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EXISTENCE OF POSITIVE SOLUTIONS OF NEUMANN BOUNDARY VALUE PROBLEM VIA A CONVEX FUNCTIONAL COMPRESSION-EXPANSION FIXED POINT THEOREM

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Abstract. This paper is devoted to study the existence of positive solutions of second-order boundary value problem

$$-u'' + Mu = h(t)f(t, u), \quad t \in \ (0, 1)$$

with Neumann boundary conditions

$$u'(0) = u'(1) = 0,$$

where $M > 0$, $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$. $h(t)$ is allowed to be singular at $t = 0$ and $t = 1$. The arguments are based only upon the positivity of the Green's function and the fixed point theorem of cone expansion and compression of convex function type.

Key Words and Phrases: Neumann BVP, positive solutions, cone, fixed point theorem.
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1. Introduction

Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations and difference equations have been studied extensively. The main tools used are fixed-point theorems. Fixed-point theorems and their applications to nonlinear problems have a long history, some of which is documented in Zeidler's book [7], and the recent book by Agarwal, O'Regan and Wong [1] contains an excellent summary of the current results and applications. The fixed point theorem of cone expansion and compression of convex function type [8] is an extension of the fixed point theorem of cone expansion and compression of norm type that is usually referred to as Guo-Krasnosel'skii fixed point theorem, a proof of which can be found in [3]. In this paper, we are concerned with the second-order two-point Neumann boundary value problem

$$-u'' + Mu = h(t)f(t, u), \quad t \in \ (0, 1), \tag{1.1}$$

$$u'(0) = u'(1) = 0, \tag{1.2}$$

where $M > 0$ and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$. Recently, Neumann boundary value problems have deserved the attention of many researchers, see [2, 4, 5, 9, 6], and the references therein. The goal of this paper is to study the existence results for second-order Neumann boundary value problem (1.1) and (1.2) under the weaker conditions by the fixed point theorem of cone expansion and compression of convex function type.

2. Preliminaries and lemmas

In Banach space $C[0,1]$ in which the norm is defined by $\|u\| = \max_{0\le t\le 1} |u(t)|$ for any $u \in C[0,1]$. We set $P = \{u \in C[0,1] | u(t) \ge 0,\ t \in\ [0,1]\}$ be a cone in $C[0,1]$. The function u is said to be a positive solution of BVP(1.1),(1.2) if $u \in C[0,1] \cap C^2(0,1)$ satisfies $(1.1), (1.2)$ and $u(t) > 0$ for $t \in (0,1)$.

Let $G(t,s)$ be the Green's function of the problem (1.1), (1.2) with $f(t,u) \equiv 0$ (see [4], [5]), that is,

$$G(t,s) = \begin{cases} \dfrac{\mathrm{ch}(m(1-t))\mathrm{ch}(ms)}{m\mathrm{sh}m}, & 0 \le s \le t \le 1, \\ \dfrac{\mathrm{ch}(m(1-s))\mathrm{ch}(mt)}{m\mathrm{sh}m}, & 0 \le t \le s \le 1, \end{cases}$$

where $m = \sqrt{M},\ \mathrm{ch}x = \dfrac{e^x + e^{-x}}{2},\ \mathrm{sh}x = \dfrac{e^x - e^{-x}}{2}$. Obviously, $G(t,s)$ is continuous on $[0,1] \times [0,1]$ and $G(t,s) \ge 0$ for $0 \le t,s \le 1$. After direct computations we get

$$0 < \frac{1}{m\mathrm{sh}m} = \alpha \le G(t,s) \le \beta = \frac{\mathrm{ch}^2 m}{m\mathrm{sh}m},\ \forall\ 0 \le t,s \le 1. \tag{2.1}$$

We make the following assumptions:

(H_1) $h : (0,1) \to [0,+\infty)$ is continuous, and

$$0 < \int_0^1 h(t)dt < +\infty;$$

(H_2) $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Let

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds, \quad t \in [0,1]. \tag{2.2}$$

We can verify that the nonzero fixed points of the operator A are positive solutions of the problem (1.1), (1.2).

Define

$$K = \{u \in P | u(t) \ge \gamma\|u\|,\ t \in [0,1]\},$$

where $0 < \gamma = \frac{\alpha}{\beta} < 1$. Then K is subcone of P.

Lemma 2.1. *Suppose that (H_1) and (H_2) are satisfied. Then $A : K \to K$ is a completely continuous operator.*

Proof. Let $u \in K$. Since $G(t,s) \ge 0,\ (t,s) \in\ [0,1] \times [0,1]$, by the definition, we have $(Au)(t) \ge 0,\ t \in [0,1]$. On the other hand, by (2.1) we have

$$(Au)(t) = \int_0^1 G(t,s)h(s)f(s,u(s))ds \ge \alpha \int_0^1 h(s)f(s,u(s))ds, \tag{2.3}$$

$$\|Au\| = \max_{t\in[0,1]} \int_0^1 G(t,s)h(s)f(s,u(s))ds \leq \beta \int_0^1 h(s)f(s,u(s))ds, \qquad (2.4)$$

for every $t \in [0,1]$. By (2.3) and (2.4) we have $(Au)(t) \geq \gamma\|Au\|$. Thus, we assert that $A : K \to K$. It follows from [10] that if $(H_1) - (H_2)$ are satisfied, $A : K \to K$ is completely continuous. □

Our main results concerning positive solutions of (1.1) and (1.2) will arise as applications of the following fixed point theorem due to Zhang and Sun [8].

Lemma 2.2. *Let K be a cone in a real Banach space E, Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : K \cap (\overline{\Omega_2}\backslash\Omega_1) \to K$ is completely continuous and $\rho : K \to [0,+\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(u) > 0$ for $u \neq \theta$. If one of the two conditions*

(A_1) $\rho(Au) \leq \rho(u),\ \forall\ u \in K \cap \partial\Omega_1$ and $\inf\limits_{u\in K\cap\partial\Omega_2} \rho(u) > 0,\ \rho(Au) \geq \rho(u),\ \forall\ u \in K \cap \partial\Omega_2$

or

(A_2) $\inf\limits_{u\in K\cap\partial\Omega_1} \rho(u) > 0,\ \rho(Au) \geq \rho(u),\ \forall\ u \in K \cap \partial\Omega_1$ and $\rho(Au) \leq \rho(u),\ \forall\ u \in K \cap \partial\Omega_2$ is satisfied, then A has at least one fixed point in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$.

3. Existence results

In this section, we impose growth conditions on f and then apply Lemma 2.2 to establish the existence of positive solutions of (1.1), (1.2).

Theorem 3.1. *Suppose that (H_1) and (H_2) are satisfied. If there exist constants a and b with $0 < a < b$ satisfying*

(H_3) $a < \gamma^2 b$;

(H_4) $f(t,u) \leq \dfrac{1}{\beta\int_0^1 h(t)dt}u,\quad \forall\ (t,u) \in [0,1] \times \left[0, \dfrac{a}{\gamma\int_0^1 h(t)dt}\right]$;

(H_5) $f(t,u) \geq \dfrac{1}{\alpha\int_0^1 h(t)dt}u,\quad \forall\ (t,u) \in [0,1] \times \left[\dfrac{\gamma b}{\int_0^1 h(t)dt}, \dfrac{b}{\gamma\int_0^1 h(t)dt}\right]$.

Then Neumann BVP (1.1), (1.2) has at least one positive solution.

Proof. Define $\rho : K \to \mathbb{R}^+$ by $\rho(u) = \int_0^1 h(t)u(t)dt$.

We observe here that, $\rho : K \to \mathbb{R}^+$ is a uniformly continuous convex function with $\rho(\theta) = 0$ and for $u \in K\backslash\{\theta\}$, from the definition of K, such that

$$\rho(u) = \int_0^1 h(t)u(t)dt \geq \gamma \int_0^1 h(t)dt\|u\| > 0$$

Set

$$\Omega_1 \cap K = \{u \in K|\rho(u) < a\},\ \Omega_2 \cap K = \{u \in K|\rho(u) < b\}.$$

It is clear that $\theta \in \Omega_1 \cap K$, $\overline{\Omega_1 \cap K} \subset \Omega_2 \cap K$. If $u \in \Omega_1 \cap K$, we have

$$a \geq \rho(u) \geq \gamma \int_0^1 h(t)dt\|u\|,$$

and thus $\|u\| \le \dfrac{a}{\gamma \int_0^1 h(t)dt}$ which implies that $\Omega_1 \cap K$ is bounded. In the same way we know that for each $u \in \Omega_2 \cap K$, $\|u\| \le \dfrac{b}{\gamma \int_0^1 h(t)dt}$.

If $u \in \partial\Omega_1 \cap K$, then $\rho(u) = a$ and $\|u\| \le \dfrac{a}{\gamma \int_0^1 h(t)dt}$. It follows from (H_4) that

$$\begin{aligned} \rho(Au) = & \int_0^1 h(t)\Big[\int_0^1 G(t,s)h(s)f(s,u(s))ds\Big]dt \\ \le & \ \beta \int_0^1 h(t)dt \int_0^1 h(s)f(s,u(s))ds \\ \le & \int_0^1 h(s)u(s)ds = \rho(u). \end{aligned}$$

If $u \in \partial\Omega_2 \cap K$, then $\rho(u) = b$ and $\|u\| \le \dfrac{b}{\gamma \int_0^1 h(t)dt}$. Note that this yields $\inf\limits_{u \in K\cap\partial\Omega_2} \rho(u) = b > 0$. Since

$$b = \int_0^1 h(t)u(t)dt \le \int_0^1 h(t)dt\|u\|,$$

we have that $\|u\| \ge \dfrac{b}{\int_0^1 h(t)dt}$ and thus for each $t \in [0,1]$, $u(t) \ge \gamma\|u\| \ge \dfrac{\gamma b}{\int_0^1 h(t)dt}$. So it follows from (H_5) that

$$\begin{aligned} \rho(Au) = & \int_0^1 h(t)\Big[\int_0^1 G(t,s)h(s)f(s,u(s))ds\Big]dt \\ \ge & \ \alpha \int_0^1 h(t)dt \int_0^1 h(s)f(s,u(s))ds \\ \ge & \int_0^1 h(s)u(s)ds = \rho(u). \end{aligned}$$

Thus the hypothesis (A_1) of Lemma 2.2 is satisfied, and therefore the proof is finished.

Theorem 3.2. *Suppose that* (H_1) *and* (H_2) *are satisfied. If there exist constants* r *and* R *with* $0 < r < R$ *satisfying*

(H_6) $r < \gamma^2 R$;

(H_7) $f(t,u) \le \dfrac{r}{\beta \int_0^1 h(t)dt}, \quad \forall\ (t,u) \in [0,1] \times \Big[0, \dfrac{r}{\gamma}\Big]$;

(H_8) $f(t,u) \geq \dfrac{R}{\alpha \displaystyle\int_0^1 h(t)dt}$, $\quad \forall\ (t,u) \in [0,1] \times [\gamma R, +\infty)$.

Then Neumann BVP (1.1), (1.2) has at least one positive solution.

Proof. Define $\widetilde{\rho} : K \to \mathbb{R}^+$ by $\widetilde{\rho}(u) = \int_0^1 u(t)dt$.

We observe here that, $\widetilde{\rho} : K \to \mathbb{R}^+$ is a uniformly continuous convex function with $\widetilde{\rho}(\theta) = 0$ and for $u \in K \backslash \{\theta\}$, from the definition of K, such that

$$\widetilde{\rho}(u) = \int_0^1 u(t)dt \geq \gamma \|u\| > 0.$$

Set

$$\Omega_1 \cap K = \{u \in K | \widetilde{\rho}(u) < r\},\ \Omega_2 \cap K = \{u \in K | \widetilde{\rho}(u) < R\}.$$

It is clear that $\theta \in \Omega_1 \cap K$, $\overline{\Omega_1 \cap K} \subset \Omega_2 \cap K$. If $u \in \Omega_1 \cap K$, we have $r \geq \widetilde{\rho}(u) \geq \gamma \|u\|$, and, thus, $\|u\| \leq \dfrac{r}{\gamma}$ which implies that $\Omega_1 \cap K$ is bounded. In the same way we know that for each $u \in \Omega_2 \cap K$, $\|u\| \leq \dfrac{R}{\gamma}$.

If $u \in \partial\Omega_1 \cap K$, then $\widetilde{\rho}(u) = r$ and $\|u\| \leq \dfrac{r}{\gamma}$. It follows from (H_7) that

$$\begin{aligned}
\widetilde{\rho}(Au) = & \int_0^1 \Big[\int_0^1 G(t,s)h(s)f(s,u(s))ds\Big]dt \\
\leq & \ \beta \int_0^1 h(s)f(s,u(s))ds \\
\leq & \ \beta \cdot \frac{r}{\beta \displaystyle\int_0^1 h(t)dt} \cdot \int_0^1 h(s)ds = r = \widetilde{\rho}(u).
\end{aligned}$$

If $u \in \partial\Omega_2 \cap K$, then $\widetilde{\rho}(u) = R$ and $\|u\| \leq \dfrac{R}{\gamma}$. Note that this yields $\inf\limits_{u \in K \cap \partial\Omega_2} \widetilde{\rho}(u) = R > 0$. Since

$$R = \int_0^1 u(t)dt \leq \int_0^1 dt\|u\| = \|u\|,$$

we have that $\|u\| \geq R$ and thus for each $t \in [0,1]$, $u(t) \geq \gamma\|u\| \geq \gamma R$. So it follows from (H_8) that

$$\begin{aligned}
\widetilde{\rho}(Au) = & \int_0^1 \int_0^1 G(t,s)h(s)f(s,u(s))dsdt \\
\geq & \ \alpha \int_0^1 h(s)f(s,u(s))ds \\
\geq & \ \alpha \cdot \frac{R}{\alpha \displaystyle\int_0^1 h(t)dt} \cdot \int_0^1 h(s)ds = R = \widetilde{\rho}(u).
\end{aligned}$$

Thus the hypothesis (A_1) of Lemma 2.2 is satisfied, and therefore the proof is finished.

4. An example

Consider the Neumann boundary value problem

$$\begin{cases} -u''(t) + u(t) = f(u(t)), & 0 < t < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.1}$$

where $f : [0, +\infty,) \to [0, +\infty)$ is defined by

$$f(u) = \begin{cases} \dfrac{\sqrt{u}}{108}, & 0 \le u \le 1, \\ \dfrac{1}{72}u^2 - \dfrac{1}{216}, & 1 \le u \le +\infty. \end{cases}$$

BVP (4.1) can be regarded as a BVP of form (1.1), (1.2), where $M = 1$, $h(t) \equiv 1$, $f(t, u) \equiv f(u)$. It is clear conditions (H_1) and (H_2) are satisfied, and $\frac{3}{4} \le G(t, s) \le 9$, $\gamma = \frac{1}{12}$.

Taking $r = \frac{1}{12}$, $R = 24$. Thus for $0 \le u \le 1$, we have $f(u) \le \frac{1}{108}$ and for $u \ge 288$, we have $f(u) \ge 32$, then conditions $(H_6) - (H_8)$ are satisfied. Consequently, Theorem 3.2 guarantees the BVP (4.1) has a positive solution.

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