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SOME ASPECTS ON EIGENVALUES AND SURJECTIVITY USING FIXED POINTS

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Abstract. In this paper, we will first prove the existence of fixed points for a weakly continuous, strictly quasi bounded operator on a reflexive Banach space and a completely continuous, strictly quasi bounded operator on any normed linear space. Using these results we can deduce the existence of eigenvalues and surjectivity of quasi bounded operator in similar situations.
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1. Introduction

In [2] G. Isac and S.Z. Nemeth have proved some interesting results on fixed points, eigenvalues and surjectivity. They proved their results for a non expansive mapping defined on a reflexive Banach space. The result was proved by applying certain conditions on the mapping and by using the help of Banach contraction principle. In [4] In Sook Kim have proved that a countably condensing operator on a closed wedge in a Banach space has a fixed point if it is strictly quasibounded, by using index theory for such operators and from this he had deduced the eigenvalues and surjectivity. Also in [1] we can find a result on surjectivity of a continuous, strictly quasi bounded function which maps each bounded subset of a normed space X into a compact subset of X. All the authors have used different methods to prove their results. Here we will prove that any strictly quasi bounded, weakly continuous operator on a reflexive Banach space has a fixed point. Also we will prove that a strictly quasi bounded, completely continuous operator on any normed linear space has a fixed point. For proving this we will use the well known Leray-Schauder alternative. Using these fixed point theorems we will prove the existence of eigenvalues and surjectivity of some quasi bounded mappings. Before proving these results let us recall some important definitions and theorems.

Definition 1.1. Let X and Y be normed spaces and $F : X \to Y$. F is weakly continuous at $x_0 \in X$ if, for any sequence $\{x_n\}$ which converges weakly to x_0,the sequence $\{Fx_n\}$ converges weakly to Fx_0.

Definition 1.2. Let X and Y be metric spaces and $F : X \to Y$. F is completely continuous if the image of each bounded set in X is contained in a compact subset of Y.

Definition 1.3. Let X be a normed space and $F : X \to X$. F is quasi bounded if $\limsup\limits_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} < \infty$ and it is strictly quasi bounded if $\limsup\limits_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} < 1$.

Theorem 1.4. ([8]) *Let X be a reflexive Banach space, K a closed convex subset of X and F a weakly continuous mapping of K into a bounded subset of K. Then F has a fixed point in K.*

Theorem 1.5. (Leray-Schauder alternative) ([3]) *Let X be a normed linear space, $C \subset E$ be a convex set and let U be open in C such that $0 \in U$. Then each compact map $F : \overline{U} \to C$ has at least one of the following two properties*

(1) *F has a fixed point*
(2) *There exists $x \in \partial U$ and $\lambda \in (0, 1)$ such that $x = \lambda\ Fx$ where ∂U denote boundary of U.*

Theorem 1.6. ([1]) *A normed space X is reflexive iff every bounded sequence in X has a weak convergent subsequence.*

Theorem 1.7. ([1]) *If a sequence $\{x_n\}$ in a normed space X is weak convergent then it is bounded.*

2. Main results

First let us prove a fixed point theorem for a strictly quasi bounded operator on a reflexive Banach space.

Theorem 2.1. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and strictly quasi bounded operator on X. Then F has a fixed point in X.*

Proof. For $n = 1, 2, 3, ...$ define $S_n = \{x \in X : \|x\| \le n\}$. We will prove that $F(S_n) \subseteq S_n$ for some n.

By contradiction, assume that $F(S_n)$ is not a subset of S_n, for all n. Then for each $n = 1, 2, 3, ...$ there exists $x_n \in S_n$ such that $\|Fx_n\| > n$ $(*)$.

Now if $\{x_n\}$ is a bounded sequence in X, since X is reflexive by Theorem 1.6, it has a weak convergent subsequence $\{x_{n_i}\}$. Then as F is weakly continuous $\{Fx_{n_i}\}$ is a weakly convergent sequence in X. Therefore, by Theorem 1.7, $\{Fx_{n_i}\}$ is a bounded sequence in X, which is a contradiction to $(*)$. Hence $\|x_n\| \to \infty$ as $n \to \infty$.
We have $\|Fx_n\| > n \ge \|x_n\|$, for all n. Therefore $\limsup\limits_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} \ge \limsup\limits_{\|x_n\|\to\infty} \frac{\|Fx_n\|}{\|x_n\|} \ge 1$ which is a contradiction with the fact that F is strictly quasi bounded. Hence $F(S_n) \subseteq S_n$ for some n.

Then, using Theorem 1.4, we have that F has a fixed point in S_n and hence in X. □

Corollary 2.2. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and quasi bounded operator on X. Let $l := \limsup\limits_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$. Then for each $\lambda > l$, λ is an eigenvalue of F provided $F(0) \ne 0$.*

Proof. For $x \in X$, define $G(x) = \frac{1}{\lambda} F(x)$ where $\lambda > l$.

Let $m = \limsup_{\|x\|\to\infty} \frac{\|Gx\|}{\|x\|}$. Then $m = \frac{l}{\lambda} < 1$. Hence G is a strictly quasi bounded, weakly continuous function from $X$to X. Then by Theorem 2.1 G has a fixed point $x_0 \in X$. Further as $F(0) \neq 0$ we have $G(0) \neq 0$. Hence $x_0 \neq 0$, i.e., there exists a non zero element $x_0 \in X$ such that $Fx_0 = \lambda x_0$. Therefore λ is an eigenvalue of F. □

Corollary 2.3. *Let X be reflexive Banach space. $F : X \to X$ be a weakly continuous and quasi bounded operator on X. Then $I - \frac{1}{\lambda}F$ is surjective for all $\lambda > l$ where* $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$.

Proof. Let $y \in X$. Define $G(x) = y + \frac{1}{\lambda}F(x)$.

Let $m = \limsup_{\|x\|\to\infty} \frac{\|Gx\|}{\|x\|}$. Then $m \leq \frac{l}{\lambda} < 1$. Hence G is a strictly quasi bounded, weakly continuous function from X to X and hence by Theorem 2.1 G has a fixed point $x_0 \in X$. Then $y + \frac{1}{\lambda}Fx_0 = x_0$. ie., $\left(I - \frac{1}{\lambda}F\right)x_0 = y$. Therefore $I - \frac{1}{\lambda}F$ is surjective. □

Before proving the next theorem let us discuss a very simple example which verifies our arguments.

Example 2.4. Define $F : R \to R$ by $Fx = \frac{2}{2x^2+1}$.

Then one can easily see that F is not a contraction, even it is not non expansive. But it is strictly quasi bounded and weakly continuous operator on R. (Remember that since R is finite dimensional both weak convergence and strong convergence coincides in R). Clearly F has a fixed point which lies in $(0, 1)$.

Further $\limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} = 0$ and any real number $\lambda > 0$ is an eigenvalue of F. □

Theorem 2.5. *Let X be normed linear space, $F : X \to X$ be a completely continuous and strictly quasi bounded operator on X. Then F has a fixed point in X.*

Proof. Since F is strictly quasi bounded, $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|} < 1$. Therefore there exists $r > 0$ such that $\frac{\|Fx\|}{\|x\|} < 1, \quad \forall x \in X$ with $\|x\| \geq r$.

If we suppose contrary, there exists $x \in X$ with $\|x\| = r$ and $\lambda \in (0, 1)$ such that $x = \lambda Fx$. Then $\|x\| = |\lambda| \|Fx\| < \|Fx\|$ which is a contradiction.

Hence by Leray-Schauder alternative of F on $\{x : \|x\| \leq r\}$ (Theorem 1.5) F has a fixed point. □

Corollary 2.6. *Let X be normed linear space, $F : X \to X$ be a completely continuous and quasi bounded operator on X. Let $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$. Then for each $\lambda > l$, λ is an eigenvalue of F provided $F(0) \neq 0$.*

Corollary 2.7. *Let X be normed linear space, $F : X \to X$ be a completely continuous and quasi bounded operator on X. Then for all $\lambda > l$, $I - \frac{1}{\lambda}F$ is surjective, where* $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x\|}$.

Theorem 2.8. *Let X be normed linear space, $F : X \to X$ be a completely continuous operator such that $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x-Fx\|} < 1$. Then F has a fixed point.*

Proof. Since $l = \limsup_{\|x\|\to\infty} \frac{\|Fx\|}{\|x-Fx\|} < 1$, there exists $r > 0$ such that $\frac{\|Fx\|}{\|x-Fx\|} < 1, \quad \forall x \in X$ with $\|x\| \geq r$.

Suppose there exists $x \in X$ with $\|x\| = r$ and $\lambda \in (0,1)$ such that $x = \lambda Fx$. Then, $\|x - Fx\| = \|\lambda Fx - Fx\| = (1-\lambda)\|Fx\| < \|Fx\|$, which is a contradiction since $x \in X$ and $\|x\| = r$.

Hence by Leray-Schauder alternative of F on $\{x : \|x\| \leq r\}$ (Theorem 1.5) F has a fixed point. □

Theorem 2.9. *Let X be normed linear space, $F : X \to X$ be a completely continuous operator such that $l = \limsup_{\|x\|\to\infty} \frac{B(Fx,x)}{B(x,x)} < 1$, where $B : X \times X \to R$ is a map which satisfies the following conditions*

(1) $B(\lambda x, y) = \lambda B(x,y)$ *for all* $\lambda > 0$ *and for all* $x, y \in E$.
(2) $B(x,x) > 0$ *for all* $x \in E$ *with* $x \neq 0$.

Then F has a fixed point in X.

Proof. Since $l < 1$, $\exists r > 0$ with $\frac{B(Fx,x)}{B(x,x)} < 1$, $\forall x \in X$ with $\|x\| \geq r$.

If we suppose by contradiction, there exists $x \in X$ with $\|x\| = r$ and $\lambda \in (0,1)$ such that $x = \lambda Fx$. Then,

$$B(Fx,x) = B\left(\tfrac{1}{\lambda}x, x\right) = \tfrac{1}{\lambda} B(x,x) > B(x,x)$$

which is a contradiction. Hence by Leray-Schauder alternative of F on $\{x : \|x\| \leq r\}$, F has a fixed point in X. □

Corollary 2.10. *Let X be normed linear space, $F : X \to X$ be a completely continuous operator such that $l = \limsup_{\|x\|\to\infty} \frac{B(Fx,x)}{B(x,x)} < \infty$ and $F(0) \neq 0$. Then if $\lambda > l$, λ is an eigenvalue of F.*

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