Fixed Point Theory, 11(2010), No. 2, 369-374
http://www.math.ubbcluj.ro/~nodeacj/sfptcj.html

ITERATES OF STANCU OPERATORS (VIA FIXED POINT PRINCIPLES) REVISITED

IOAN A. RUS

Babeş-Bolyai University, Department of Applied Mathematics
Kogălniceanu Street, No. 1, 400084, Cluj-Napoca, Romania
E-mail: iarus@math.ubbcluj.ro

Abstract. In this paper we use some fixed point principles to study the convergence of the iterates of Stancu operators. To do this we shall investigate the existence of an invariant expression and a Volterra interval for some classes of positive linear operators on $C[0,1]$, preserving constant and only constant functions. Stancu's operators are relevant examples of such operators.
Key Words and Phrases: Fixed point, weakly Picard operator, linear positive operator, invariant expression, Volterra interval.
2010 Mathematics Subject Classification: 47H10, 41A36, 54H25.

1. Introduction

Let $\alpha, \beta \in \mathbb{R}$, $0 \le \alpha \le \beta$ and $n \in \mathbb{N}^*$. In the paper [9], D.D. Stancu studied the following linear positive operators (see also [1] and [2])

$$S_{n,\alpha,\beta} : C[0,1] \to C[0,1]$$

defined by

$$S_{n,\alpha,\beta}(f)(x) := \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

In [7] we established the following results on the convergence of the sequence of iterates of Stancu operators.

Theorem 1.1. *Let $n \in \mathbb{N}^*$ and $\beta > 0$. Then for all $f \in C[0,1]$*

$$S^m_{n,0,\beta}(f)(x) \to f(0) \text{ as } m \to \infty,$$

uniformly with respect to $x \in [0, \frac{n}{n+\beta}]$.

Theorem 1.2. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$. Then for all $f \in C[0,1]$*

$$S^m_{n,\alpha,\alpha}(f)(x) \to f(1) \text{ as } m \to \infty,$$

uniformly with respect to $x \in [\frac{\alpha}{n+\alpha}, 1]$.

Relative to the above results the authors of the paper [3] have the following opinion: ,, ... due to the fact that the operators considered neither reproduce linear functions nor interpolate the function at the endpoints, the results formulated by Rus were limited to proper compact subinterval of $[0,1]$." On the other hand we established in [6] the following abstract result (see also [8]):

Theorem 1.3. *Let X be a nonempty set and $A : X \to X$ be an operator. The following statements are equivalent:*

(i) $F_A = F_{A^n} \neq \emptyset$, $\forall n \in \mathbb{N}^*$;

(ii) *there exists an L-space structure on X, $\to$, such that, $A : (X,\to) \to (X,\to)$ is weakly Picard operator;*

(iii) *there exists a metric d on X such that, $A : (X,d) \to (X,d)$ is weakly Picard operator;*

(iv) *there exists $\alpha \in]0,1[$, a complete metric d on X and a partition of X, $X = \bigcup\limits_{\lambda\in\Lambda} X_\lambda$, such that:*
  (a) $A(X_\lambda) \subset X_\lambda$, $F_A \cap X_\lambda = \{x_\lambda^*\}$, $\forall \lambda \in \Lambda$;
  (b) $A\big|_{X_\lambda} : X_\lambda \to X_\lambda$ *is an α-contraction with respect to d;*

(v) *there exists $\alpha \in]0,1[$ and a complete metric d on X such that:*
  (a) $A : (X,d) \to (X,d)$ *is orbitally continuous;*
  (b) $d(A^2(x), A(x)) \leq \alpha d(x, A(x))$, $\forall x \in X$.

The aim of this paper is to use this abstract result, as an intuition, to realize a complete study of the convergence of the iterates of Stancu's operators. To do this we shall study the existence of an invariant expression and of a Volterra interval for these operators.

2. Invariant expression of positive linear operators preserving constant and only constant functions

Let $\varphi_k \in C([0,1],\mathbb{R}_+)$, $k = \overline{0,n}$ and $0 \leq a_0 < a_1 < \ldots < a_{n-1} < a_n \leq 1$. We suppose that:

(1_φ) $\{\varphi_0, \ldots, \varphi_n\}$ is linearly independent;

(2_φ) $\sum\limits_{k=0}^{n} \varphi_k(x) = 1$, $\forall x \in [0,1]$.

Let us consider the following linear positive operator

$$A : C[0,1] \to C[0,1],\ A(f) := \sum_{k=0}^{n} f(a_k)\varphi_k.$$

From (2_φ) it follows that: the constant functions are fixed points of the operator A. Moreover we have

Lemma 2.1. *Let us suppose that the conditions (1_φ) and (2_φ) are satisfied. Then the following statements are equivalent:*

$(3_{\varphi,a})$ $f \in C[0,1]$, $A(f) = f \Rightarrow f$ *is a constant function;*

$(4_{\varphi,a})$ $rank\big([\varphi_i(a_k)] - I_{n+1}\big) = n$.

Proof. First we remark that if f is a fixed point of A, then $f = \sum_{i=0}^{n} p_i\varphi_i$, with $p_i \in \mathbb{R}$, $i = \overline{0,n}$. Since $p_0 = 1, \dots, p_n = 1$ is a solution, we get that $rank\big([\varphi_i(a_k)] - I_{n+1}\big) \le n$. Thus, we have $(3_{\varphi,a})$ iff we have $(4_{\varphi,a})$. $\square$

Definition 2.1. *Let $c = (c_0, \dots, c_n) \in \mathbb{R}^{n+1}$, $c \neq 0$. By definition the expression, $\sum_{i=0}^{n} c_i f(a_i)$, is an invariant expression by A iff*

$$\sum_{i=0}^{n} c_i A(f)(a_i) = \sum_{i=0}^{n} c_i f(a_i),\ \forall f \in C[0,1].$$

Lemma 2.2. *Let the conditions (1_φ), (2_φ) and $(3_{\varphi,a})$ be satisfied. Then there exists a unique $c^* \in \mathbb{R}^{n+1}$, such that:*

(1) $c^* \ge 0$, $\sum_{i=0}^{n} c_i^* = 1$;

(2) $\sum_{i=0}^{n} c_i^* A(f)(a_i) = \sum_{i=0}^{n} c_i^* f(a_i)$, $\forall f \in C[0,1]$.

Proof. Let us determine $c \in \mathbb{R}^{n+1}$, $c \neq 0$, such that

(1_I) $\sum_{i=0}^{n} c_i A(f)(a_i) = \sum_{i=0}^{n} c_i f(a_i)$, $\forall f \in C[0,1]$.

It is clear that we have (1_I) if and only if

(2_I) $\sum_{i=0}^{n} c_i \varphi_k(a_i) = c_k$, $k = \overline{0,n}$.

Let $K := \{c \in \mathbb{R}^{n+1} \mid c_i \ge 0, \sum_{i=0}^{n} c_i = 1\} \subset \mathbb{R}^{n+1}$. We consider the function

$$T : K \to K,\ T(c) := \Big(\sum_{i=0}^{n} c_i\varphi_0(a_i), \dots, \sum_{i=0}^{n} c_i\varphi_n(a_i)\Big).$$

The invariance of K with respect to T follows from the fact that the matrix $[\varphi_k(a_i)]$ is a stochastic matrix (see, for example, Chapter 9 in [4]). From the Brouwer fixed point theorem there exists $c^* \in K$ such that $T(c^*) = c^*$. From Lemma 2.1 it follows that there exists such a unique fixed point. $\square$

Remark 2.1. *If $\varphi_k(a_i) > 0$, for all $k, i \in \{0, 1, \dots, n\}$, then $c_i^* > 0$ for $i = \overline{0,n}$.*

Remark 2.2. *If A is as in Lemma 2.2 and is a weakly Picard operator in $(C[0,1], \xrightarrow{unif})$ (see [8]), then*

$$A^\infty(f) = \sum_{i=0}^{n} c_i^* f(a_i).$$

Indeed, from (1_I) it follows that

$$\sum_{i=0}^{n} c_i^* A^m(f)(a_i) = \sum_{i=0}^{n} c_i^* f(a_i),\ \forall m \in \mathbb{N}^*,$$

which implies that

$$\sum_{i=0}^{n} c_i^* A^\infty(f)(a_i) = \sum_{i=0}^{n} c_i^* f(a_i).$$

Since $A^\infty \in F_A$, we have that $A^\infty(f)$ is a constant function and $A^\infty(f) = \sum_{i=0}^{n} c_i^* f(a_i)$.

Now the problem to study is the following one: In which conditions on φ_i and a_i, $i = \overline{0,n}$, the operator A is weakly Picard operator?

3. Operators with a Volterra interval and with values in Π_n

Let $E \subset \mathbb{R}$ be an interval. We denote by $\Pi_n(E)$ the set of all polynomial functions on E with values in $\mathbb{R}$. Let $A : C[a,b] \to C[a,b]$ be an operator.

Definition 3.1. *A non-degenerate interval $E \subset [a,b]$ is a Volterra interval for A iff $f, g \in C[a,b]$, $f\big|_E = g\big|_E \Rightarrow A(f) = A(g)$.*

If A is an operator with a Volterra interval E, then we define the operator $A_E : C(E) \to C(E)$ by $A_E(f) := A(\tilde{f})$, where $\tilde{f} \in C[0,1]$ such that $\tilde{f}\big|_E = f$.

Example 3.1. *For the Stancu operators, $S_{n,0,\beta}$, the interval $E := [0, \frac{n}{n+\beta}]$ is a Volterra interval.*

Example 3.2. *For the Stancu operators, $S_{n,\alpha,\alpha}$, the interval $E := [\frac{\alpha}{n+\alpha}, 1]$ is a Volterra interval.*

We have

Lemma 3.1. *Let $A : C[a,b] \to C[a,b]$ be an operator. We suppose that:*

(i) *the operator A has a Volterra interval $E \subset [a,b]$;*
(ii) *$A(C[a,b]) \subset \Pi_n[a,b]$.*

Then the following statements are equivalent:

(a) *A is weakly Picard operator on $(C[a,b], \stackrel{unif}{\longrightarrow})$;*
(b) *A_E is weakly Picard operator on $(C(E), \stackrel{unif}{\longrightarrow})$.*

4. Iterates of the operator A in §2

It is well known that if a linear positive operator $A : C[0,1] \to C[0,1]$ is with $F_A = \Pi_1[0,1]$ then $A(f)(0) = f(0)$ and $A(f)(1) = f(1)$, for all $f \in C[0,1]$ (see, for example [5]). For the operator A in §2 in the case that $F_A = \Pi_0[0,1]$ we have

Theorem 4.1. *We suppose that the conditions (1_φ), (2_φ), $(3_{\varphi,a})$ and the following are satisfied:*

$(5_{\varphi,a})$ $\varphi_i(x) > 0$ *for all* $i = \overline{0,n}$ *and* $x \in [a_0, a_n]$*;*
(6_φ) φ_i*,* $i = \overline{0,n}$*, are polynomial functions.*

Then:

(a) *there exists in* K *(see the proof of Lemma 2.2) a unique* c^* *such that*

$$\sum_{i=0}^{n} c_i^* A(f)(a_i) = c_i^* f(a_i),\ \forall f \in C[0,1].$$

Moreover, $c^* \in \overset{\circ}{K}$*.*

(b) A *is weakly Picard operator on* $(C[0,1], \xrightarrow{unif})$ *and* $A^\infty(f) = \sum_{i=0}^{n} c_i^* f(a_i)$*.*

Proof. From Lemma 2.2 it follows (a). Let us prove (b). To do this we consider the Banach space $(C[0,1], \|\cdot\|_\infty)$. Let for $\alpha \in \mathbb{R}$,

$$X_\alpha := \{f \in C[0,1] \mid \sum_{i=0}^{n} c_i^* f(a_i) = \alpha\}.$$

We remark that:

(1) X_α is a closed subset of $C[0,1]$, $\forall \alpha \in \mathbb{R}$;
(2) $A(X_\alpha) \subset X_\alpha$, $\forall \alpha \in \mathbb{R}$;
(3) $C[0,1] = \bigcup_{\alpha \in \mathbb{R}} X_\alpha$, is a partition of $C[0,1]$.

Now we shall prove that the restriction of A to X_α, $A\big|_{X_\alpha} : X_\alpha \to X_\alpha$ is a Picard operator. Since φ_i, $i = \overline{0,n}$, are polynomial functions, it is sufficient (see Lemma 3.1) to prove that $(A^m(f))_{m\in\mathbb{N}^*}$ converges on $[a_0, a_n]$. Let $Y_\alpha := \{f\big|_{[a_0,a_n]} \mid f \in X_\alpha\}$. Let $I := [a_0, a_n]$. Let $A_I : Y_\alpha \to Y_\alpha$ be defined by $A_I(f)(x) := A(f)(x),\ x \in I$.
Now we shall prove that the operator A_I is a contraction with respect to $d_{\|\cdot\|_\infty}$. Since $f, g \in Y_\alpha$ implies that $f - g \in Y_0$, one need to prove that there exists $l \in]0,1[$ such that $\|A_I(f)\|_\infty \le l\|f\|_\infty,\ \forall f \in Y_0$. Since $\sum_{i=0}^{n} c_i^* f(a_i) = 0$, for $f \in Y_0$ and $c_i^* > 0$, we have $|A_I(f)(x)| = \left|\sum_{k=0}^{n} f(a_k)\varphi_k(x)\right| \le \max_{\substack{x\in I\\ 0\le k\le n}} \big(1 - \varphi_k(x)\big)\|f\|_\infty$. Thus,

$$\|A_I(f)\|_\infty \le l\|f\|_\infty,\ \forall f \in Y_0,\ \text{with } l := \max_{\substack{x\in I\\ 0\le k\le n}} \big(1 - \varphi_k(x)\big).$$

□

5. Iterates of Stancu operators

From Theorem 1.1 and the Lemma 3.1, we have the following theorem.

Theorem 5.1. *Let* $n \in \mathbb{N}^*$ *and* $\beta > 0$*. Then the operator* $S_{n,0,\beta}$ *is weakly Picard operator on* $(C[0,1], \xrightarrow{unif})$ *and*

$$S_{n,0,\beta}^\infty(f) = f(0),\ \forall f \in C[0,1].$$

From Theorem 1.2 and the Lemma 3.1 we have the folloing result.

Theorem 5.2. *Let $n \in \mathbb{N}^*$ and $\alpha > 0$. Then the operator $S_{n,\alpha,\alpha}$ is weakly Picard operator on $(C[0,1], \stackrel{unif}{\longrightarrow})$ and*

$$S^{\infty}_{n,\alpha,\alpha}(f) = f(1), \ \forall f \in C[0,1].$$

For the case $0 < \alpha < \beta$ we have the following result.

Theorem 5.3. *The operator $S_{n,\alpha,\beta}$ is weakly Picard operator on $(C[0,1], \stackrel{unif}{\longrightarrow})$ and*

$$S^{\infty}_{n,\alpha,\beta}(f) = \sum_{i=0}^{n} c_i^* f\left(\frac{i+\alpha}{n+\beta}\right),$$

where $c^* \in \mathbb{R}^{n+1}$ is the unique solution in K of the following system

$$\sum_{i=0}^{n} c_i \left(\frac{i+\alpha}{n+\beta}\right)^n \left(1 - \frac{i+\alpha}{n+\beta}\right)^{n-k} = c_k, \ k = \overline{0,n}.$$

Proof. With $a_i := \frac{i+\alpha}{n+\beta}$ and $\varphi_i(x) := x^i(1-x)^{n-i}$, $i = \overline{0,n}$, we are in the conditions of Theorem 4.1. □

Remark 5.1. *By some simple calculations we have*

$$S^{\infty}_{1,\alpha,\beta}(f) = \left(1 - \frac{\alpha}{\beta}\right) f\left(\frac{\alpha}{1+\beta}\right) + \frac{\alpha}{\beta} f\left(\frac{1+\alpha}{1+\beta}\right).$$

Thus, Theorem 5.1 corrects Proposition 2.5.5 in [2].

Remark 5.2. *Let $D \subset \mathbb{R}^p$ be a compact set with $\mathring{D} \neq \emptyset$. In a joint paper with O. Agratini, we extend the above results for a linear positive operator A on $(C(D), \stackrel{unif}{\longrightarrow})$ with $F_A = \Pi_0(D)$. For the case of a sequence $(A_n)_{n\in\mathbb{N}}$ of linear positive operators with $A_n \to 1_{C(D)}$ as $n \to \infty$ we also study the limit of the coupled iterates A_n^m.*

References

[1] O. Agratini, *Aproximare prin operatori liniari*, Presa Univ. Clujeană, Cluj-Napoca, 2000.

[2] F. Altomare, M. Campiti, *Korovkin-type Approximation Theory and its Applications*, de Gruyter, Berlin, 1994.

[3] H. Gonska, P. Piţul, I. Raşa, *Over-iterates of Bernstein-Stancu operators*, Calcolo, **44** (2007), 117-125.

[4] L. Hogben (ed.), *Handbook of Linear Algebra*, Chapman and Hall / CRC, 2007.

[5] I. Raşa, *Positive linear operators preserving linear functions*, Ann. T. Popoviciu Seminar of Funct. Eq. Approx. Conv., **7** (2009), 105-109.

[6] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, **34** (1993), No. 4, 769-773.

[7] I.A. Rus, *Iterates of Stancu operators, via contraction principle*, Studia Univ. Babeş-Bolyai, Math., **47** (2002), No. 4, 101-104.

[8] I.A. Rus, *Picard operators and applications*, Sc. Math. Japonicae, **58** (2003), No. 1, 191-219.

[9] D.D. Stancu, *Asupra unei generalizări a polinoamelor lui Bernstein*, Studia Univ. Babeş-Bolyai, **14** (1969), No. 2, 31-45.

Received: November 21, 2009; Accepted: March 10, 2010.