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ON EKELAND'S VARIATIONAL PRINCIPLE IN b-METRIC SPACES

MONICA BOTA, ANDREA MOLNÁR AND CSABA VARGA

Department of Mathematics, Babeş-Bolyai University
1, Kogălniceanu Str., 400084, Cluj-Napoca, Romania.
E-mails: bmonica@math.ubbcluj.ro mr_andi16@yahoo.com csvarga@cs.ubbcluj.ro

Abstract. In this paper we prove a version of Ekeland's variational principle in b-metric spaces and, as a consequence, we obtain a Caristi type fixed point theorem.
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1. Introduction

The concept of b-metric space or generalizations of it appeared in works of N. Bourbaki [11], I.A. Bakhtin [2], S. Czerwik [15], J. Heinonen [19], etc. Some examples of b-metric spaces and some fixed and strict fixed point theorems in b-metric spaces can also be found in M. Boriceanu, A. Petruşel and I.A.Rus [5], M. Boriceanu [8], [6], M. Bota [7]. I. Ekeland in 1974 [16] formulated a variational principle, which has applications in many domains of mathematics, including fixed point theory. Later Borwein and Preiss [9] gave another form of this principle suitable for applications in subdifferential theory. Ekeland's variational principle has many generalizations, see [23], [24] and the very recent books of Borwein and Zhu [10], Meghea [21] and their references. Ekeland's variational principle is the main tool in proving the so-called Caristi fixed point theorem in complete metric spaces, see [10] and [21]. In this paper we give a version of Ekeland's variational principle in b-metric spaces and, as consequence, we will also obtain a Caristi type fixed point theorem in a complete b-metric space. Another recent generalizations of Caristi fixed point theorem in metric spaces can be found in T. Cardinali, P. Rubbioni [12], A. Amini-Harandi [1].

2. Preliminaries

We will first give the definition of a b-metric space.

Definition 1.1. (Bakhtin [2], Czerwik [15]) Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;
(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

The class of b-metric spaces is larger than the class of metric spaces, since a b-metric space is a metric space, when $s = 1$ in the third assumption of the above definition. Some examples of b-metric spaces are given by V. Berinde [3], S. Czerwik [15], J. Heinonen [19].

If (X, d) is a b-metric space and Y is a nonempty subset of X, then we denote (as in metric spaces) by $diam(Y)$ the diameter of the set Y, i.e., $diam(Y) := \sup\{d(a, b) \mid a, b \in Y\}$.

Example 1.2. Let X be a set with the cardinal $card(X) \geq 3$. Suppose that $X = X_1 \cup X_2$ is a partition of X such that $card(X_1) \geq 2$. Let $s > 1$ be arbitrary. Then, the functional $d : X \times X \to \mathbb{R}_+$ defined by:

$$d(x, y) := \begin{cases} 0, & x = y \\ 2s, & x, y \in X_1 \\ 1, & \text{otherwise} \end{cases}$$

is a b-metric on X with coefficient $s > 1$.

Example 1.3. The set $l^p(\mathbb{R})$ (with $0 < p < 1$), where $l^p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) := (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

(where $x = (x_n), y = (y_n) \in l^p(\mathbb{R})$) is a b-metric space with coefficient $s = 2^{1/p} > 1$. Notice that the above result holds for the general case $l^p(X)$ with $0 < p < 1$, where X is a Banach space.

Example 1.4. The space $L^p[0, 1]$ (where $0 < p < 1$) of all real functions $x(t)$, $t \in [0, 1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, together with the functional

$$d(x, y) := (\int_0^1 |x(t) - y(t)|^p dt)^{1/p}, \text{ for each } x, y \in L^p[0, 1],$$

is a b-metric space. Notice that $s = 2^{1/p}$.

We will present now the notions of convergence, compactness, closedness and completeness in a b-metric space.

Definition 1.5. Let (X, d) be a b-metric space. Then a sequence $(x_n)_{n \in \mathbb{N}}$ in X is called:

(a) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to +\infty$. In this case, we write $\lim\limits_{n \to \infty} x_n = x$.

(b) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to +\infty$.

Remark 1.6. *Notice that in a b-metric space (X, d) the following assertions hold:*
(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy;

(iii) $(X, \xrightarrow{d})$ *is an L-space (see Fréchet* [18], *Blumenthal* [4]*);*

(iv) in general, a b-metric is not continuous;

(v) in general, a b-metric does not induce a topology on X*.*

Taking into account of (iii), we have the following concepts.

Definition 1.7. Let (X, d) be a b-metric space. A subset $Y \subset X$ is called:

(i) closed if and only if for each sequence $(x_n)_{n\in\mathbb{N}}$ in Y which converges to an element x, we have $x \in Y$;

(ii) compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y.

The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Example 1.8. Let E be a Banach space, let P be a cone in E with $intP \neq \emptyset$ and let $\leq$ be a partial ordering with respect to P. A mapping $d : X \times X \to E$ is called a cone metric on the nonempty set X if the following axioms are satisfies:

1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

2) $d(x, y) = d(y, x)$, for all $x, y \in X$

3) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

The pair (X, d), where X is a nonempty set and d is a cone metric is called a cone metric space.

Notice that (see Lemma 5 in [20]) that if the cone P is normal with constant K, then the cone metric $d : X \times X \to E$ is continuous, i.e. if $(x_n), (y_n)$ are sequences in X with $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $d(x_n, y_n) \to d(x, y)$, as $n \to \infty$.

Let E be a Banach space and P be a normal cone in E with the coefficient of normality denoted by K. Let $D : X \times X \to \mathbb{R}$ be defined by $D(x, y) = \|d(x, y)\|$, where $d : X \times X \to E$ is a cone metric space. Then (X, D) is a b-metric space with constant $s := K \geq 1$.

Moreover, since the topology τ_d generated by the cone metric d coincides with the topology τ_D generated by the b-metric D, (see [22], Theorem 2.4), the b-metric D is continuous.

3. Ekeland variational principle and consequences

We begin this section with Cantor's intersection theorem in b-metric spaces.

Lemma 2.1. *Let* (X, d) *be a b-metric space. Suppose that* (X, d) *is complete. Then, for every descending sequence* $\{F_n\}_{n\geq 1}$ *of nonempty closed subsets of* X *such that* $diam(F_n) \to 0$ *as* $n \to \infty$*. Then the intersection* $\bigcap_{n=1}^{\infty} F_n$ *contains one and only one point.*

Proof. We suppose that (X, d) is complete. For each positive integer n, let x_n be any point in F_n. Then by the hypothesis, $x_n, x_{n+1}, x_{n+2}, \ldots$ all lie in F_n. Given $\varepsilon > 0$, there exists some integer n_0 such that $diam(F_{n_0}) < \varepsilon$. Now, $x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots$ all lie in F_{n_0}. For $m, n \geq n_0$, we have that $d(x_m, x_n) \leq diam(F_{n_0}) < \varepsilon$. This shows that the sequence $\{x_n\}_{n\geq 1}$ is a Cauchy sequence in the complete b-metric space X. So, it is convergent. Let $x \in X$ be such that $\lim_{n\to\infty} x_n = x$. Now for any given n, we have thet $x_n, x_{n+1}, ... \subset F_n$. In view of this, $x = \lim_{n\to\infty} x_n \in \bar{F}_n = F_n$, since F_n is closed. Hence, $x \in \bigcap_{n=1}^{\infty} F_n$. If $y \in \bigcap_{n=1}^{\infty} F_n$ and $y \neq x$, then $d(y,x) = \alpha > 0$. There exists $n \in \mathbb{N}$ large enough such that $diam(F_n) < \alpha = d(y,x)$, which ensures that $y \notin F_n$. Hence, y cannot be in $\bigcap_{n=1}^{\infty} F_n$. Thus, the intersection contains only one point. □

We present now Ekeland's variational principle in b-metric spaces.

Theorem 2.2. *Let (X,d) be a complete b-metric space (with $s > 1$), such that the b-metric d is continuous and let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous, proper and lower bounded mapping. Then, for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$f(x_0) \leq \inf_{x\in X} f(x) + \varepsilon,$$

there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x_\varepsilon \in X$ such that:

$$(i)\ x_n \to x_\varepsilon \text{ as } n \to \infty \tag{1}$$

$$(ii)\ d(x_\varepsilon, x_n) \leq \frac{\varepsilon}{2^n}, \quad n \in \mathbb{N} \tag{2}$$

$$(iii)\ f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x_\varepsilon, x_n) \leq f(x_0) \leq \inf_{x\in X} f(x) + \varepsilon \tag{3}$$

$$(iv)\ f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x, x_n) > f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n} d(x_\varepsilon, x_n), \text{ for every } x \neq x_\varepsilon. \tag{4}$$

Proof. We consider the set

$$T(x_0) = \{x \in X | f(x) + d(x, x_0) \leq f(x_0)\}. \tag{5}$$

Using the fact that f is a lower semicontinuous mapping and $x_0 \in T(x_0)$, we obtain that $T(x_0)$ is nonempty and closed in (X,d) and for every $y \in T(x_0)$

$$d(y, x_0) \leq f(x_0) - f(y) \leq f(x_0) - \inf_{x\in X} f(x) \leq \varepsilon. \tag{6}$$

We choose $x_1 \in T(x_0)$ such that $f(x_1) + d(x_1, x_0) \leq \inf_{x\in T(x_0)}\{f(x) + d(x, x_0)\} + \frac{\varepsilon}{2s}$ and let

$$T(x_1) := \Big\{x \in T(x_0) | f(x) + \sum_{i=0}^{1} \frac{1}{s^i} d(x, x_i) \leq f(x_1) + d(x_1, x_0)\Big\}.$$

Inductively, we can suppose that $x_{n-1} \in T(x_{n-2})$ was already chosen and we consider

$$\begin{aligned} T(x_{n-1}) \quad &:= \quad \Big\{x \in T(x_{n-2}) | f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i} d(x, x_i) \leq \\ &\leq \quad f(x_{n-1}) + \sum_{i=0}^{n-2} \frac{1}{s^i} d(x_{n-1}, x_i)\Big\}. \end{aligned} \tag{7}$$

Let us choose $x_n \in T(x_{n-1})$ such that

$$\begin{aligned} f(x_n) &\quad + \sum_{i=0}^{n-1} \frac{1}{s^i}\, d(x_n, x_i) \leq \\ &\leq \qquad \inf_{x \in T(x_{n-1})} \left\{ f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i}\, d(x, x_i) \right\} + \frac{\varepsilon}{2^n\, s^i}. \end{aligned} \tag{8}$$

and define the set

$$\begin{aligned} T(x_n) \quad &:= \quad \{x \in T(x_{n-1}) | f(x) + \sum_{i=0}^{n} \frac{1}{s^i} d(x, x_i) \leq \\ &\leq \quad f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s^i} d(x_n, x_i)\}. \end{aligned} \tag{9}$$

As before, the set $T(x_n)$ is nonempty and closed. From the relations (8) and (9) it follows that for each $y \in T(x_n)$ we have

$$\begin{aligned} \frac{1}{s^n}\, d(y, x_n) \quad &\leq \quad \left[f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s_i}\, d(x_n, x_i) \right] - \left[f(y) + \sum_{i=0}^{n-1} \frac{1}{s_i}\, d(y, x_i) \right] \\ &\leq \quad \left[f(x_n) + \sum_{i=0}^{n-1} \frac{1}{s_i}\, d(x_n, x_i) \right] - \inf_{x \in T(x_{n-1})} \left\{ f(x) + \sum_{i=0}^{n-1} \frac{1}{s^i}\, d(x, x_i) \right\} \\ &\leq \quad \frac{\varepsilon}{2^n\, s^n}, \end{aligned}$$

therefore, for all $y \in T(x_n)$ we have

$$d(y, x_n) \leq \frac{\varepsilon}{2^n}. \tag{10}$$

We can observe that $d(y, x_n) \to 0$ as $n \to \infty$, so diam $T(x_n) \to 0$. Because (X, d) is a complete b-metric space, from Cantor's intersection theorem (see Lemma 2.1) we have that $\bigcap_{n=0}^{\infty} T(x_n) = \{x_\varepsilon\}$. From (6), (10) we obtain that $x_\varepsilon \in X$ satisfies (2). Thus $x_n \to x_\varepsilon$ as $n \to \infty$.

Moreover, for all $x \neq x_\varepsilon$ we have $x \notin \bigcap_{n=0}^{\infty} T(x_n)$, so there exists $m \in \mathbb{N}$ such that

$$f(x) + \sum_{i=0}^{m} \frac{1}{s^i}\, d(x, x_i) > f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s^i}\, d(x_m, x_i).$$

From (5), (7) and (8), for every $q \geq m$, we have that

$$\begin{aligned} f(x_0) \geq f(x_m) + \sum_{i=0}^{m-1} \frac{1}{s_i} d(x_m, x_i) \geq f(x_q) + \sum_{i=0}^{q-1} \frac{1}{s_i} d(x_q, x_i) \\ \geq f(x_\varepsilon) + \sum_{i=0}^{q} \frac{1}{s_i} d(x_\varepsilon, x_i). \end{aligned}$$

Thus (3) and (4) hold. □

We have the following consequence of Ekeland's variational principle in b-metric spaces.

Corollary 2.3. *Let (X,d) be a complete b-metric space (with $s>1$), such that the b-metric d is continuous and let $f : X \to \overline{\mathbb{R}}$ be a lower semicontinuous, proper and lower bounded mapping. Then, for every $\varepsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ and $x^* \in X$ such that:*

$$(i)\ x_n \to x_\varepsilon, x_\varepsilon \in X, n \to \infty$$

$$(ii)\ f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x_\varepsilon, x_n) \le \inf_{x\in X} f(x) + \varepsilon$$

$$(iii)\ f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x, x_n) \ge f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x_\varepsilon, x_n),\ \textit{for any } x \in X.$$

Next, we will give an extension of Caristi's fixed point theorem.

Theorem 2.4. Let (X,d) be a complete b-metric space (with $s>1$), such that the b-metric d is continuous. Let $T : X \to X$ be an operator for which there exists a lower semicontinuous mapping $f : X \to \overline{\mathbb{R}}$, such that

$$d(u,v) + s\, d(u,T(u)) \ge d(T(u),v) \tag{11}$$

$$\frac{s^2}{s-1}\, d(u,T(u)) \le f(u) - f(T(u)),\ \text{for any } u,v \in X \tag{12}$$

Then T has at least one fixed point.

Proof. We suppose that for all $x \in X$ we have that $T(x) \ne x$. Using Corollary 2.3 for f, we obtain that, for each $\varepsilon > 0$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X, such that $x_n \to x_\varepsilon$ as $n \to \infty, x_\varepsilon \in X$ and

$$f(x) + \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x, x_n) \ge f(x_\varepsilon) + \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x_\varepsilon, x_n),$$

for every $x \in X$. If, in the above inequality, we put $x := T(x_\varepsilon)$, since $T(x_\varepsilon) \ne x_\varepsilon$, we get that

$$f(x_\varepsilon) - f(T(x_\varepsilon)) < \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(T(x_\varepsilon), x_n) - \sum_{n=0}^{\infty} \frac{1}{s^n}\, d(x_\varepsilon, x_n).$$

Using (11) for $u = x_\varepsilon, v = x_n$ we have

$$f(x_\varepsilon) - f(T(x_\varepsilon)) < \sum_{n=0}^{\infty} \frac{s}{s^n}\, d(x_\varepsilon, T(x_\varepsilon)). \tag{13}$$

In (12) we choose $u = x_\varepsilon$. Then

$$\frac{s^2}{s-1}\, d(x_\varepsilon, T(x_\varepsilon)) \le f(x_\varepsilon) - f(T(x_\varepsilon)). \tag{14}$$

Moreover, from (13) we get that

$$f(x_\varepsilon) - f(T(x_\varepsilon)) < \frac{s^2}{s-1}\, d(x_\varepsilon, T(x_\varepsilon)). \tag{15}$$

If we compare the inequalities (14) and (15), we obtain that

$$\frac{s^2}{s-1}\, d(x_\varepsilon, T(x_\varepsilon)) \leq f(x_\varepsilon) - f(T(x_\varepsilon)) < \frac{s^2}{s-1}\, d(x_\varepsilon, T(x_\varepsilon)), \tag{16}$$

which is a contradiction. Thus, there exists $x^* \in X$ such that $x^* \in T(x^*)$. $\square$

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