Theorem 3.2]). *Let $\{x_n\}$ be generated by the algorithm (3.1). Assume that the following conditions hold*

(i) $\sum_{n=0}^{\infty}\|e_n\| < \infty$;

(ii) $0 < \liminf_{n\to\infty}\alpha_n \le \limsup_{n\to\infty}\alpha_n < 1$;

(iii) $c_n \ge c$, *where c is some positive constant;*

(iv) $c_{n+1} - c_n \to 0$.
Then $\{x_n\}$ *converges weakly to a point in* Ω.
Proof. In Theorem 3.1, put $A = I$ the identity mapping of H, and $\lambda_n = 0$ for all $n \geq 0$. Then we have

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)) \\ &= (1-\beta_n)x_n + \beta_n x_n = x_n. \end{aligned}$$

Hence algorithm (3.9) reduces to (3.1). Therefore, from Theorem 3.2 we immediately obtain the desired result. □

We observe that in general the proximal point algorithms have only weak convergence. Recently some modified proximal point algorithms with strong convergence have been proposed (see, e.g., [1,10,14,15,18,27,32]). Very recently, motivated by Marino and Xu [12], Yao and Noor [19] suggested the following contraction proximal point algorithm: for given $u \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n u + \gamma_n x_n + \delta_n J_{c_n}(x_n) + e_n, \quad \forall n \geq 0, \tag{3.16}$$

where $\alpha_n, \gamma_n, \delta_n \in [0,1]$ and $\alpha_n + \gamma_n + \delta_n = 1$ $(\forall n \geq 0)$, $c_n > 0$ and e_n is an error. We remark that algorithm (3.16) includes the algorithm of Marino and Xu [12] as a special case. Now we introduce the following hybrid contraction proximal point algorithm in the sense of Marino and Xu: for given $u \in H$, let the sequence $\{x_n\}$ be generated iteratively by

$$\begin{cases} x_{n+1} = \alpha_n u + \gamma_n x_n + (1-\alpha_n-\gamma_n)J_{c_n}(y_n) + e_n, \\ y_n = (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n\mu F(Ax_n)), \quad \forall n \geq 0, \end{cases} \tag{3.17}$$

where $\mu \in (0, 2\eta/\kappa^2)$, $\lambda_n, \alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\alpha_n + \gamma_n \leq 1$ $(\forall n \geq 0)$, $c_n > 0$ and e_n is an error. The convergence result for algorithm (3.17) is given as follows:

Theorem 3.3. *Let* $F : H \to H$ *be a mapping such that for some constants* $\kappa, \eta > 0$, F *is* κ*-Lipschitzian and* η*-strongly monotone. Let* $A : H \to H$ *be nonexpansive such that* $\mathrm{Fix}(A) \cap \Omega \neq \emptyset$. *Let* $\mu \in (0, 2\eta/\kappa^2)$, *let* $x_0, u \in H$, $\{c_n\} \subset (0,\infty)$, $\{e_n\} \subset H$ *and* $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ *satisfying the conditions:*
(i) $\sum_{n=0}^{\infty} \lambda_n < \infty$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$;
(iv) $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ *and* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(v) $c_n \geq c$, *where* c *is some positive constant;*
(vi) $\lim_{n\to\infty} |c_{n+1} - c_n| = 0$;
(vii) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.
Then the sequence $\{x_n\}$ *generated by (3.17) converges strongly to* $P_{\mathrm{Fix}(A)\cap\Omega}u$, *i.e., the nearest point projection of* u *onto* $\mathrm{Fix}(A) \cap \Omega$.
Proof. Take $p \in \mathrm{Fix}(A) \cap \Omega$. Noting that each resolvent J_r is nonexpansive for $r > 0$, we have

$$\begin{aligned} \|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n\mu F(Ax_n)) - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|(Ax_n - \lambda_n\mu F(Ax_n)) - p\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[\|A^{\lambda_n}x_n - A^{\lambda_n}p\| + \|A^{\lambda_n}p - p\|] \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n[(1-\lambda_n\tau)\|x_n - p\| + \lambda_n\mu\|F(p)\|] \\ &\leq \|x_n - p\| + \lambda_n\mu\|F(p)\|, \end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n u+\gamma_n x_n+(1-\alpha_n-\gamma_n)J_{c_n}(y_n)+e_n-p\|\\ &\le \alpha_n\|u-p\|+\gamma_n\|x_n-p\|+(1-\alpha_n-\gamma_n)\|J_{c_n}(y_n)-p\|+\|e_n\|\\ &\le \alpha_n\|u-p\|+\gamma_n\|x_n-p\|+(1-\alpha_n-\gamma_n)\|y_n-p\|+\|e_n\|\\ &\le \alpha_n\|u-p\|+\gamma_n\|x_n-p\|+(1-\alpha_n-\gamma_n)[\|x_n-p\|+\lambda_n\mu\|F(p)\|]+\|e_n\|\\ &\le \alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|+\lambda_n\mu\|F(p)\|+\|e_n\|.\end{aligned}$$

By induction we obtain

$$\|x_{n+1}-p\|\le \max\{\|u-p\|,\|x_0-p\|\}+\mu\|F(p)\|\cdot\sum_{i=0}^{n}\lambda_i+\sum_{i=0}^{n}\|e_i\|,\quad \forall n\ge 0.$$

Hence $\{x_n\}$ is bounded.

Next we show that $\omega_w(x_n)\subset \mathrm{Fix}(A)\cap\Omega$. To this end we first prove that $\|x_{n+1}-x_n\|\to 0$. Put $x_{n+1}=\gamma_n x_n+(1-\gamma_n)z_n$. Then we have

$$\begin{aligned}z_{n+1}-z_n &= \tfrac{x_{n+2}-\gamma_{n+1}x_{n+1}}{1-\gamma_{n+1}}-\tfrac{x_{n+1}-\gamma_n x_n}{1-\gamma_n}\\ &= \tfrac{\alpha_{n+1}u+(1-\alpha_{n+1}-\gamma_{n+1})J_{c_{n+1}}(y_{n+1})+e_{n+1}}{1-\gamma_{n+1}}-\tfrac{\alpha_n u+(1-\alpha_n-\gamma_n)J_{c_n}(y_n)+e_n}{1-\gamma_n}\\ &= (\tfrac{\alpha_{n+1}}{1-\gamma_{n+1}}-\tfrac{\alpha_n}{1-\gamma_n})u+\tfrac{1-\alpha_{n+1}-\gamma_{n+1}}{1-\gamma_{n+1}}(J_{c_{n+1}}(y_{n+1})-J_{c_n}(y_n))\\ &\quad +(\tfrac{1-\alpha_{n+1}-\gamma_{n+1}}{1-\gamma_{n+1}}-\tfrac{1-\alpha_n-\gamma_n}{1-\gamma_n})J_{c_n}(y_n)+\tfrac{e_{n+1}}{1-\gamma_{n+1}}-\tfrac{e_n}{1-\gamma_n}.\end{aligned}\tag{3.18}$$

By the same argument as (3.11), we also have

$$\|J_{c_{n+1}}(y_{n+1})-J_{c_n}(y_n)\|\le\|y_{n+1}-y_n\|+\frac{K_1}{c}|c_{n+1}-c_n|,$$

where K_1 is some constant such that

$$\sup\{\|J_{c_{n+1}}(y_{n+1})-y_n\|,\|J_{c_n}(y_n)-y_{n+1}\|,n\ge 0\}\le K_1.$$

Then, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &\le |\tfrac{\alpha_{n+1}}{1-\gamma_{n+1}}-\tfrac{\alpha_n}{1-\gamma_n}|\|u\|+\tfrac{1-\alpha_{n+1}-\gamma_{n+1}}{1-\gamma_{n+1}}\|y_{n+1}-y_n\|+\\ &\quad +\tfrac{1-\alpha_{n+1}-\gamma_{n+1}}{1-\gamma_{n+1}}\tfrac{K_1}{c}|c_{n+1}-c_n|\\ &\quad +|\tfrac{1-\alpha_{n+1}-\gamma_{n+1}}{1-\gamma_{n+1}}-\tfrac{1-\alpha_n-\gamma_n}{1-\gamma_n}|\|J_{c_n}(y_n)\|\\ &\quad +\tfrac{\|e_{n+1}\|}{1-\gamma_{n+1}}+\tfrac{\|e_n\|}{1-\gamma_n},\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|y_{n+1}-y_n\|)\le 0.\tag{3.19}$$

Also, observe that

$$\begin{aligned}\|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_n}x_n\| &\le \|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_{n+1}}x_n\|+\|A^{\lambda_{n+1}}x_n-A^{\lambda_n}x_n\|\\ &\le (1-\lambda_{n+1}\tau)\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\mu\|F(Ax_n)\|,\end{aligned}$$

and hence

$$\begin{aligned}\|y_{n+1}-y_n\| &\le |\beta_n-\beta_{n+1}|\|x_{n+1}\|+(1-\beta_n)\|x_{n+1}-x_n\|\\ &\quad +|\beta_{n+1}-\beta_n|\|A^{\lambda_{n+1}}x_{n+1}\|+\beta_n\|A^{\lambda_{n+1}}x_{n+1}-A^{\lambda_n}x_n\|\end{aligned}$$

$$\begin{aligned}&\leq |\beta_n - \beta_{n+1}|\|x_{n+1}\| + (1-\beta_n)\|x_{n+1} - x_n\| + |\beta_{n+1} - \beta_n|\|A^{\lambda_{n+1}}x_{n+1}\| \\ &\quad +\beta_n[(1-\lambda_{n+1}\tau)\|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\mu\|F(Ax_n)\|] \\ &\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|\mu\|F(Ax_n)\| \\ &\quad +|\beta_n - \beta_{n+1}|(\|x_{n+1}\| + \|A^{\lambda_{n+1}}x_{n+1}\|).\end{aligned}$$

Since A and F are Lipschitzian and $\{x_n\}$ is bounded, we know that both $\{F(Ax_n)\}$ and $\{A^{\lambda_n}x_n\}$ are bounded. Hence from $\lambda_n \to 0$ and $|\beta_{n+1} - \beta_n| \to 0$ it follows that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

This together with (3.19) implies that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0. \tag{3.20}$$

It follows from (3.20) and Lemma 2.5 that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.21}$$

Note that

$$\begin{aligned}\|x_n - J_{c_n}(y_n)\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - J_{c_n}(y_n)\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n\|u - J_{c_n}(y_n)\| + \gamma_n\|x_n - J_{c_n}(y_n)\| + \|e_n\|,\end{aligned}$$

that is,

$$\|x_n - J_{c_n}(y_n)\| \leq \frac{1}{1-\gamma_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\gamma_n}\|u - J_{c_n}(y_n)\| + \frac{1}{1-\gamma_n}\|e_n\|.$$

This together with (ii), (iii), (vii) and (3.21) implies that

$$\lim_{n\to\infty}\|x_n - J_{c_n}(y_n)\| = 0. \tag{3.22}$$

Furthermore, observe that for $p \in \mathrm{Fix}(A) \cap \Omega$

$$\begin{aligned}\|y_n - p\|^2 &= \|(1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n\mu F(Ax_n)) - p\|^2 \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p) - \beta_n\lambda_n\mu F(Ax_n)\|^2 \\ &\leq \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p)\|^2 + 2\|(1-\beta_n)(x_n - p) \\ &\quad +\beta_n(Ax_n - p)\|\|\beta_n\lambda_n\mu F(Ax_n)\| + \|\beta_n\lambda_n\mu F(Ax_n)\|^2 \\ &\leq \|(1-\beta_n)(x_n - p) + \beta_n(Ax_n - p)\|^2 + \lambda_n M\mu\|F(Ax_n)\| \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n\|Ax_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2 \\ &\quad +\lambda_n M\mu\|F(Ax_n)\| \\ &\leq \|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2 + \lambda_n M\mu\|F(Ax_n)\|,\end{aligned} \tag{3.23}$$

where M is a constant such that $M > 2\|x_n - p\| + \mu\|F(Ax_n)\|$. In the meantime, observe also that

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(u-p) + \gamma_n(x_n - p) + (1-\alpha_n-\gamma_n)(J_{c_n}(y_n) - p) + e_n\|^2 \\ &\leq \|\alpha_n(u-p) + \gamma_n(x_n - p) + (1-\alpha_n-\gamma_n)(J_{c_n}(y_n) - p)\|^2 + 2\langle e_n, x_{n+1} - p\rangle \\ &\leq \alpha_n\|u-p\|^2 + \gamma_n\|x_n - p\|^2 + (1-\alpha_n-\gamma_n)\|y_n - p\|^2 + 2\|e_n\|\|x_{n+1} - p\| \\ &\leq \alpha_n\|u-p\|^2 + \gamma_n\|x_n - p\|^2 + (1-\alpha_n-\gamma_n)[\|x_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - Ax_n\|^2\end{aligned}$$

$$
\begin{aligned}
&+\lambda_n M\mu\|F(Ax_n)\|] + 2\|e_n\|\|x_{n+1}-p\| \\
&\le \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 - (1-\alpha_n-\gamma_n)\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\
&\quad +\lambda_n M\mu\|F(Ax_n)\| + 2\|e_n\|\|x_{n+1}-p\| \\
&\le \alpha_n\|u-p\|^2 + \|x_n-p\|^2 - (1-\alpha_n-\gamma_n)\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\
&\quad +M(\lambda_n M + \|e_n\|),
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \le{}& \alpha_n\|u-p\|^2 + \|x_n-p\|^2 - (1-\alpha_n-\gamma_n)\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\
&+M(\lambda_n M + \|e_n\|),
\end{aligned}
$$

and so

$$
\begin{aligned}
&(1-\alpha_n-\gamma_n)\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\
&\le \alpha_n\|u-p\|^2 + \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + M(\lambda_n M + \|e_n\|) \\
&\le \alpha_n\|u-p\|^2 + |\|x_n-p\| - \|x_{n+1}-p\||(\|x_n-p\| + \|x_{n+1}-p\|) + M(\lambda_n M + \|e_n\|) \\
&\le \alpha_n\|u-p\|^2 + \|x_n-x_{n+1}\|(\|x_n-p\| + \|x_{n+1}-p\|) + M(\lambda_n M + \|e_n\|).
\end{aligned}
$$

Since

$$
\begin{aligned}
0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1, \\
0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1,
\end{aligned}
$$

$\alpha_n \to 0,\ \lambda_n \to 0$ and $\|e_n\| \to 0$, we deduce from $\|x_{n+1}-x_n\| \to 0$ that

$$\lim_{n\to\infty}\|x_n - Ax_n\| = 0.$$

This together with (3.17) implies that

$$\|y_n - x_n\| \le \beta_n\|x_n-Ax_n\| + \beta_n\lambda_n\mu\|F(Ax_n)\| \le \|x_n-Ax_n\| + \lambda_n M \to 0.$$

That is,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Thus, we get from (3.22)

$$
\begin{aligned}
\|J_{c_n}(x_n) - x_n\| &\le \|J_{c_n}(x_n) - J_{c_n}(y_n)\| + \|J_{c_n}(y_n) - x_n\| \\
&\le \|x_n - y_n\| + \|J_{c_n}(y_n) - x_n\| \to 0.
\end{aligned}
$$

That is,

$$\lim_{n\to\infty}\|J_{c_n}(x_n) - x_n\| = 0. \tag{3.24}$$

It follows from Lemma 2.4 and (3.24) that

$$\lim_{n\to\infty}\|J_c(x_n) - x_n\| = 0.$$

By the same argument as that in the proof of Theorem 3.1 we obtain $\{x_n\}$ converges weakly to a point $\hat{x} \in \mathrm{Fix}(A) \cap \Omega$.

Now let $x^* = P_{\mathrm{Fix}(A)\cap\Omega}u$. Since $\hat{x} \in \mathrm{Fix}(A)\cap\Omega$ and $x_n \rightharpoonup \hat{x}$, we get that

$$\limsup_{n\to\infty}\langle u - x^*, x_n - x^*\rangle = \langle u - P_{\mathrm{Fix}(A)\cap\Omega}u, \hat{x} - P_{\mathrm{Fix}(A)\cap\Omega}u\rangle \le 0. \tag{3.25}$$

Finally, applying Lemma 2.8 (ii), we have from (3.23) (take $p = x^*$)

$$\|x_{n+1}-x^*\|^2 = \|\alpha_n(u-x^*) + \gamma_n(x_n-x^*) + (1-\alpha_n-\gamma_n)(J_{c_n}(y_n)-x^*) + e_n\|^2$$

$$
\begin{aligned}
&\le \|\alpha_n(u-x^*)+\gamma_n(x_n-x^*)+(1-\alpha_n-\gamma_n)(J_{c_n}(y_n)-x^*)\|^2+2\langle e_n, x_{n+1}-x^*\rangle \\
&\le \|\alpha_n(u-x^*)+\gamma_n(x_n-x^*)+(1-\alpha_n-\gamma_n)(J_{c_n}(y_n)-x^*)\|^2+2\|x_{n+1}-x^*\|\|e_n\| \\
&\le \|\gamma_n(x_n-x^*)+(1-\alpha_n-\gamma_n)(J_{c_n}(y_n)-x^*)\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*-e_n\rangle \\
&\quad +2\|x_{n+1}-x^*\|\|e_n\| \\
&\le \gamma_n\|x_n-x^*\|^2+(1-\alpha_n-\gamma_n)\|y_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle \\
&\quad +(2\alpha_n\|u-x^*\|+2\|x_{n+1}-x^*\|)\|e_n\| \\
&\le \gamma_n\|x_n-x^*\|^2+(1-\alpha_n-\gamma_n)[\|x_n-x^*\|^2-\beta_n(1-\beta_n)\|x_n-Ax_n\|^2 \\
&\quad +\lambda_n M\mu\|F(Ax_n)\|]+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle+(2\alpha_n\|u-x^*\| \\
&\quad +2\|x_{n+1}-x^*\|)\|e_n\| \\
&\le (1-\alpha_n)\|x_n-x^*\|^2+\lambda_n M\mu\|F(Ax_n)\|+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle \\
&\quad +(2\alpha_n\|u-x^*\|+2\|x_{n+1}-x^*\|)\|e_n\| \\
&\le (1-\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle u-x^*, x_{n+1}-x^*\rangle \\
&\quad +\lambda_n M^2+(2\|u-x^*\|+M)\|e_n\|, \qquad (3.26)
\end{aligned}
$$

where the constant $M > 2\|x_n - x^*\| + \mu\|F(Ax_n)\|$ for all $n \ge 0$. Put $t_n = 2\langle u - x^*, x_{n+1} - x^*\rangle$ and $\delta_n = \lambda_n M^2 + (2\|u - x^*\| + M)\|e_n\|$ for all $n \ge 0$. Then the inequality (3.26) is rewritten as

$$
\|x_{n+1}-x^*\|^2 \le (1-\alpha_n)\|x_n-x^*\|^2+\alpha_n t_n+\delta_n, \quad \forall n \ge 0. \qquad (3.27)
$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$ (due to (ii)), $\limsup_{n\to\infty} t_n \le 0$ (due to (3.25)) and $\sum_{n=0}^{\infty}\delta_n < \infty$ (due to (i), (vii)), it follows from Lemma 2.6 and (3.27) that $\{x_n\}$ converges strongly to $x^* = P_{\mathrm{Fix}(A)\cap\Omega}u$. The proof is therefore complete. □

Corollary 3.3. *Let $F : H \to H$ be a mapping such that for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone. Let $A : H \to H$ be nonexpansive such that* $\mathrm{Fix}(A) \cap \Omega \ne \emptyset$. *Let $\mu \in (0, 2\eta/\kappa^2)$, let $x_0, u \in H$, $c > 0$, $\{e_n\} \subset H$ and $\{\lambda_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [0,1]$ satisfying the conditions:*

(i) $\sum_{n=0}^{\infty}\lambda_n < \infty$;
(ii) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$;
(iv) $\lim_{n\to\infty}|\beta_{n+1}-\beta_n| = 0$ *and* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$;
(v) $\sum_{n=0}^{\infty}\|e_n\| < \infty$.

Then the sequence $\{x_n\}$ generated by the algorithm

$$
\begin{cases} x_{n+1} = \alpha_n u + \gamma_n x_n + (1-\alpha_n-\gamma_n)J_c(y_n) + e_n, \\ y_n = (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n\mu F(Ax_n)), \quad \forall n \ge 0, \end{cases}
$$

converges strongly to $P_{\mathrm{Fix}(A)\cap\Omega}u$, i.e., the nearest point projection of u onto $\mathrm{Fix}(A) \cap \Omega$.

Corollary 3.4 ([19, Theorem 3.3]). *Let $\{x_n\}$ be generated by the contraction proximal point algorithm (3.16). Assume that*

(i) $\lim_{n\to\infty}\alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$;
(iv) $c_n \ge c$, where c is some positive constant;
(v) $c_{n+1} - c_n \to 0$;

(vi) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ *converges strongly to* $P_\Omega u$*, i.e., the nearest point projection of* u *onto* Ω.

Proof. In Theorem 3.3, put $A = I$ the identity mapping of H, and $\lambda_n = 0$ for all $n \geq 0$. Then we have

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n(Ax_n - \lambda_n \mu F(Ax_n)) \\ &= (1-\beta_n)x_n + \beta_n x_n = x_n. \end{aligned}$$

Hence algorithm (3.17) reduces to (3.16) with $\delta_n = 1 - \alpha_n - \gamma_n$. Therefore, from Theorem 3.3 we immediately obtain the desired result. □

Corollary 3.5. ([19, Corollary 3.1]). *Let* $\{x_n\}$ *be generated by the algorithm*

$$x_{n+1} = \alpha_n u + \gamma_n x_n + \delta_n J_c(x_n) + e_n, \quad \forall n \geq 0,$$

where $c > 0$ *is a constant. Assume that*

(i) $\lim_{n\to\infty} \alpha_n = 0$*;*

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$*;*

(iv) $\sum_{n=0}^{\infty} \|e_n\| < \infty$.

Then $\{x_n\}$ *converges strongly to* $P_\Omega u$*, i.e., the nearest point projection of* u *onto* Ω.

Remark 3.1. Let $\phi : H \to R \cup \{\infty\}$ be a proper lower-semicontinuous convex function. Let $\partial\phi$ be the subdifferential of ϕ; that is,

$$\partial\phi(x) = \{z \in H : \phi(y) \geq \phi(x) + \langle y - x, z\rangle,\ \forall y \in H\}, \quad \forall x \in \mathrm{dom}(\partial\phi).$$

It is well-known [1] that $\partial\phi$ is a maximal monotone operator on H. The inclusion $0 \in \partial\phi(x)$ is equivalent to $x = \mathrm{argmin}_{v\in H}\phi(v)$. Let $T = \partial\phi$. Assume that the set Ω of minimizers of ϕ over H is nonempty. If H is infinite dimensional, then Rockafellar's proximal point algorithm only has weak convergence, in general. However, our hybrid contraction proximal point algorithm (3.17) in the sense of Marino and Xu, always has strong convergence. In the meantime, there is no doubt that our problem of finding an element of $\mathrm{Fix}(A) \cap \Omega$ is more general than the one of finding a point of Ω.

Remark 3.2. (1) It is clear that $\sum_{n=1}^{\infty} |c_{n+1} - c_n| < \infty$ implies $c_{n+1} - c_n \to 0$; (2) We remove the assumption in that $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ or $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$ (see [20,22]), but we add the very weak restrictions that $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

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