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STRUCTURE OF COMMON FIXED POINT SET OF DEMICONTINUOUS ASYMPTOTICALLY S-NONEXPANSIVE SEMIGROUPS

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Abstract. Every asymptotically nonexpansive mapping is uniformly continuous, but this fact is not true for asymptotically S-nonexpansive mappings in general. In a Banach space X, by constructing a sequence $\{x_n\}$ defined in Browder's technique for a demicontinuous asymptotically S-nonexpansive semigroup $\mathcal{T} = \{T(t) : t \geq 0\}$ of mappings from $C \subset X$ into itself with function $k(\cdot)$, we prove that the common fixed point set $F(S) \cap \cap_{t>0} F(T(t))$ is the sunny nonexpansive retract of $F(S)$, where $F(S)$ is the fixed point set of a weakly continuous mapping S form C into itself. Under the assumption on uniform convexity of X, we prove that the common fixed point set $F(S) \cap \cap_{t>0} F(T(t))$ is closed and convex.
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1. Introduction

Let C be a nonempty subset of a normed space X and let S and T be two self-mappings of C. Then T is said to be

(i) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$;
(ii) *S- nonexpansive* ([13]) if $\|Tx - Ty\| \leq \|Sx - Sy\|$ for all $x, y \in C$;
(iii) *S- contraction* if $\|Tx - Ty\| \leq k\|Sx - Sy\|$ for all $x, y \in C$ and for some $k \in [0, 1)$;
(iv) *asymptotically S-nonexpansive* ([19]) if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that
$$\|T^n x - T^n y\| \leq k_n\|Sx - Sy\|$$
for all $x, y \in C$ and $n \in \mathbb{N}$.

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An asymptotically S-nonexpansive is called *asymptotically nonexpansive* ([6]) if $S = I$. Here I is the identity operator.

In 1967, Browder [2] proved the following strong convergence theorem for nonexpansive mappings: Let C be a nonempty closed convex bounded subset of a Hilbert space H and $T : C \to C$ a nonexpansive self-mapping. Let $u \in C$ and for each $t \in (0, 1)$, let

$$G_t x = tu + (1 - t)Tx, \ x \in C.$$

Then G_t has a unique fixed point x_t in C and $\{x_t\}$ strongly converges, as $t \to 0$, to a fixed point of T in C.

Consequently, considerable research effects have been devoted, especially within the past 40 years, to generalize this result for the classes of nonexpansive mappings, pseudocontractive mappings and asymptotically nonexpansive mappings in Banach spaces (see, e.g. [1, 10, 11, 14, 15, 20]).

Recently, Beg, author and Diwan [5] studied the strong convergence of a sequence $\{x_n\}$ defined in a reflexive Banach space by

$$Sx_n = x_n = \mu_n T^n x_n + (1 - \mu_n)u \tag{1.1}$$

where T is asymptotically S-nonexpansive, $\mu_n = \lambda_n / k_n$ and $\lambda_n \in (0, 1)$ such that $\lim\limits_{n\to\infty} \lambda_n = 1$.

In this paper, an important existence result as a significant improvement of a result of Jungck [8] for non-continuous semigroup of S-contraction mappings is proved. This result is applied for existence of a sequence $\{y_n\}$ similar to $\{x_n\}$ defined by (1.1) and existence of common fixed points of a semigroup of non-continuous asymptotically S-nonexpansive mappings. Finally, we prove prove strong convergence of sequence $\{y_n\}$ in a reflexive Banach space with a weakly continuous duality mapping. Our results are significant improvements of corresponding results of Al-Thagafi [4], Dotson [7], Jungck [9] and Shahzad [17]. One of our results is an extension of celebrated result of Browder [2] from Hilbert space to Banach space for a semigroup of demicontinuous asymptotically S-nonexpansive mappings.

2. Preliminaries

Let C be a nonempty subset of a normed space X. The set C is called *q-starshaped* with $q \in C$ if for all $x \in C$, the line segment $[x, q]$ joining x to q is contained in C, that is, $tx + (1 - t)q \in C$ for all $x \in C$ and $0 \le t \le 1$. Note that if C is q-starshaped for every $q \in C$, then C is *convex*.

Let C be a nonempty subset of a normed space X. Then a mapping $S : C \to C$ is said to be

(I) *affine* if C is convex and $S(tx + (1 - t)y) = tSx + (1 - t)Sy$ for all $x, y \in C$ and $0 \le t \le 1$.

(II) *q-affine* if C is q-starshaped with $q = Sq$ and $S(tx + (1 - t)q) = tSx + (1 - t)q$ for all $x \in C$ and $0 \le t \le 1$.

Let C be a nonempty subset of a normed space X and $T : C \to C$ a mapping. We adopt the following notations:

$$F(T) = \{u \in C : Tu = u\}.$$

When $\{x_n\}$ is a sequence in X, we denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$. T is said to be *demicontinuous* if $\{x_n\}$ is a sequence in X such that $x_n \to x$, then $Tx_n \rightharpoonup Tx$. The mapping T is said to be *weakly continuous* if $\{x_n\}$ is a sequence in X such that $x_n \rightharpoonup x$, then $Tx_n \rightharpoonup Tx$. Note that every continuous mapping is weakly continuous but not conversely (see [18]).

Let X be a Banach space with dual space X^*. A mapping T with domain $D(T)$ and range $R(T)$ in X is said to be *demiclosed* at a point $y \in R(T)$ if whenever $\{x_n\}$ is a sequence in $D(T)$ which converges weakly to a point $u \in D(T)$ and $\{Tx_n\}$ converges strongly to y, then $Tu = y$.

A Banach space X is said to satisfy *the Opial condition* ([12]) if for each sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in X, y \neq x$.

Let X be a Banach space. Then a mapping $J : X \to 2^{X^*}$ defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\| = \|u\|\}, u \in X$$

is called the normalized duality mapping. Suppose that J is single-valued. Then J is said to be *weakly sequentially continuous* if, for each $\{x_n\}$ in X with $x_n \rightharpoonup x$, $J(x_n) \rightharpoonup^* J(x)$. It is well known that if X admits a weakly sequentially continuous duality mapping, then X satisfies the Opial condition.

By a *gauge* we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. We associate with a gauge φ a (generally multi-valued) *duality mapping* $J_\varphi : X \to X^*$ defined by

$$J_\varphi(x) := \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\varphi(\|x\|) \textit{ and } \|x^*\| = \varphi(\|x\|)\}, x \in X.$$

Clearly the (normalized) duality mapping J corresponds to the gauge $\varphi(t) = t$. Note that

$$J_\varphi(x) = \frac{\varphi(\|x\|)}{\|x\|} J(x), \quad x \neq 0.$$

Browder [3] initiated the study of certain classes of nonlinear operators by means of duality mappings of the form of J_φ. For $t \geq 0$, let

$$\Phi(t) := \int_0^t \varphi(r)dr.$$

It is known that $J_\varphi(x)$ is the subdifferential of the convex function $\Phi(\|\cdot\|)$ at x.

Now let us recall that X is said to have a *weakly continuous duality mapping* if there exists a gauge φ such that the duality mapping J_φ is single-valued and continuous from X with the weak topology to X^* with the weak* topology. Every ℓ^p $(1 < p < \infty)$ space has a weakly continuous duality mapping with the gauge $\varphi(t) = t^{p-1}$.

One sees that Φ is a convex function and

$$J_\varphi(x) = \partial\Phi(\|x\|), \quad x \in X,$$

where ∂ denotes the subdifferential in the sense of convex analysis. We need the subdifferential inequality:

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, j(x+y)\rangle \quad \text{for all } x, y \in X \text{ and } \; j(x+y) \in J_\varphi(x+y).$$

For a smooth X, we have

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle \quad \text{for all } x, y \in X;$$

or considering the normalized duality mapping J, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle \quad \text{for all } x, y \in X.$$

Let C be a closed convex subset of X, D a nonempty subset of C and P_D a retraction from C into D that is, $P_D x = x$ for all $x \in D$. A retraction P_D is said to be *sunny* if $P_D(P_D x + t(x - P_D x)) = P_D x$ for all $x \in C$ and $t \geq 0$. The set D is said to be a *sunny nonexpansive retract of* C if there exists a sunny nonexpansive retraction of C onto D.

In the sequel, we shall need the following results.

Lemma 2.1. ([1]) *Let C be a nonempty convex subset of a smooth Banach space, D a nonempty subset of C and P_D a retraction from C onto D. Then P_D is sunny and nonexpansive if and only if*

$$\langle x - P_D x, J(z - P_D x)\rangle \leq 0 \quad \textit{for all } x \in C \textit{ and } z \in D.$$

3. Auxiliary results

First, we introduce the concept of semigroup of asymptotically nonexpansive mappings with respect to S.

Definition 3.1. *Let X be a metric space, $S : C \to C$ a mapping and $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ a family of mappings from C into itself. Then $\mathcal{T}$ is said to be a strongly continuous semigroup of S-Lipschitzian semigroup with $k(\cdot) : \mathbb{R}^+ \to (0, \infty)$ if the following conditions are satisfied:*

(S_1) *$T(0)x = x$ for all $x \in C$;*
(S_2) *$T(s+t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;*
(S_3) *for each $t > 0$, there exists a function $k(\cdot) : \mathbb{R}^+ \to (0, \infty)$ such that*

$$d(T(t)x, T(t)y) \leq k(t)d(Sx, Sy) \textit{ for all } x, y \in C;$$

(S_4) *for each $x \in C$, the mapping $T(\cdot)x = x$ from $\mathbb{R}^+$ into C is continuous.*

A strongly continuous semigroup of S-Lipschitzian semigroup with $k(\cdot) : \mathbb{R}^+ \to (0, \infty)$ is said to be

(i) asymptotically S-nonexpansive semigroup if $k(\cdot) : \mathbb{R}^+ \to [1, \infty)$ with $\lim_{t\to\infty} k(t) = 1$,
(ii) S-contraction semigroup if $k(t) < 1$ for all $t > 0$,
(iii) S-nonexpansive semigroup if $k(t) = 1$ for all $t > 0$.

The following lemma is an improvement of Jungck [8, Lemma 2.1] in the following ways:

(i) C is not necessarily closed,

(ii) T is not necessarily continuous,

(ii) location of unique common fixed point is given.

Proposition 3.2. *Let X be a metric space, $S : C \to C$ a mapping and $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ a family of mappings from C into itself. Let $\mathcal{T}$ be a strongly continuous semigroup of S-contraction mappings such that*

(i) *$T(t)(C) \subseteq S(C)$ for all $t > 0$,*
(ii) *for each $t > 0$, $\{S, T(t)\}$ is commuting on C.*

If S(C) is complete, then $F(S) \cap F(\mathcal{T}) \cap S(C)$ is a singleton.

Proof. Pick $x_0 \in C$ and $t > 0$. Since $T(t)(C) \subseteq S(C)$, we can construct a sequence $\{x_n\}$ in C such that $Sx_n = T(t)x_{n-1}$ for all $n \in \mathbb{N}$. Then

$$d(Sx_{n+1}, Sx_n) = d(T(t)x_n, T(t)x_{n-1}) \le k(t)d(Sx_n, Sx_{n-1}) \text{ for all } n \in \mathbb{N},$$

it follows that $\{Sx_n\}$ is a Cauchy sequence in $S(C)$.

Suppose $S(C)$ is complete. Then $Sx_n \to z$ for some $z \in S(C)$ and there exists $u \in C$ such that $z = Su$. By the S-contractivity of $\mathcal{T}$, we have

$$d(T(t)u, T(t)x_n) \le k(t)d(Su, Sx_n).$$

Taking limit as $n \to \infty$ yields

$$d(T(t)u, z) \le k(t)d(z, z) = 0.$$

Thus, $Su = T(t)u = z$. Since $\{S, T(t)\}$ is commuting on C, it follows that $Sz = T(t)z$. Note that

$$d(T(t)z, T(t)x_n) \quad \le \quad k(t)d(Sz, Sx_n).$$

Letting limit as $n \to \infty$, we obtain

$$d(T(t)z, z) \quad \le \quad k(t)d(Sz, z) = k(t)d(T(t)z, z).$$

It shows that $Sz = T(t)z = z$. Hence for each $r > 0$, there exists a unique element $z_r \in C$ such that $Sz_r = T(r)z_r = z_r$. For $s > 0$, we obtain that $T(s)z_r = T(s)T(r)z_r = T(r)T(s)z_r$ and $ST(s)z_r = T(s)Sz_r = T(s)z_r$. Therefore, there exists a unique point $z \in C$ such that $Sz = T(t)z = z$ for all $t > 0$. □

Corollary 3.3. *Let X be a metric space and let $S, T : C \to C$ be two mappings such that T is S-contraction, $T(C) \subseteq S(C)$, and $\{S, T\}$ is commuting on C. If $S(C)$ is complete, then $F(S) \cap F(T) \cap S(C)$ is a singleton.*

Using Proposition 3.2, we have

Proposition 3.4. *Let C be a nonempty subset of a normed space X. Let $S : C \to C$ be a mapping with $F(S) \neq \emptyset$ and let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous* *semigroup of asymptotically S-nonexpansive mappings from C into itself with function $k(\cdot)$ satisfying the conditions:*

(i) $S(C)$ is q-starshaped and S is q-affine with $q \in F(S)$,

(ii) for each $t > 0$, $T(t)(C) \subseteq S(C)$,

(iii) for each $t > 0$, $\{S, T(t)\}$ is commuting on C,

(iv) $S(C)$ is complete.

Then there exists exactly one point $x_n \in S(C)$ such that

$$x_n = Sx_n = (1 - \mu_n)q + \mu_n T(t_n)x_n, \ n \in \mathbb{N}, \tag{3.1}$$

where $\mu_n = \lambda_n / k(t_n)$, $\lambda_n \in (0, 1)$ and $t_n \in (0, \infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} t_n = \infty$.

Proof. Let $\{\lambda_n\}$ be a sequence in (0,1) such that $\lambda_n \to 1$ and let $\{t_n\}$ be a sequence in (0, ∞) such that $\lim_{n\to\infty} t_n = \infty$. For each $n \in \mathbb{N}$, define a mapping T_n by

$$T_n x = (1 - \mu_n)q + \mu_n T(t_n)x \text{ for all } x \in C.$$

Note that each $T_n : C \to C$ is an S-contraction on C. Indeed,

$$\|T_n x - T_n y\| \ = \ \mu_n \|T(t_n)x - T(t_n)y\| \leq \lambda_n \|Sx - Sy\| \text{ for all } x, y \in C.$$

Since $\{S, T(t_n)\}$ is commuting and S is q-affine, we have

$$ST_n x = (1 - \mu_n)q + \mu_n ST(t_n)x = (1 - \mu_n)q + \mu_n T(t_n)Sx = T_n Sx, \ x \in C.$$

Thus, the pair $\{S, T_n\}$ is commuting on C.

For $x \in C$ and $n \in \mathbb{N}$, we have $T(t_n)x \in T(t_n)(C) \subseteq S(C)$, i.e., there exists a point $y \in C$ such that $T(t_n)x = Sy \in S(C)$. Observe that

$$T_n x \ = \ (1 - \mu_n)q + \mu_n T(t_n)x = (1 - \mu_n)q + \mu_n Sy \in S(C).$$

It follows that $T_n(C) \subseteq S(C)$, for all $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we conclude that

$(i)'$ $T_n(C) \subseteq S(C)$,

$(ii)'$ T_n is S-contraction,

$(iii)'$ $\{S, T_n\}$ is commuting on C,

$(iv)'$ $S(C)$ is complete.

Therefore, Proposition 3.2 implies that there exists exactly one point $x_n \in S(C)$ such that $x_n = Sx_n = (1 - \mu_n)q + \mu_n T(t_n)x_n$. □

4. Existence

Before presenting main result of this section, we need the following:

Proposition 4.1 *Let C be a nonempty subset of a normed space $(X, \|\cdot\|)$. Let $S : C \to C$ be a weakly continuous mapping and let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of demicontinuous asymptotically S-nonexpansive mappings from C into itself. Let $\{t_n\}$ be a sequence in (0, ∞) such that $\lim_{n\to\infty} t_n = \infty$*

and let $\{y_n\}$ *be a sequence in* C *such that* $\lim_{n\to\infty} y_n = z$, $\lim_{n\to\infty}(y_n - Sy_n) = 0$ *and* $\lim_{n\to\infty}(y_n - T(t_n)y_n) = 0$. *Then* $z = Sz = T(t)z$ *for all* $t > 0$.

Proof. Since S is weakly continuous and $Sy_n \rightharpoonup z$, we obtain that $Sz = z$. Since $\lim_{n\to\infty} y_n = z$, it gives that $\lim_{n\to\infty} T(t_n)y_n = z$. By asymptotic S-nonexpansiveness of $T(t_n)$, we have

$$\|T(t_n)y_n - T(t_n)z\| \le k(t_n)\|Sy_n - Sz\| = k(t_n)\|Sy_n - z\|.$$

Taking limit as $n \to \infty$ yields $\lim_{n\to\infty} T(t_n)z = z$. Let $t > 0$. Note $T(t_n)z \rightharpoonup z$, it follows from the demicontinuity of $T(t)$ that $T(t)T(t_n)z \rightharpoonup T(t)z$. Observe that $T(t)T(t_n)z = T(t+t_n)z \rightharpoonup z$. By the uniqueness of weak limit of $\{T(t)T(t_z)z\}_{n\in N}$, we have $T(t)z = z$. Therefore, $T(t)z = Sz = z$ for all $t > 0$. □

Theorem 4.2. *Let* C *be a nonempty subset of a normed space* $(X, \|\cdot\|)$. *Let* $S : C \to C$ *be a mapping with* $F(S) \ne \emptyset$ *and let* $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ *be a strongly continuous semigroup of demicontinuous asymptotically S-nonexpansive mappings from* C *into itself with function* $k(\cdot)$ *satisfying the conditions (i)* $\sim$ *(iii) of Proposition 3.4. Suppose that* S *is weakly continuous and* $S(C)$ *is compact. Then there exists* $y^* \in S(C)$ *such that* $y^* = Sy^* = T(t)y^*$ *for all* $t > 0$.

Proof. By Proposition 3.4, there exists exactly one point $x_n \in S(C)$ such that $x_n = Sx_n = (1-\mu_n)q + \mu_n T(t_n)x_n$ for all $n \in \mathbb{N}$, where $\mu_n = \lambda_n/k(t_n)$, $\lambda_n \in (0,1)$ and $t_n \in (0,\infty)$ with $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} t_n = \infty$. By the compactness of $S(C)$, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $\lim_{m\to\infty} Sx_m = y \in S(C)$. Thus, $y = Su$ for some $u \in C$. The assumption (ii) implies that $\{T(t_m)x_m\}$ is bounded. It follows that

$$\|x_m - T(t_m)x_m\| \quad \le \quad (1-\lambda_m)\|q - T(t_m)x_m\| \to 0 \text{ as } m \to \infty.$$

By Proposition 4.1, we conclude that $T(t)y = Sy = y$ for all $t > 0$. □

Remark 4.3. The mapping S in Theorem 4.2 is not necessarily linear. Therefore, Theorem 4.2 improves Al-Thagafi [4, Theorem 2.2], Dotson [7, Theorem 1], Jungck [9, Theorem 3.1] and Shahzad [17, Lemma 2.2].

Theorem 4.4. *Let* C *be a nonempty closed convex subset of a uniformly convex Banach space* X, $S : C \to C$ *an affine weakly continuous mapping and* $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ *a strongly continuous semigroup of demicontinuous asymptotically S-nonexpansive mappings form* C *into itself. Then* $F(S) \cap F(\mathcal{T})$ *is closed and convex.*

Proof. Closedness of $F(S) \cap F(\mathcal{T})$:

Let $\mathcal{F} := F(S) \cap F(\mathcal{T})$. Let $\{z_n\}$ be a sequence in $\mathcal{F}$ such that $z_n \to z$. Then it remains to show that $z \in \mathcal{F}$. Note $z_n = Sz_n$ and S is weakly continuous, we have $z = Sz$. For $t \in \mathbb{R}^+$, we have

$$\|z_n - T(t)z\| = \|T(t)z_n - T(t)z\| \le k(t)\|z_n - z\|,$$

which implies that $z = T(t)z$.

Convexity of $F(T)$: Let $x, y \in F(S) \cap F(\mathcal{T})$ such that $x \neq y$. Let $z = \frac{1}{2}(x+y)$. Then, for $t > 0$, we have

$$\|T(t)z - x\| = \|T(t)z - T(t)x\| \leq k(t)\|Sz - Sx\| = k(t)\frac{1}{2}\|x - y\|.$$

Thus,

$$\begin{aligned} \|T(t)z - z\| &= \|\frac{1}{2}(T(t)z - x) + \frac{1}{2}(T(t)z - y)\| \\ &\leq k(t)(\frac{1}{2}\|x - y\|)\left\{1 - \delta_X\left(\frac{2}{k(t)}\right)\right\} \end{aligned}$$

for all $t \in \mathbb{R}^+$, where δ_X is the modulus of convexity of X. It follows that $\lim_{t\to\infty} T(t)z = z$ and hence, by demicontinuity of $T(r)$ $(r > 0)$, we have $T(r)z = z$. Thus, $z \in F(\mathcal{T})$. □

Corollary 4.5. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings form C into itself. Then $F(\mathcal{T})$ is closed and convex.*

5. Browder's type strong convergence theorem

The following result extends Browder's strong convergence theorem for S-nonexpansive mappings.

Theorem 5.1. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty subset of X, $S : C \to C$ a affine weakly continuous mapping with $F(S) \neq \emptyset$ such that $S(C)$ is closed and convex. Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of demicontinuous asymptotically S-nonexpansive mappings form C into itself with function $k(\cdot)$ and $T(t)(C) \subseteq S(C)$ for all $t > 0$. Let $\{x_n\}$ be a bounded sequence in $S(C)$ defined by (3.1). Here $\lambda_n \in (0,1)$ and $t_n \in (0,\infty)$ such that*

$$\lim_{n\to\infty} \lambda_n = 1, \quad \lim_{n\to\infty} t_n = \infty,$$

$$\lim_{n\to\infty} \frac{k(t_n) - 1}{k(t_n) - \lambda_n} = 0$$

and

$$\lim_{n\to\infty} \|x_n - T(t_m)x_n\| = 0 \text{ for all } m \in \mathbb{N}.$$

Then we have the following:

(a) $F(S) \cap F(\mathcal{T}) \neq \emptyset$

(b) $\lim_{n\to\infty} x_n = P_{F(S)\cap F(\mathcal{T})}(q)$, *where $P_{F(S)\cap F(\mathcal{T})}$ is a sunny nonexpansive retraction from $F(S)$ onto $F(S) \cap F(\mathcal{T})$.*

Proof. (a) Note X is a reflexive and $\{x_n\} \subset S(C)$ is bounded. Denote $\{x_n\}$ a subsequence of a sequence $\{x_n\}$ defined by (3.1) such that $x_n \rightharpoonup x \in S(C)$ since $S(C)$ is weakly closed. Since S is weakly continuous, $Sx_n \rightharpoonup Sx \in S(C)$. By $x_n = Sx_n$, we obtain that $x = Sx$.

Note $\lim_{n\to\infty} \|x_n - T(t_m)x_n\| = 0$ for all $m \in \mathbb{N}$, it follows that $T(t_m)x_n \rightharpoonup x$ for all $m \in \mathbb{N}$. Set

$$r_m := \limsup_{n\to\infty} \|T(t_m)x_n - x\|,\ m \in \mathbb{N}.$$

Let $m, s \in \mathbb{N}$. Since $T(t_m + t_s)x_n \rightharpoonup x$ as $n \to \infty$, by Opial's condition, we have

$$\begin{aligned} r_{m+s} = \limsup_{n\to\infty} \|T(t_m + t_s)x_n - x\| &< \limsup_{n\to\infty} \|T(t_m + t_s)x_n - T(t_s)x\| \\ &\leq \limsup_{n\to\infty} k(t_s)\|ST(t_m)x_n - Sx\| \\ &= \limsup_{n\to\infty} k(t_s)\|T(t_m)Sx_n - Sx\| \\ &= k(t_s)r_m. \end{aligned}$$

It follows that $\limsup_{s\to\infty} r_s \leq r_m$, which implies that

$$\limsup_{s\to\infty} r_s \leq \liminf_{m\to\infty} r_m.$$

Thus, $\lim_{m\to\infty} r_m$ exists. Suppose $\lim_{m\to\infty} r_m = r$ for some $r > 0$. Noting by Agarwal, O'Regan and Sahu [1, Theorem 2.5.23] that

$$\Phi(\|x + y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\varphi(x + ty)\rangle dt \text{ for all } x, y \in X.$$

For $m, s \in \mathbb{N}$, we have

$$\begin{aligned} \Phi(\|T(t_m + t_s)x_n - x\|) &= \Phi(\|T(t_m + t_s)x_n - T(t_m)x + T(t_m)x - x\|) \\ &= \Phi(\|T(t_m + t_s)x_n - T(t_m)x\|) \\ &\quad + \int_0^1 \langle T(t_m)x - x, J_\varphi(T(t_m + t_s)x_n - T(t_m)x \\ &\quad + t(T(t_m)x - x))\rangle dt \\ &\leq \Phi(k(t_m)\|ST(t_s)x_n - Sx\|) \\ &\quad + \int_0^1 \langle T(t_m)x - x, J_\varphi(T(t_m + t_s)x_n - T(t_m)x \\ &\quad + t(T(t_m)x - x))\rangle dt. \end{aligned}$$

Since $T(t_m + t_s)x_n \rightharpoonup x$ as $n \to \infty$, we obtain

$$\begin{aligned} \Phi(r_{m+s}) &= \Phi(\limsup_{n\to\infty} \|T(t_m + t_s)x_n - x\|) \\ &\leq \Phi(k(t_m)r_s) \\ &\quad - \int_0^1 \langle T(t_m)x - x, J_\varphi((1 - t)(T(t_m)x - x))\rangle dt \\ &\leq \Phi(k(t_m)r_s - \int_0^1 \|T(t_m)x - x\|\varphi(t\|T(t_m)x - x\|)dt \\ &= \Phi(k(t_m)r_s - \Phi(\|T(t_m)x - x\|), \end{aligned}$$

which implies that

$$\Phi(\|T(t_m)x - x\|) \quad \leq \quad \Phi(k(t_m)r_s) - \Phi(r_{m+s}).$$

Since $\lim_{s\to\infty} r_s$ exists, we have

$$\Phi(\|T(t_m)x - x\|) \quad \leq \quad \Phi(k(t_m)r) - \Phi(r) \to 0 \text{ as } m \to \infty.$$

By the demicontinuity of $T(t)$, we have

$$T(t_m)x \rightharpoonup x \Rightarrow T(t)T(t_m)x \rightharpoonup T(t)x \text{ for all } t > 0.$$

Note $T(t)T(t_m)x = T(t+t_m)x \rightharpoonup x$ for all $t > 0$. Thus, $T(t)x = x$ for all $t > 0$ and hence $F(S) \cap F(\mathcal{T})$ is nonempty.

(b) For $u \in F(S) \cap F(\mathcal{T})$, we have

$$\begin{aligned}
& \langle x_n - T(t_n)x_n, J_\varphi(x_n - u)\rangle \\
= \; & \langle x_n - u + T(t_n)u - T(t_n)x_n, J_\varphi(x_n - u)\rangle \\
= \; & \|x_n - u\|\varphi(\|x_n - u\|) - \langle T(t_n)x_n - T(t_n)u, J_\varphi(x_n - u)\rangle \\
\geq \; & \|x_n - u\|\varphi(\|x_n - u\|) - \|T(t_n)x_n - T(t_n)u\| \; \varphi(\|x_n - u\|) \\
\geq \; & \|x_n - u\|\varphi(\|x_n - u\|) - k(t_n)\|Sx_n - Su\| \; \varphi(\|x_n - u\|) \\
= \; & -(k(t_n) - 1)\|x_n - u\|\varphi(\|x_n - u\|). \qquad (5.1)
\end{aligned}$$

Since $x_n - T(t_n)x_n = \frac{k(t_n)-\lambda_n}{\lambda_n}(q - x_n)$, it follows from (5.1) that

$$\langle x_n - q, J_\varphi(x_n - u)\rangle \leq \lambda_n \frac{k(t_n) - 1}{k(t_n) - \lambda_n}\|x_n - u\|\varphi(\|x_n - u\|) \qquad (5.2)$$

for all $u \in F(S) \cap F(\mathcal{T})$. It is proved in part (a) that $x_n \rightharpoonup x \in F(S) \cap \cap_{t>0} F(T(t))$. Note $\{\|x_n - x\|\}$ is bounded. Using (5.2), we get

$$\langle x_n - q, J_\varphi(x_n - x)\rangle \leq \frac{k(t_n) - 1}{k(t_n) - \lambda_n} M \qquad (5.3)$$

for some constant $M \geq 0$. From (5.3), we have

$$\begin{aligned}
\|x_n - x\|^2 \; &= \; \langle x_n - x, J_\varphi(x_n - x)\rangle \\
&= \; \langle x_n - q, J_\varphi(x_n - x)\rangle + \langle q - x, J_\varphi(x_n - x)\rangle \\
&\leq \; \frac{k(t_n) - 1}{k(t_n) - \lambda_n} M + \langle q - x, J_\varphi(x_n - x)\rangle. \qquad (5.4)
\end{aligned}$$

Since J_φ is weakly continuous, it follows from (5.4) that $x_n \to x$ as $n \to \infty$.

In summary, we have proved that $\{x_n\}$ is sequentially compact and each cluster point of $\{x_n\}$ $(n \to \infty)$ equals x. Therefore, $x_n \to x$ as $n \to \infty$.

(c) Set $\mathcal{F} := F(S) \cap F(\mathcal{T})$. We can define a mapping $P_{\mathcal{F}}$ from $F(S)$ onto $\mathcal{F}$ by $\lim_{n\to\infty} x_n = P_{\mathcal{F}} v$, since v is an arbitrary point of $F(S)$. From (5.3), we know that

$$\langle v - P_{\mathcal{F}} v, J_\varphi(x - P_{\mathcal{F}} v)\rangle \leq 0 \quad \text{for all } v \in F(S) \text{ and } x \in \mathcal{F}.$$

This proves that $P_{\mathcal{F}}$ is a sunny nonexpansive retraction from $F(S)$ onto $\mathcal{F}$ by Lemma 2.1. □

Corollary 5.2. *Let X be a reflexive Banach space with a weakly continuous duality mapping $J_\varphi : X \to X^*$ with gauge function φ. Let C be a nonempty subset of X, $S : C \to C$ a affine weakly continuous mapping with $F(S) \neq \emptyset$ such that $S(C)$ is* *closed and convex. Let* $\mathcal{T} = \{T(t) : t \in \mathbb{R}^+\}$ *be a strongly continuous semigroup of demicontinuous S-nonexpansive mappings form* C *into itself and* $T(t)(C) \subseteq S(C)$ *for all* $t > 0$. *Assume that* $\lambda_n \in (0,1)$ *and* $t_n \in (0,\infty)$ *such that* $\lim_{n\to\infty} \lambda_n = 1$ *and* $\lim_{n\to\infty} t_n = \infty$. *Then we have the following:*

(a) *There exists sequence* $\{x_n\}$ *in* $S(C)$ *such that*

$$x_n = Sx_n = (1-\lambda_n)q + \lambda_n T(t_n)x_n \text{ for all } n \in \mathbb{N}.$$

(b) *If* $\{x_n\}$ *is bounded and* $\lim_{n\to\infty} \|x_n - T(t_m)x_n\| = 0$ *for all* $m \in \mathbb{N}$, *then* $F(S) \cap F(\mathcal{T}) \neq \emptyset$

(c) $\lim_{n\to\infty} x_n = P_{F(S)\cap F(\mathcal{T})}(q)$, *where* $P_{F(S)\cap F(\mathcal{T})}$ *is a sunny nonexpansive retraction from* $F(S)$ *onto* $F(S) \cap F(\mathcal{T})$.

Theorem 5.1 generalizes Schu [16, Theorem 1.7] from the class of asymptotically nonexpansive mappings to a semigroup of demicontinuous asymptotically S-nonexpansive mappings. Corollary 5.2 is a generalization Browder [2, Theorem 2], who assumes in addition that X is uniformly convex and that T is a nonexpansive mapping defined in the whole space X.

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