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EXISTENCE THEORY FOR IMPULSIVE PARTIAL HYPERBOLIC DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER AT VARIABLE TIMES

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Abstract. In this paper, we investigate the existence and uniqueness of solutions of a class of partial hyperbolic differential equations with impulses at variable times involving the Caputo fractional derivative. Our results are based on suitable fixed point theorems.
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1. Introduction

In this paper, we shall concern with the existence and uniqueness of solutions for the following impulsive partial hyperbolic fractional order differential equations at variable times:

$$({}^cD_0^r u)(x,y) = f(x,y,u(x,y)), \quad \text{if } (x,y) \in J, \quad x \neq x_k(u(x,y)),\ k = 1,\dots,m, \tag{1.1}$$

$$u(x^+,y) = I_k(u(x,y)), \quad \text{if } (x,y) \in J,\ x = x_k(u(x,y)),\ k = 1,\dots,m, \tag{1.2}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ x \in [0,a],\ y \in [0,b] \tag{1.3}$$

where $J = [0,a] \times [0,b]$, $a,b > 0$, ${}^cD_0^r$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = x_0 < x_1 < \cdots < x_m < x_{m+1} = a$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1,\dots,m$ are given functions and $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are absolutely continuous functions with $\varphi(0) = \psi(0)$.

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions was studied in numerous works (see [34, 48]), a similar problem in spaces of continuous functions was studied in [49]. We can find numerous applications of differential equations of

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fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [26, 28, 41, 42]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [36], Lakshmikantham *et al.* [38], Podlubny [45], Samko *et al.* [46], the papers of Abbas and Benchohra [1, 2], Agarwal *et al.* [3], Belarbi *et al.* [9], Benchohra *et al.* [12, 13, 14, 15], Diethelm [22], Kilbas and Marzan [35], Mainardi [41], Vityuk and Golushkov [50], Zhang [51] and the references therein.

The theory of impulsive differential equations have become important in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. There has been a significant development in impulse theory in recent years, especially in the area of impulsive differential equations and inclusions with fixed moments; see the monographs of Bainov and Simeonov [7], Benchohra *et al.* [14], Lakshmikantham *et al.* [37], Samoilenko and Perestyuk [47], and the references therein. The theory of impulsive differential equations and inclusions with variable time is relatively less developed due to the difficulties created by the state-dependent impulses. Some interesting extensions to impulsive differential equations with variable times have been done by Bajo and Liz [6], Belarbi and Benchohra [8], Benchohra *et al.* [10, 14, 16], Frigon and O'Regan [23, 24, 25], Kaul *et al.* [29], Kaul and Liu [32, 33], Lakshmikantham *et al.* [39], and the references cited therein.

Very recently, some extensions to impulsive fractional order differential equations have been obtained by Agarwal *et al.* [4], Ait Dads *et al.* [5], Benchohra and Slimani [17], and Mophou [43].

In this paper, we present two results for the problem (1.1)-(1.3). The first one is based on Schaefer's fixed point (Theorem 3.5) and the second one on Banach's contraction principle (Theorem 3.6) As an extension to nonlocal problems, we present two similar results for the problem (4.1)-(4.3). The present results initiate the study of impulsive hyperbolic differential equations with fractional order and variable times. In particular our results extend those with integer order derivative [10, 11, 18, 19, 20, 21, 30, 31, 40, 44] and those with fractional derivative and without impulses [1, 2, 35].

2. Preliminaries

In this section, we introduce notations and definitions which are used throughout this paper. By $L^1(J,\mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $f : J \to \mathbb{R}^n$ with the norm

$$\|f\|_1 = \int_0^a \int_0^b \|f(x,y)\| dydx,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.

To define the solutions of problems (1.1)-(1.3), we shall consider the space

$$\begin{aligned}\Omega = \big\{ & u : J \to \mathbb{R}^n : \text{ there exist } 0 = x_0 < x_1 < x_2 < ... < x_m < x_{m+1} = a \\ & \text{such that } x_k = x_k(u(x_k,.)), \text{ and } u(x_k^-,.),\ u(x_k^+,.) \text{ exist with } u(x_k^-,.) = u(x_k,.); \\ & k = 1,\dots,m, \text{ and } u \in C(J_k,\mathbb{R}^n); k = 0,\dots,m \big\},\end{aligned}$$

where $J_k := (x_k, x_{k+1}] \times [0, b]$. This set is a Banach space with the norm

$$\|u\|_\Omega = \max\{\|u_k\|,\ k = 0, ..., m\},$$

where u_k is the restriction of u to $J_k,\ k = 0, ..., m$.

Let $a_1 \in [0, a],\ z^+ = (a_1^+, 0) \in J,\ J_z = [a_1, a] \times [0, b],\ r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J_z, \mathbb{R}^n)$, the expression

$$(I^r_{z^+} f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{a_1^+}^{x} \int_0^y (x - s)^{r_1 - 1}(y - t)^{r_2 - 1} f(s, t) dt ds,$$

where $\Gamma(.)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Definition 2.1. *([50]). For $f \in L^1(J_z, \mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression $({}^cD^r_{z^+} f)(x, y) = (I^{1-r}_{z^+} \frac{\partial^2}{\partial x \partial y} f)(x, y)$.*

3. Existence of solutions

Let us define what we mean by a solution of problem (1.1)-(1.3). Set

$$J' := J \backslash \{(x_1, y), \ldots , (x_m, y),\ y \in [0, b]\}.$$

Definition 3.1. A function $u \in \Omega$ whose r-derivative exists on J' is said to be a solution of (1.1)-(1.3) if u satisfies $({}^cD^r_0 u)(x, y) = f(x, y, u(x, y))$ on J' and conditions (1.2) and (1.3) are satisfied.

Let $h \in C([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n),\ z_k = (x_k, 0)$,

$$\mu_k(x, y) = u(x, 0) + u(x^+_k, y) - u(x^+_k, 0), \quad k = 0, \ldots , m,$$

and $D^2_{xy} := \frac{\partial^2}{\partial x \partial y}$ denotes the mixed second order partial derivative.

For the existence of solutions for the problem $(1.1) - (1.3)$, we need the following lemma:

Lemma 3.2. *A function $u \in AC([x_k, x_{k+1}] \times [0, b], \mathbb{R}^n);\ k = 0, \ldots , m$ is a solution of the differential equation*

$$({}^cD^r_{z_k} u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b],$$

if and only if $u(x, y)$ satisfies

$$u(x, y) = \mu_k(x, y) + (I^r_{z_k} h)(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b]. \tag{3.1}$$

Proof. Let $u(x, y)$ be a solution of $({}^cD^r_{z_k} u)(x, y) = h(x, y);\ (x, y) \in [x_k, x_{k+1}] \times [0, b]$. Then, taking into account the definition of the derivative $({}^cD^r_{z_k^+} u)(x, y)$, we have

$$I^{1-r}_{z_k^+}(D^2_{xy} u)(x, y) = h(x, y).$$

Hence, we obtain

$$I^r_{z_k^+}(I^{1-r}_{z_k} D^2_{xy} u)(x, y) = (I^r_{z_k^+} h)(x, y),$$

then

$$I^1_{z_k^+} D^2_{xy} u(x, y) = (I^r_{z_k^+} h)(x, y).$$

Since

$$I^1_{z_k^+}(D^2_{xy}u)(x,y) = u(x,y) - u(x,0) - u(x_k^+,y) + u(x_k^+,0),$$

we have

$$u(x,y) = \mu_k(x,y) + (I^r_{z_k^+}h)(x,y).$$

Now let $u(x,y)$ satisfies (3.1). It is clear that $u(x,y)$ satisfy

$$({}^cD^r_0 u)(x,y) = h(x,y), \text{ on } [x_k,x_{k+1}]\times[0,b].$$

Lemma 3.3. *Let $0 < r_1, r_2 \leq 1$ and let $h : J \to \mathbb{R}^n$ be continuous, denote $\mu(x,y) := \mu_0(x,y)$; $(x,y) \in J$. A function u is a solution of the fractional integral equation*

$$u(x,y) = \begin{cases} \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \ \textit{if } (x,y)\in[0,x_1]\times[0,b], \\ \varphi(x) + I_k(u(x_k,y)) - I_k(u(x_k,0)) \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_k}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds; \\ \ \textit{if } (x,y)\in(x_k,x_{k+1}]\times[0,b],\ k=1,\dots,m, \end{cases} \tag{3.2}$$

if and only if u is a solution of the fractional IVP

$$^cD^r u(x,y) = h(x,y), \quad (x,y)\in J', \tag{3.3}$$

$$u(x_k^+,y) = I_k(u(x_k,y)), \quad k=1,\dots,m. \tag{3.4}$$

Proof. Assume u satisfies (3.3)-(3.4). If $(x,y)\in[0,x_1]\times[0,b]$ then

$$^cD^r u(x,y) = h(x,y).$$

Lemma 3.2 implies

$$u(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds.$$

If $(x,y)\in(x_1,x_2]\times[0,b]$ then Lemma 3.2 implies

$$\begin{aligned} u(x,y) &= \mu_1(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds \\ &= \varphi(x) + u(x_1^+,y) - u(x_1^+,0) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds \\ &= \varphi(x) + I_1(u(x_1,y)) - I_1(u(x_1,0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds. \end{aligned}$$

If $(x,y)\in(x_2,x_3]\times[0,b]$ then from Lemma 3.2 we get

$$\begin{aligned}u(x,y)&=\mu_2(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds\\&=\varphi(x)+u(x_2^+,y)-u(x_2^+,0)\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds\\&=\varphi(x)+I_2(u(x_2,y))-I_2(u(x_2,0))\\&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_2}^{x}\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}h(s,t)dtds.\end{aligned}$$

If $(x,y)\in(x_k,x_{k+1}]\times[0,b]$ then again from Lemma 3.2 we get (3.2).

Conversely, assume that u satisfies the impulsive fractional integral equation (3.2). If $(x,y)\in[0,x_1]\times[0,b]$ and using the fact that ${}^cD^r$ is the left inverse of I^r we get

$${}^cD^r u(x,y)=h(x,y),\quad \text{for each } (x,y)\in[0,x_1]\times[0,b].$$

If $(x,y)\in[x_k,x_{k+1})\times[0,b]$, $k=1,\dots,m$ and using the fact that ${}^cD^rC=0$, where C is a constant, we get

$${}^cD^r u(x,y)=h(x,y),\text{for each } (x,y)\in[x_k,x_{k+1})\times[0,b].$$

Also, we can easily show that

$$u(x_k^+,y)=I_k(u(x_k,y)),\quad y\in[0,b],k=1,\dots,m.$$

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 3.4. *([27]) Let $\upsilon,\omega:J\to[0,\infty)$ be nonnegative, locally integrable functions on J. If there are constants $c>0$ and $0<r_1,r_2<1$ such that*

$$\upsilon(x,y)\le\omega(x,y)+c\int_0^x\int_0^y\frac{\upsilon(s,t)}{(x-s)^{1-r_1}(y-t)^{1-r_2}}dtds,$$

then, for every $(x,y)\in J$,

$$\upsilon(x,y)\le\omega(x,y)+\int_0^x\int_0^y\sum_{j=1}^{\infty}\frac{(c\Gamma(r_1)\Gamma(r_2))^j}{\Gamma(jr_1)\Gamma(jr_2)}\frac{\omega(s,t)}{(x-s)^{1-jr_1}(y-t)^{1-jr_2}}dtds.\quad(3.5)$$

If $\omega(x,y)=\omega$ constant on J, then the inequality (3.5) is reduced to

$$\upsilon(x,y)\le\omega E_{(r_1,r_2)}(c\Gamma(r_1)\Gamma(r_2)x^{r_1}y^{r_2}),$$

where $E_{(r_1,r_2)}$ is the Mittag-Leffler function [36]*, defined by*

$$E_{(r_1,r_2)}(z):=\sum_{k=1}^{\infty}\frac{z^k}{\Gamma(kr_1+1)\Gamma(kr_2+1)};\quad r_j,z\in\mathbb{C},\ \Re e(r_j)>0;\ j=1,2.$$

We are now in a position to state and prove our existence result for our problem based on Schaefer's fixed point theorem.

Theorem 3.5. *Assume that*

(H1) *The function* $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous.*

(H2) *There exists a constant* $M > 0$ *such that*

$$\|f(x,y,u)\| \leq M(1+\|u\|), \text{ for each } (x,y) \in J, \text{ and each } u \in \mathbb{R}^n.$$

(H3) *The function* $x_k \in C^1(\mathbb{R}^n, \mathbb{R})$ *for* $k = 1, \ldots, m$. *Moreover,*

$$0 = x_0(u) < x_1(u) < \ldots < x_m(u) < x_{m+1}(u) = a, \quad \text{for all } u \in \mathbb{R}^n.$$

(H4) *There exists a constant* $M^* > 0$ *such that*

$$\|I_k(u)\| \leq M^*(1+\|u\|), \text{ for each } u \in \mathbb{R}^n, \ k = 1, \ldots, m.$$

(H5) *For all* $u \in \mathbb{R}^n$, $x_k(I_k(u)) \leq x_k(u) < x_{k+1}(I_k(u))$ *for* $k = 1, \ldots, m$.

(H6) *For all* $(s,t,u) \in J \times \mathbb{R}^n$, *we have*

$$x_k'(u)[\varphi'(s) + \frac{r_1 - 1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_k}^{s} \int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1} f(\theta, \eta, u(\theta,\eta)) d\eta d\theta] \neq 1,$$

$k = 1, \ldots, m$. *Then* (1.1)-(1.3) *has at least one solution on* J.

Proof. The proof will be given in several steps.
Step 1. Consider the following problem

$$({}^cD_0^r u)(x,y) = f(x,y,u(x,y)), \text{ if } (x,y) \in J, \tag{3.6}$$

$$u(x,0) = \varphi(x), \ u(0,y) = \psi(y), \ x \in [0,a], \ y \in [0,b]. \tag{3.7}$$

Transform problem (3.6)-(3.7) into a fixed point problem. Consider the operator $N : C(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n)$ defined by

$$N(u)(x,y) = \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u(s,t)) dt ds.$$

Lemma 3.2 implies that the fixed points of operator N are solutions of problem (3.6)-(3.7) . We shall show that the operator N is continuous and completely continuous.
Claim 1. *N is continuous.*

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, \mathbb{R}^n)$. Let $\eta > 0$ be such that $\|u_n\| \leq \eta$. Then for each $(x,y) \in J$, we have

$$\begin{aligned}
&\|N(u_n)(x,y) - N(u)(x,y)\| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t,u_n(s,t)) - f(s,t,u(s,t))\| dt ds \\
&\leq \frac{\|f(.,.,u_n(.,.)) - f(.,.,u(.,.))\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b (x-s)^{r_1-1}(y-t)^{r_2-1} dt ds \\
&\leq \frac{a^{r_1} b^{r_2} \|f(.,.,u_n(.,.)) - f(.,.,u(.,.))\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)}.
\end{aligned}$$

Since f is a continuous function, we have

$$\|N(u_n) - N(u)\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Claim 2. *N maps bounded sets into bounded sets in $C(J,\mathbb{R}^n)$.*

Indeed, it is enough to show that for any $\eta^* > 0$, there exists a positive constant ℓ such that for each $u \in B_{\eta^*} = \{u \in C(J,\mathbb{R}^n) : \|u\|_\infty \leq \eta^*\}$, we have $\|N(u)\|_\infty \leq \ell$. By (H2) we have for each $(x,y) \in J$, we have

$$\begin{aligned}\|N(u)(x,y)\| &\leq \|\mu(x,u)\| \\ &\quad + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))\|dtds \\ &\leq \|\mu(x,u)\| + \frac{M(1+\eta^*)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.\end{aligned}$$

Thus

$$\|N(u)\|_\infty \leq \|\mu\|_\infty + \frac{M(1+\eta^*)a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \ell.$$

Claim 3. *N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R}^n)$.*

Let $(\tau_1,y_1),(\tau_2,y_2) \in J$, $\tau_1 < \tau_2$ and $y_1 < y_2$, B_{η^*} be a bounded set of $C(J,\mathbb{R})$ as in Claim 2, and let $u \in B_{\eta^*}$. Then for each $(x,y) \in J$, we have

$$\begin{aligned}&\|N(u)(\tau_2,y_2) - N(u)(\tau_1,y_1)\| = \Big\|\mu(\tau_1,y_1) - \mu(\tau_2,y_2) \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (\tau_1-s)^{r_1-1} \\ &\qquad\times(y_1-t)^{r_2-1}]f(s,t,u(s,t))dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}f(s,t,u(s,t))dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}f(s,t,u(s,t))dtds \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}f(s,t,u(s,t))dtds\Big\| \\ &\leq \|\mu(\tau_1,y_1) - \mu(\tau_2,y_2)\| \\ &+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_0^{y_1}[(\tau_1-s)^{r_1-1}(y_1-t)^{r_2-1} - (\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}]dtds \\ &+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\tau_1}\int_{y_1}^{y_2}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &+\frac{M(1+\eta^*)}{\Gamma(r_1)\Gamma(r_2)}\int_{\tau_1}^{\tau_2}\int_0^{y_1}(\tau_2-s)^{r_1-1}(y_2-t)^{r_2-1}dtds \\ &\leq \|\mu(\tau_1,y_1) - \mu(\tau_2,y_2)\| \\ &+\frac{M(1+\eta^*)}{\Gamma(r_1+1)\Gamma(r_2+1)}[2y_2^{r_2}(\tau_2-\tau_1)^{r_1} + 2\tau_2^{r_1}(y_2-y_1)^{r_2} \\ &\qquad+\tau_1^{r_1}y_1^{r_2} - \tau_2^{r_1}y_2^{r_2} - 2(\tau_2-\tau_1)^{r_1}(y_2-y_1)^{r_2}].\end{aligned}$$

As $\tau_1 \longrightarrow \tau_2$ and $y_1 \longrightarrow y_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : C(J,\mathbb{R}^n) \to C(J,\mathbb{R}^n)$ is completely continuous.

Claim 4. *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \{u \in C(J,\mathbb{R}^n) : u = \lambda N(u) \ \text{ for some } \ 0 < \lambda < 1\}$$

is bounded. Let $u \in \mathcal{E}$, then $u = \lambda N(u)$ for some $0 < \lambda < 1$. Thus, for each $(x, y) \in J$, we have

$$\begin{aligned}
\|u(x,y)\| &\leq \|\mu(x,y)\| \\
&+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))\|dtds \\
&\leq \|\mu\|_\infty + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\
&+ \frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)\|dtds.
\end{aligned}$$

Set

$$\omega = \|\mu\|_\infty + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}, \ c = \frac{M}{\Gamma(r_1)\Gamma(r_2)}.$$

Then Lemma 3.4 implies that for each $(x, y) \in J$,

$$\|u(x,y)\| \leq \omega E_{(r_1,r_2)}(c\Gamma(r_1)\Gamma(r_2)a^{r_1}b^{r_2}) := R.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that N has a fixed point which is a solution of the problem (3.6)-(3.7). Denote this solution by u_1. Define the functions

$$r_{k,1}(x,y) = x_k(u_1(x,y)) - x, \qquad \text{for } x \geq 0, y \geq 0.$$

Hypothesis (H3) implies that $r_{k,1}(0,0) \neq 0$ for $k = 1,\ldots,m$.
If $r_{k,1}(x,y) \neq 0$ on J for $k = 1,\ldots,m$; i.e.

$$x \neq x_k(u_1(x,y)) \quad \text{on } J, \quad \text{for } k = 1,\ldots,m,$$

then u_1 is a solution of the problem (1.1)-(1.3).

It remains to consider the case when $r_{1,1}(x,y) = 0$ for some $(x,y) \in J$. Now since $r_{1,1}(0,0) \neq 0$ and $r_{1,1}$ is continuous, there exists $x_1 > 0, y_1 > 0$ such that $r_{1,1}(x_1,y_1) = 0$, and $r_{1,1}(x,y) \neq 0$, for all $(x,y) \in [0,x_1) \times [0,y_1)$.
Thus by (H6) we have

$$r_{1,1}(x_1,y_1) = 0 \text{ and } r_{1,1}(x,y) \neq 0, \text{ for all } (x,y) \in [0,x_1) \times [0,y_1] \cup (y_1,b].$$

Suppose that there exist $(\bar{x},\bar{y}) \in [0,x_1) \times [0,y_1] \cup (y_1,b]$ such that $r_{1,1}(\bar{x},\bar{y}) = 0$. The function $r_{1,1}$ attains a maximum at some point $(s,t) \in [0,x_1) \times [0,b]$. Since

$$({}^cD_0^r u_1)(x,y) = f(x,y,u_1(x,y)), \text{ for } (x,y) \in J,$$

then

$$\frac{\partial u_1(x,y)}{\partial x} \text{ exists, and } \frac{\partial r_{1,1}(s,t)}{\partial x} = x_1'(u_1(s,t))\frac{\partial u_1(s,t)}{\partial x} - 1 = 0.$$

Since

$$\frac{\partial u_1(x,y)}{\partial x} = \varphi'(x) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1}f(s,t,u_1(s,t))dtds,$$

then

$$x_1'(u_1(s,t))[\varphi'(s) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)}\int_0^s\int_0^t (s-\theta)^{r_1-2}(t-\eta)^{r_2-1}f(\theta,\eta,u_1(\theta,\eta))d\eta d\theta] = 1,$$

which contradicts (H6). From (H3) we have

$$r_{k,1}(x,y) \neq 0 \text{ for all } (x,y) \in [0,x_1]\times[0,b] \text{ and } k = 1,...m.$$

Step 2. In what follows set

$$X_k := [x_k, a]\times[0,b];\ k = 1,...,m.$$

Consider now the problem

$$({}^cD_0^r u)(x,y) = f(x,y,u(x,y)),\ \text{if } (x,y) \in X_1, \tag{3.8}$$

$$u(x_1^+, y) = I_1(u_1(x_1,y)),\ \text{if } y \in [0,b]. \tag{3.9}$$

Consider the operator $N_1 : C(X_1,\mathbb{R}^n) \to C(X_1,\mathbb{R}^n)$ defined as

$$\begin{aligned} N_1(u) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t,u(s,t))dtds. \end{aligned}$$

As in Step 1 we can show that N_1 is completely continuous. Now it remains to show that the set

$$\mathcal{E}^* = \{u \in C(X_1,\mathbb{R}^n) : u = \lambda N_1(u) \text{ for some } 0 < \lambda < 1\}$$

is bounded. Let $u \in \mathcal{E}^*$, then $u = \lambda N_1(u)$ for some $0 < \lambda < 1$. Thus, from (H2) and (H4) we get for each $(x,y) \in X_1$,

$$\begin{aligned} \|u(x,y)\| &\le \|\varphi(x)\| + \|I_1(u_1(x_1,y))\| + \|I_1(u_1(x_1,0))\| \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{x_1}^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|f(s,t,u(s,t))\|dtds \\ &\le \|\varphi\|_\infty + 2M^*(1+\|u_1\|) + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} \\ &+ \frac{M}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|u(s,t)\|dtds. \end{aligned}$$

Set

$$\omega^* = \|\varphi\|_\infty + 2M^*(1+\|u_1\|) + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)},\ c = \frac{M}{\Gamma(r_1)\Gamma(r_2)}.$$

Then Lemma 3.4 implies that for each $(x,y) \in X_1$,

$$\|u(x,y)\| \le \omega^* E_{(r_1,r_2)}(c\Gamma(r_1)\Gamma(r_2)a^{r_1}b^{r_2}) := R^*.$$

This shows that the set $\mathcal{E}^*$ is bounded. As a consequence of Schaefer's theorem, we deduce that N_1 has a fixed point u which is a solution to problem (3.8)-(3.9). Denote this solution by u_2. Define

$$r_{k,2}(x,y) = x_k(u_2(x,y)) - x, \quad \text{for } (x,y) \in X_1.$$

If $r_{k,2}(x,y) \neq 0$ on $(x_1, a] \times [0,b]$ and for all $k = 1, \dots, m$, then

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in [0,x_1) \times [0,b], \\ u_2(x,y), & \text{if } (x,y) \in [x_1,a] \times [0,b], \end{cases}$$

is a solution of the problem (1.1)-(1.3). It remains to consider the case when $r_{2,2}(x,y) = 0$, for some $(x,y) \in (x_1,a] \times [0,b]$. By (H5), we have

$$\begin{aligned} r_{2,2}(x_1^+, y_1) &= x_2(u_2(x_1^+, y_1) - x_1 \\ &= x_2(I_1(u_1(x_1,y_1))) - x_1 > x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $x_2 > x_1,\ y_2 > y_1$ such that $r_{2,2}(x_2,y_2) = 0$, and $r_{2,2}(x,y) \neq 0$ for all $(x,y) \in (x_1,x_2) \times [0,b]$.
It is clear by (H3) that

$$r_{k,2}(x,y) \neq 0 \quad \text{for all } (x,y) \in (x_1,x_2) \times [0,b],\ k = 2, \dots, m.$$

Now suppose that there are $(s,t) \in (x_1,x_2) \times [0,b]$ such that $r_{1,2}(s,t) = 0$. From (H5) it follows that

$$\begin{aligned} r_{1,2}(x_1^+, y_1) &= x_1(u_2(x_1^+, y_1) - x_1 \\ &= x_1(I_1(u_1(x_1,y_1))) - x_1 \leq x_1(u_1(x_1,y_1)) - x_1 = r_{1,1}(x_1,y_1) = 0. \end{aligned}$$

Thus $r_{1,2}$ attains a nonnegative maximum at some point $(s_1,t_1) \in (x_1,a) \times [0,x_2) \cup (x_2,b]$. Since

$$({}^cD_0^r u_2)(x,y) = f(x,y,u_2(x,y)), \text{ for } (x,y) \in X_1,$$

then we get

$$\begin{aligned} u_2(x,y) &= \varphi(x) + I_1(u_1(x_1,y)) - I_1(u_1(x_1,0)) \\ &+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t,u_2(s,t))dtds, \end{aligned}$$

hence

$$\frac{\partial u_2}{\partial x}(x,y) = \varphi'(x) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x} \int_0^y (x-s)^{r_1-2}(y-t)^{r_2-1} f(s,t,u_2(s,t))dtds,$$

then

$$\frac{\partial r_{1,2}(s_1,t_1)}{\partial x} = x_1'(u_2(s_1,t_1)) \frac{\partial u_2}{\partial x}(s_1,t_1) - 1 = 0.$$

Therefore

$$x_1'(u_2(s_1,t_1))[\varphi'(s_1) + \frac{r_1-1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{s_1} \int_0^{t_1} (s_1-\theta)^{r_1-2}(t_1-\eta)^{r_2-1} f(\theta,\eta,u_2(\theta,\eta))d\eta d\theta] = 1,$$

which contradicts (H6).

Step 3. We continue this process and take into account that $u_{m+1} := u\Big|_{X_m}$ is a solution to the problem

$$\begin{aligned}&({}^cD_0^r u)(x,y) = f(x,y,u(x,y)), \quad \text{a.e. } (x,y) \in (x_m, a] \times [0,b],\\ &u(x_m^+, y) = I_m(u_{m-1}(x_m, y)), \text{ if } y \in [0,b].\end{aligned}$$

The solution u of the problem (1.1)-(1.3) is then defined by

$$u(x,y) = \begin{cases} u_1(x,y), & \text{if } (x,y) \in [0,x_1] \times [0,b],\\ u_2(x,y), & \text{if } (x,y) \in (x_1,x_2] \times [0,b],\\ \dots & \\ u_{m+1}(x,y), & \text{if } (x,y) \in (x_m,a] \times [0,b].\end{cases}$$

We give now (without proof) a uniqueness result for the problem (1.1)-(1.3) using the Banach contraction principle.

Theorem 3.6. *Assume* $(H1), (H3), (H5), (H6)$ *and the following conditions*

(H7) *There exists* $d > 0$ *such that*

$$\|f(x,y,u) - f(x,y,\bar{u})\| \le d\|u - \bar{u}\|, \text{ for each } (x,y) \in J, \ u, \bar{u} \in \mathbb{R}^n,$$

(H8) *There exists* $c_k > 0; \ k = 1, 2, ...m$ *such that*

$$\|I_k(x,y,u) - I_k(x,y,\bar{u})\| \le c_k\|u - \bar{u}\|, \text{ for each } (x,y) \in J, \ u, \bar{u} \in \mathbb{R}^n,$$

hold. If

$$2c_k + \frac{d a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then the IVP (1.1)-(1.3) has a unique solution.

4. Nonlocal impulsive partial differential equations

This section is concerned with a generalization of the result presented in the previous section to nonlocal impulsive partial hyperbolic differential equations. We shall present existence results for the following nonlocal initial value problem

$$({}^cD_0^r u)(x,y) = f(x,y,u(x,y)), \text{ if } (x,y) \in J, \ \ x \neq x_k(u(x,y)), \ k = 0, \dots, m, \quad (4.1)$$

$$u(x^+, y) = I_k(u(x,y)), \text{ if } (x,y) \in J, \ x = x_k(u(x,y)), \ k = 0, \dots, m, \quad (4.2)$$

$$u(x,0) + Q(u) = \varphi(x), \ u(0,y) + K(u) = \psi(y), \ x \in [0,a], \ y \in [0,b], \quad (4.3)$$

where $f, \ \varphi, \ \psi, \ I_k; \ k = 1, ...m$, are as in problem (1.1)-(1.3) and $Q, K : \Omega \to \mathbb{R}^n$ are continuous functions.

Definition 4.1. *A function* $u \in \Omega$ *whose* r*-derivative exists on* J' *is said to be a solution of (4.1)-(4.3) if* u *satisfies* $({}^cD_0^r u)(x,y) = f(x,y,u(x,y))$ *on* J' *and conditions* (4.2) *and* (4.3) *are satisfied.*

Theorem 4.2. *Assume* $(H1) - (H6)$ *and the following conditions hold:*

(H_2') *There exists* $\tilde{L} > 0$ *such that* $\|Q(u)\| \le \tilde{L}(1 + \|u\|_\infty)$, *for any* $u \in \Omega$,

(H_2'') *There exists* $L^* > 0$ *such that* $\|K(u)\| \le L^*(1 + \|u\|_\infty)$, *for any* $u \in \Omega$. *Then there exists at leat one solution for* IVP *(4.1)-(4.3) on* J.

Theorem 4.3. *Assume* $(H1), (H3), (H5) - (H8)$ *and the following conditions hold:*

(H_3') *There exists* $l > 0$ *such that* $\|Q(u) - Q(v)\| \le l\|u - v\|_\infty$, *for any* $u, v \in \Omega$,

(H_3'') *There exists* $l^* > 0$ *such that* $\|K(u) - K(v)\| \le l^*\|u - v\|_\infty$, *for any* $u, v \in \Omega$.

If

$$l + l^* + 2c_k + \frac{da^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} < 1,$$

then there exists a unique solution for IVP *(4.1)-(4.3) on* J.

5. An Example

As an application of our results we consider the following impulsive partial hyperbolic differential equations of the form

$$({}^cD_0^r u)(x,y) = \frac{1+u(x,y)}{9+e^{x+y}}, \ \text{ if } (x,y) \in J, \ x \neq x_k(u(x,y)); \ k = 1, \dots, m, \tag{5.1}$$

$$u(x_k^+, y) = d_k u(x_k, y); \ y \in [0,1], \ k = 1, \dots, m, \tag{5.2}$$

$$u(x,0) = x, \ u(0,y) = y^2, \ x \in [0,1], \ y \in [0,1], \tag{5.3}$$

where $J = [0,1] \times [0,1]$, $r = (r_1, r_2)$, $0 < r_1, r_2 \le 1$, $x_k(u) = 1 - \frac{1}{2^k(1+u^2)}$ and $\frac{\sqrt{2}}{2} < d_k \le 1$, for $k = 1, \dots, m$.

Denote $f(x,y,u) = \frac{1+u}{9+e^{x+y}}$, $(x,y,u) \in [0,1] \times [0,1] \times \mathbb{R}$ and $I_k(u) = d_k u$, $u \in \mathbb{R}$, and $k = 1, \dots, m$. Let $u \in \mathbb{R}$ then we have

$$x_{k+1}(u) - x_k(u) = \frac{1}{2^{k+1}(1+u^2)} > 0 \text{ for } k = 1, \dots, m.$$

Hence $0 < x_1(u) < x_2(u) < \ldots < x_m(u) < 1$, for each $u \in \mathbb{R}$.

Also, for each $u \in \mathbb{R}$ we have

$$x_{k+1}(I_k(u)) - x_k(u) = \frac{1 + (2d_k^2 - 1)u^2}{2^{k+1}(1+u^2)(1+d_k^2)} > 0.$$

Thus, for all $(x,y) \in J$ and each $u \in \mathbb{R}$ we get $|I_k(u)| = |d_k u| \le |u| \le 3(1+|u|)$ for $k = 1, \dots, m$, and $|f(x,y,u)| = \frac{|1+u|}{9+e^{x+y}} \le \frac{1}{10}(1+|u|)$.

Since all conditions of Theorem 3.5 are satisfied, the problem (5.1)-(5.3) has at least one solution on $[0,1] \times [0,1]$.

References

[1] S. Abbas, M. Benchohra, *Partial hyperbolic differential equations with finite delay involving the Caputo fractional derivative*, Commun. Math. Anal., **7**(2009), 62-72.

[2] S. Abbas, M. Benchohra, *Darboux problem for perturbed partial differential equations of fractional order with finite delay*, Nonlinear Anal. Hybrid Syst., **3**(2009), 597-604.

[3] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta. Appl. Math., **109**(2010), no. 3, 973-1033.

[4] R.P. Agarwal, M. Benchohra, B.A. Slimani, *Existence results for differential equations with fractional order and impulses*, Mem. Differential Equations Math. Phys., **44**(2008), 1-21.

[5] E. Ait Dads, M. Benchohra, S. Hamani, *Impulsive fractional differential inclusions involving the Caputo fractional derivative*, Fract. Calc. Appl. Anal., **12**(2009), no. 1, 15-38.

[6] I. Bajo, E. Liz, *Periodic boundary value problem for first order differential equations with impulses at variable times*, J. Math. Anal. Appl., **204**(1996), 65-73.
[7] D.D. Bainov, P.S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.
[8] A. Belarbi, M. Benchohra, *Existence theory for perturbed impulsive hyperbolic differential inclusions with variable times*, J. Math. Anal. Appl., **327**(2007), 1116-1129.
[9] A. Belarbi, M. Benchohra, A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces*, Appl. Anal., **85**(2006), 1459-1470.
[10] M. Benchohra, L. Górniewicz, S.K. Ntouyas, A. Ouahab, *Impulsive hyperbolic differential inclusions with variable times*, Topol. Methods Nonlinear Anal., **22**(2003), 319-329.
[11] M. Benchohra, L. Górniewicz, S.K. Ntouyas, A. Ouahab, *Existence results for impulsive hyperbolic differential inclusions*, Appl. Anal., **82**(2003), 1085-1097.
[12] M. Benchohra, J.R. Graef, S. Hamani, *Existence results for boundary value problems with nonlinear fractional differential equations*, Appl. Anal., **87**(2008), no. 7, 851-863.
[13] M. Benchohra, S. Hamani, S.K. Ntouyas, *Boundary value problems for differential equations with fractional order*, Surv. Math. Appl., **3**(2008), 1-12.
[14] M. Benchohra, J. Henderson, S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol. 2, New York, 2006.
[15] M. Benchohra, J. Henderson, S.K. Ntouyas, A. Ouahab, *Existence results for functional differential equations of fractional order*, J. Math. Anal. Appl., **338**(2008), 1340-1350.
[16] M. Benchohra, J.A. Ouahab, *Impulsive neutral functional differential inclusions with variable times*, Electron. J. Differential Equations, **2003**(2003), no. 67, 1-12.
[17] M. Benchohra, B.A. Slimani, *Existence and uniqueness of solutions to impulsive fractional differential equations*, Electron. J. Differential Equations, **2009**(2009), no. 10, 11 pp.
[18] L. Byszewski, *Theorem about existence and uniqueness of continuous solutions of nonlocal problem for nonlinear hyperbolic equation*, Appl. Anal., **40**(1991), 173-180.
[19] L. Byszewski, *Differential and functional-differential problems with nonlocal conditions*, (in Polish), Cracow University of Technology Monograph, **184**, Krakow, 1995.
[20] L. Byszewski, *Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation* $u_{xt} = F(x, t, u, u_x)$, J. Appl. Math. Stochastic Anal., **3**(1990), 163-168.
[21] M. Dawidowski, I. Kubiaczyk, *An existence theorem for the generalized hyperbolic equation* $z''_{xy} \in F(x, y, z)$ *in Banach space*, Ann. Soc. Math. Pol. Ser. I, Comment. Math., **30**(1990), no. 1, 41-49.
[22] K. Diethelm, N.J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl., **265**(2002), 229-248.
[23] M. Frigon, D. O'Regan, *Impulsive differential equations with variable times*, Nonlinear Anal., **26**(1996), 1913-1922.
[24] M. Frigon, D. O'Regan, *First order impulsive initial and periodic problems with variable moments*, J. Math. Anal. Appl., **233**(1999), 730-739.
[25] M. Frigon, D. O'Regan, *Second order Sturm-Liouville BVP's with impulses at variable moments*, Dynam. Contin. Discrete Impuls. Systems, **8** (2001), no. 2, 149-159.
[26] W.G. Glockle, T.F. Nonnenmacher, *A fractional calculus approach of selfsimilar protein dynamics*, Biophys. J., **68**(1995), 46-53.
[27] D. Henry, *Geometric Theory of Semilinear Parabolic Partial Differential Equations*, Springer-Verlag, Berlin-New York, 1989.
[28] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.
[29] S.K. Kaul, V. Lakshmikantham, S. Leela, *Extremal solutions, comparison principle and stability criteria for impulsive differential equations with variable times*, Nonlinear Anal., **22**(1994), 1263-1270.
[30] Z. Kamont, *Hyperbolic Functional Differential Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
[31] Z. Kamont, K. Kropielnicka, *Differential difference inequalities related to hyperbolic functional differential systems and applications*, Math. Inequal. Appl., **8**(2005), no. 4, 655–674.
[32] S.K. Kaul, X.Z. Liu, *Vector Lyapunov functions for impulsive differential systems with variable times*, Dynam. Contin. Discrete Impuls. Systems, **6**(1999), 25-38.

[33] S.K. Kaul, X.Z. Liu, *Impulsive integro-differential equations with variable times*, Nonlinear Stud., **8**(2001), 21-32.

[34] A.A. Kilbas, B. Bonilla, J. Trujillo, *Nonlinear differential equations of fractional order in a space of integrable functions*, Dokl. Ross. Akad. Nauk, **374**(2000), no. 4, 445-449.

[35] A.A. Kilbas, S.A. Marzan, *Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions*, Differential Equations, **41**(2005), 84-89.

[36] A.A. Kilbas, Hari M. Srivastava, Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[37] V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[38] V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[39] V. Lakshmikantham, N.S. Papageorgiou, J. Vasundhara, *The method of upper and lower solutions and monotone technique for impulsive differential equations with variable moments*, Appl. Anal., **15**(1993), 41-58.

[40] V. Lakshmikantham, S.G. Pandit, *The method of upper, lower solutions and hyperbolic partial differential equations*, J. Math. Anal. Appl., **105**(1985), 466-477.

[41] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, in "Fractals and Fractional Calculus in Continuum Mechanics" (A. Carpinteri and F. Mainardi, Eds.), Springer-Verlag, Wien, 1997, 291-348.

[42] F. Metzler, W. Schick, H.G. Kilian, T.F. Nonnenmacher, *Relaxation in filled polymers: A fractional calculus approach*, J. Chem. Phys., **103**(1995), 7180-7186.

[43] G. Mophou, *Existence and uniqueness of mild solutions to impulsive fractional differential equations*, Nonlinear Anal., **72**(2010), 1604-1615.

[44] S.G. Pandit, *Monotone methods for systems of nonlinear hyperbolic problems in two independent variables*, Nonlinear Anal., **30**(1997), 235-272.

[45] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[46] S.G. Samko, A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[47] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[48] N.P. Semenchuk, *On one class of differential equations of noninteger order*, Differents. Uravn., **10**(1982), 1831-1833.

[49] A.N. Vityuk, *Existence of solutions of partial differential inclusions of fractional order*, Izv. Vyssh. Uchebn., Ser. Mat., **8**(1997), 13-19.

[50] A.N. Vityuk, A.V. Golushkov, *Existence of solutions of systems of partial differential equations of fractional order*, Nonlinear Oscil., **7**(2004), no. 3, 318-325.

[51] S. Zhang, *Positive solutions for boundary-value problems of nonlinear fractional diffrential equations*, Electron. J. Differential Equations, **2006**(2006), no. 36, 12 pp.

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