A STRONG QUADRATIC FUNCTIONAL EQUATION IN
$C^\ast$-ALGEBRAS

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Abstract. In this paper, we use a fixed point method to investigate the problem of stability on
$C^\ast$-algebras of the strong quadratic functional equation

$$f(x) + f(y) = f(\sqrt{xx^* + yy^*}).$$

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ralized metric space, fixed point.

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1. Introduction and preliminaries

In 1940, S.M. Ulam [41] posed the following question concerning the stability of
group homomorphisms: Under what conditions does there exist a group homomor-
phism near an approximately group homomorphism?

In 1941, D.H. Hyers [15] considered the case of approximately additive functions
$f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon, \text{ for all } x, y \in E.$$ (1.1)

for additive and linear mappings, respectively, by allowing the Cauchy difference to
be unbounded (see also [4]).

Theorem 1.1. (Th.M. Rassias). Let $f : E \to E'$ be a mapping from a normed vector
space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$ (1.1)

for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$.

Then the limit $L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the
unique additive mapping which satisfies, for all $x \in E$, the relation

$$\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} ||x||^p.$$ (1.2)
If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then $L$ is linear.

Theorem 1.1 has been generalized by G.L. Forti [11, 12] and P. Găvruta [13] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [5], [6], [7], [10], [14], [18], [19], [21], [23], [24], [26]-[34] and [36]-[38]). We also refer the readers to the books [1], [8], [17], [22] and [39].

The functional equation
\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \]  
(1.3)

is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.3) is said to be a quadratic function. Quadratic functional equations were used to characterize inner product spaces. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function $B$ such that $f(x) = B(x, x)$ for all $x$ (see [1, 2, 20, 24]). The bi-additive function $B$ is given by
\[ B(x, y) = \frac{1}{4} \left[ f(x + y) - f(x - y) \right]. \]

The Hyers-Ulam stability problem for the quadratic functional equation (1.3) was proved by Skof [40] for functions $f : E_1 \to E_2$, where $E_1$ is a normed space and $E_2$ is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group. In the paper [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.3). Grabiec [14] has generalized these results mentioned above. Jun and Lee [21] proved the generalized Hyers-Ulam stability of a Pexiderized quadratic equation.

Let $E$ be a set. A function $d : E \times E \to [0, \infty]$ is called a generalized metric on $E$ if $d$ satisfies the usual axioms of a metric.

We recall the following theorem by Margolis and Diaz.

**Theorem 1.2.** [25] Let $(E, d)$ be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either $d(J^n x, J^{n+1} x) = \infty$ for all non-negative integers $n$ or there exists a non-negative integer $n_0$ such that

1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. the sequence $\{J^n x\}$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{ y \in E : d(J^{n_0} x, y) < \infty \}$;
4. $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Throughout this paper $A$ will be a $C^*$-algebra. We denote by $\sqrt{a}$ the unique positive element $b \in A$ such that $b^2 = a$. Also, we denote by $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Q}$ the set of real, complex and rational numbers, respectively. In this paper, we use a fixed point
method (see [5, 23, 26]) to investigate the problem of stability of the strong quadratic (or simply s-quadratic) functional equation
\[ f(x) + f(y) = f(\sqrt{xx^* + yy^*}) \] (1.4)
on $C^*$-algebras. In particular, every solution of the s-quadratic equation (1.4) is said to be a s-quadratic function. For some results on fixed point theorems in nonlinear analysis we refer the reader to [9, 16, 19, 42].

2. Solutions of Eq. (1.4)

**Theorem 2.1.** Let $X$ be a linear space. If a function $f : A \to X$ satisfies the functional equation (1.4), then $f$ is quadratic.

**Proof.** Letting $x = y = 0$ in (1.4), we get $f(0) = 0$. Replacing $x$ and $y$ by $x + y$ and $x - y$ in (1.4), respectively, we get
\[ f(x + y) + f(x - y) = f(\sqrt{2xx^* + 2yy^*}) \] (2.1)
for all $x, y \in A$. It follows from (1.4) that $f(\sqrt{2x}) + f(\sqrt{2y}) = f(\sqrt{2xx^* + 2yy^*})$ for all $x, y \in A$. Therefore we have from (2.1) that
\[ f(x + y) + f(x - y) = f(\sqrt{2x}) + f(\sqrt{2y}) \] (2.2)
for all $x, y \in A$. Setting $y = 0$ in (2.2), we get
\[ f(\sqrt{2}) = 2f(x) \] (2.3)
for all $x \in A$. It follows from (2.2) and (2.3) that $f(x + y) + f(x - y) = 2f(x) + 2f(y)$ for all $x \in A$. Hence $f$ is quadratic. □

**Remark 2.1.** Let $f : A \to A$ be the mapping defined by $f(x) = x^2$ for all $x \in A$. It is clear that $f$ is quadratic. Let $a \neq 0$ be a positive element of $A$. Hence $f$ does not satisfy in (1.4) for $x = y = i\sqrt{a}$. Therefore $f$ is not s-quadratic.

**Corollary 2.2.** Let $X$ be a linear space. If a function $f : A \to X$ satisfies the functional equation (1.4), then there exists a symmetric bi-additive function $B : A \times A \to X$ such that $f(x) = B(x, x)$ for all $x \in A$.

3. Generalized Hyers-Ulam stability of Eq. (1.4) on $C^*$-algebras

In this section, we use a fixed point method (see [5, 23, 26]) to investigate the problem of stability of the functional equation (1.4) on $C^*$-algebras. For convenience, we use the following abbreviation for a given function $f : A \to X$:
\[ Df(x, y) := f(x) + f(y) - f(\sqrt{xx^* + yy^*}) \]
for all $x, y \in A$, where $X$ is a normed linear space.

**Theorem 3.1.** Let $X$ be a linear space and let $f : A \to X$ be a function with $f(0) = 0$ for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that
\[ \|Df(x, y)\| \leq \varphi(x, y) \] (3.1)
for all $x, y \in A$. If there exists a constant $0 < L < 1$ such that
\[ \varphi(2x, 2y) \leq 4L\varphi(x, y) \] (3.2)
for all \( x, y \in A \), then there exists a unique \( s \)-quadratic function \( Q : A \to X \) such that
\[
\| f(x) - Q(x) \| \leq \frac{1}{4 - 4L} \phi(x)
\] (3.3)
for all \( x \in A \), where \( \phi(x) := \varphi(\sqrt{2}x, \sqrt{2}x) + \varphi(2x, 0) + 2\varphi(\sqrt{2}x, 0) + 2\varphi(x, x) \).

Moreover, if \( f(x) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then \( Q \) is \( \mathbb{R} \)-quadratic, i.e., \( Q(tx) = t^2 Q(x) \) for all \( x \in A \) and all \( t \in \mathbb{R} \).

**Proof.** It follows from (3.1) and (3.2) that
\[
\| f(\sqrt{2}x) + f(\sqrt{2}y) - f(\sqrt{2}x^2 + 2yy^2) \| \leq \varphi(\sqrt{2}x, \sqrt{2}y),
\] (3.4)
\[
\lim_{k \to \infty} \frac{1}{4^k} \varphi(2^k x, 2^k y) = 0
\] (3.5)
for all \( x, y \in A \). Replacing \( x \) and \( y \) by \( x + y \) and \( x - y \) in (3.1), respectively, we get
\[
\| f(x + y) + f(x - y) - f(\sqrt{2}x) - f(\sqrt{2}y) \| \leq \varphi(x + y, x - y)
\] (3.6)
for all \( x, y \in A \). It follows from (3.4) and (3.6) that
\[
\| f(x + y) + f(x - y) - f(\sqrt{2}x) - f(\sqrt{2}y) \| \\
\leq \varphi(\sqrt{2}x, \sqrt{2}y) + \varphi(x + y, x - y)
\] (3.7)
for all \( x, y \in A \). Letting \( y = 0 \) in (3.7), we get
\[
\| 2f(x) - f(\sqrt{2}x) \| \leq \varphi(\sqrt{2}x, 0) + \varphi(x, x)
\] (3.8)
for all \( x \in A \). Therefore we have from (3.7) and (3.8) that
\[
\| f(x + y) + f(x - y) - 2f(x) - 2f(y) \| \\
\leq \varphi(\sqrt{2}x, \sqrt{2}y) + \varphi(x + y, x - y)
\] (3.9)
\[+ \varphi(\sqrt{2}x, 0) + \varphi(x, x) + \varphi(\sqrt{2}y, 0) + \varphi(y, y)
\]
for all \( x, y \in A \). Setting \( x = y \) in (3.9), we get
\[
\| f(2x) - 4f(x) \| \leq \phi(x)
\] (3.10)
for all \( x \in A \). By (3.2) we have \( \phi(2x) \leq 4L \phi(x) \) for all \( x \in A \). Let \( E \) be the set of all functions \( g : A \to X \) with \( g(0) = 0 \) and introduce a generalized metric on \( E \) as follows:
\[
d(g, h) := \inf \{ C \in [0, \infty] : \| g(x) - h(x) \| \leq C \phi(x) \quad \text{for all} \ x \in A \}.
\]

It is easy to show that \( (E, d) \) is a generalized complete metric space [5].

Now we consider the function \( A : E \to E \) defined by
\[
(Ag)(x) = \frac{1}{4} g(2x), \quad \text{for all} \ g \in E \text{ and } x \in A.
\]
Let \( g, h \in E \) and let \( C \in [0, \infty] \) be an arbitrary constant with \( d(g, h) \leq C \). From the definition of \( d \), we have \( \| g(x) - h(x) \| \leq C \phi(x) \), for all \( x \in A \). By the assumption and the last inequality, we have
\[
\|(Ag)(x) - (Ah)(x)\| = \frac{1}{4} \| g(2x) - h(2x) \| \leq \frac{1}{4} C \phi(2x) \leq C \phi(x), \quad \text{for all} \ x \in A.
\]
Thus \( d(Ag, Ah) \leq Ld(g, h) \), for any \( g, h \in E \). It follows from (3.10) that \( d(Af, f) \leq \frac{1}{4} \). Therefore according to Theorem 1.2, the sequence \( \{A^k f\} \) converges to a fixed point \( Q \) of \( \Lambda \), i.e.,

\[
Q : A \to X, \quad Q(x) = \lim_{k \to \infty} (A^k f)(x) = \lim_{k \to \infty} \frac{1}{4^k} f(2^k x)
\]

and \( Q(2x) = 4Q(x) \) for all \( x \in A \). Also \( Q \) is the unique fixed point of \( \Lambda \) in the set \( E^* = \{ g \in E : d(f, g) < \infty \} \) and

\[
d(Q, f) \leq \frac{1}{1 - L} d(Af, f) \leq \frac{1}{4 - 4L},
\]

i.e., inequality (3.3) holds true for all \( x \in A \). It follows from the definition of \( Q \), (3.1) and (3.5) that

\[
\left\| DQ(x, y) \right\| = \lim_{k \to \infty} \frac{1}{4^k} \left\| Df(2^k x, 2^k y) \right\| \leq \lim_{k \to \infty} \frac{1}{4^k} \varphi(2^k x, 2^k y) = 0
\]

for all \( x, y \in A \). So \( Q \) is \( s \)-quadratic. By Theorem 2.1, the function \( Q : A \to X \) is quadratic. Finally it remains to prove the uniqueness of \( Q \). Let \( T : A \to X \) be another \( s \)-quadratic function satisfying (3.3). Since \( d(f, T) \leq \frac{1}{1 - ||T||} \) and \( T \) is quadratic, we get \( T \in E^* \) and \( (\Lambda T)(x) = \frac{1}{2} T(2x) = T(x) \) for all \( x \in A \), i.e., \( T \) is a fixed point of \( \Lambda \). Since \( Q \) is the unique fixed point of \( \Lambda \) in \( E^* \), then \( T = Q \). Moreover, if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then by the same reasoning as in the proof of [35] \( Q \) is \( \mathbb{R} \)-quadratic.

**Corollary 3.2.** Let \( 0 < r < 2 \) and \( \theta, \delta \) be non-negative real numbers and let \( f : A \to X \) be a function with \( f(0) = 0 \) such that

\[
\left\| Df(x, y) \right\| \leq \delta + \theta(\|x\|^r + \|y\|^r)
\]

for all \( x, y \in A \). Then there exists a unique \( s \)-quadratic function \( Q : A \to X \) such that

\[
\left\| f(x) - Q(x) \right\| \leq \frac{6\delta}{4 - 2^r} + \frac{4 + 4(\sqrt{2})^r + 2^r}{4 - 2^r} \theta \|x\|^r
\]

for all \( x \in A \). Moreover, if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then \( Q \) is \( \mathbb{R} \)-quadratic.

The following theorem is an alternative result of Theorem 3.1 and we leave its proof to the reader.

**Theorem 3.3.** Let \( f : A \to X \) be a function for which there exists a function \( \varphi : A^2 \to [0, \infty) \) satisfying (3.1) for all \( x, y \in A \). If there exists a constant \( 0 < L < 1 \) such that

\[
4\varphi(x, y) \leq L\varphi(2x, 2y)
\]

for all \( x, y \in A \), then there exists a unique \( s \)-quadratic function \( Q : A \to X \) such that

\[
\left\| f(x) - Q(x) \right\| \leq \frac{L}{4 - 4L}\varphi(x)
\]

for all \( x \in A \), where \( \varphi(x) \) is defined as in Theorem 3.1. Moreover, if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then \( Q \) is \( \mathbb{R} \)-quadratic.
Corollary 3.4. Let \( r > 2 \) and \( \theta \) be non-negative real numbers and let \( f : A \to X \) be an even function such that
\[
\| Df(x, y) \| \leq \theta (\| x \|^r + \| y \|^r)
\]
for all \( x, y \in A \). Then there exists a unique \( s \)-quadratic function \( Q : A \to X \) such that
\[
\| f(x) - Q(x) \| \leq \frac{4 + 4(\sqrt{2})^r + 2^r}{2r - 4} \theta \| x \|^r
\]
for all \( x \in A \). Moreover, if \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then \( Q \) is \( \mathbb{R} \)-quadratic.

For the case \( r = 2 \) we have the following counterexample which is a modification of the example of S. Czwerwik [7].

Example 3.1. Let \( \phi : \mathbb{C} \to \mathbb{C} \) be defined by
\[
\phi(x) := \begin{cases} 
|x|^2 & \text{for } |x| < 1; \\
1 & \text{for } |x| \geq 1.
\end{cases}
\]
Consider the function \( f : \mathbb{C} \to \mathbb{C} \) by the formula
\[
f(x) := \sum_{n=0}^{\infty} \frac{1}{4^n} \phi(2^n x).
\]
It is clear that \( f \) is continuous and bounded by \( \frac{1}{4} \) on \( \mathbb{C} \). We prove that
\[
|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2})| \leq 16(|x|^2 + |y|^2)
\]
for all \( x, y \in \mathbb{C} \). To see this, if \( |x|^2 + |y|^2 = 0 \) or \( |x|^2 + |y|^2 \geq \frac{1}{4} \), then
\[
|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2})| \leq 4 \leq 16(|x|^2 + |y|^2).
\]
Now suppose that \( |x|^2 + |y|^2 < \frac{1}{4} \). Then there exists a positive integer \( k \) such that
\[
\frac{1}{4^{k+1}} \leq |x|^2 + |y|^2 < \frac{1}{4^k}.
\]
(3.12)
Then \( 2^k |x|, 2^k |y|, 2^k \sqrt{|x|^2 + |y|^2} \in (-1, 1) \) and \( 2^m |x|, 2^m |y|, 2^m \sqrt{|x|^2 + |y|^2} \in (-1, 1) \), for all \( m = 0, 1, \ldots, k \).

From the definition of \( f \) and (3.12), we have
\[
|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2})| = \left| \sum_{n=k+1}^{\infty} \frac{1}{4^n} \left( \phi(2^n x) + \phi(2^n y) + \phi(2^n \sqrt{|x|^2 + |y|^2}) \right) \right|
\]
\[
\leq 3 \sum_{n=k+1}^{\infty} \frac{1}{4^n} = \frac{4}{4^{k+1}} \leq 4(|x|^2 + |y|^2).
\]
Therefore \( f \) satisfies (3.11). Let \( Q : \mathbb{C} \to \mathbb{C} \) be a quadratic function such that
\[
\| f(x) - Q(x) \| \leq \beta |x|^2 , \text{ for all } x \in \mathbb{C}, \text{ where } \beta \text{ is a positive constant.}
\]
Then there exists a constant \( c \in \mathbb{C} \) such that \( Q(x) = cx^2 \) for all \( x \in \mathbb{Q} \). So we have
\[
|f(x)| \leq (\beta + |c|)|x|^2
\]
(3.13)
for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If $x_0 \in (0, 2^{-m}) \cap \mathbb{Q}$, then $2^n x_0 \in (0, 1)$ for all $n = 0, 1, \ldots, m - 1$. So

$$f(x_0) \geq \sum_{n=0}^{m-1} \frac{1}{4^n} \phi(2^n x_0) = mx_0^2 > (\beta + |c|)x_0^2$$

which contradicts (3.13).

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**References**


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