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A STRONG QUADRATIC FUNCTIONAL EQUATION IN C^*-ALGEBRAS

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Abstract. In this paper, we use a fixed point method to investigate the problem of stability on C^*-algebras of the strong quadratic functional equation

$$f(x) + f(y) = f(\sqrt{xx^* + yy^*}\ ).$$

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1. Introduction and preliminaries

In 1940, S.M. Ulam [41] posed the following question concerning the stability of group homomorphisms: *Under what conditions does there exist a group homomorphism near an approximately group homomorphism?*

In 1941, D.H. Hyers [15] considered the case of approximately additive functions $f : E \to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon, \text{ for all } x, y \in E.$$

T. Aoki [3] and Th.M. Rassias [35] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded (see also [4]).

Theorem 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

*for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$.
Then the limit $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies, for all $x \in E$, the relation*

$$\|f(x) - L(x)\| \le \frac{2\epsilon}{2 - 2^p}\|x\|^p. \tag{1.2}$$

If $p < 0$ then inequality (1.1) *holds for $x, y \neq 0$ and* (1.2) *for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.*

Theorem 1.1 has been generalized by G.L. Forti [11, 12] and P. Găvruta [13] who permitted the Cauchy difference to be bounded by a general control function. During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers-Ulam stability to a number of functional equations and mappings (see [5], [6], [7], [10], [14], [18], [19], [21], [23], [24], [26]-[34] and [36]-[38]). We also refer the readers to the books [1], [8], [17], [22] and [39]. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic equation (1.3) is said to be a *quadratic function.* Quadratic functional equations were used to characterize inner product spaces. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B such that $f(x) = B(x, x)$ for all x (see [1, 2, 20, 24]. The bi-additive function B is given by

$$B(x, y) = \frac{1}{4}\Big[f(x+y) - f(x-y)\Big].$$

The Hyers-Ulam stability problem for the quadratic functional equation (1.3) was proved by Skof [40] for functions $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [6] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. In the paper [7], Czerwik proved the generalized Hyers-Ulam stability of the quadratic functional equation (1.3). Grabiec [14] has generalized these results mentioned above. Jun and Lee [21] proved the generalized Hyers-Ulam stability of a Pexiderized quadratic equation.

Let E be a set. A function $d : E \times E \to [0, \infty]$ is called a *generalized metric* on E if d satisfies the usual axioms of a metric.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [25] *Let (E, d) be a complete generalized metric space and let $J : E \to E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a non-negative integer n_0 such that

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{\, y \in E : d(J^{n_0} x, y) < \infty \,\}$;*
(4) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

Throughout this paper A will be a C^*-algebra. We denote by $\sqrt{a}$ the unique positive element $b \in A$ such that $b^2 = a$. Also, we denote by $\mathbb{R}, \mathbb{C}$ and $\mathbb{Q}$ the set of real, complex and rational numbers, respectively. In this paper, we use a fixed point method (see [5, 23, 26]) to investigate the problem of stability of the strong quadratic (or simply s-quadratic) functional equation

$$f(x) + f(y) = f(\sqrt{xx^* + yy^*}\ ) \tag{1.4}$$

on C^*-algebras. In particular, every solution of the s-quadratic equation (1.4) is said to be a *s-quadratic function.* For some results on fixed point theorems in nonlinear analysis we refer the reader to [9, 16, 19, 42].

2. Solutions of Eq. (1.4)

Theorem 2.1. *Let X be a linear space. If a function $f : A \longrightarrow X$ satisfies the functional equation* (1.4)*, then f is quadratic.*

Proof. Letting $x = y = 0$, in (1.4), we get $f(0) = 0$. Replacing x and y by $x + y$ and $x - y$ in (1.4), respectively, we get

$$f(x + y) + f(x - y) = f(\sqrt{2xx^* + 2yy^*}\ ) \tag{2.1}$$

for all $x, y \in A$. It follows from (1.4) that $f(\sqrt{2}x) + f(\sqrt{2}y) = f(\sqrt{2xx^* + 2yy^*}\ )$ for all $x, y \in A$. Therefore we have from (2.1) that

$$f(x + y) + f(x - y) = f(\sqrt{2}x) + f(\sqrt{2}y) \tag{2.2}$$

for all $x, y \in A$. Setting $y = 0$ in (2.2), we get

$$f(\sqrt{2}x) = 2f(x) \tag{2.3}$$

for all $x \in A$. It follows from (2.2) and (2.3) that $f(x+y)+f(x-y) = 2f(x)+2f(y)$ for all $x \in A$. Hence f is quadratic. □

Remark 2.1. Let $f : A \to A$ be the mapping defined by $f(x) = x^2$ for all $x \in A$. It is clear that f is quadratic. Let $a \neq 0$ be a positive element of A. Hence f does not satisfy in (1.4) for $x = y = i\sqrt{a}$. Therefore f is not s-quadratic.

Corollary 2.2. *Let X be a linear space. If a function $f : A \longrightarrow X$ satisfies the functional equation* (1.4)*, then there exists a symmetric bi-additive function $B : A \times A \to X$ such that $f(x) = B(x, x)$ for all $x \in A$.*

3. Generalized Hyers-Ulam stability of Eq. (1.4) on C^*-algebras

In this section, we use a fixed point method (see [5, 23, 26]) to investigate the problem of stability of the functional equation (1.4) on C^*-algebras. For convenience, we use the following abbreviation for a given function $f : A \to X$:

$$Df(x, y) := f(x) + f(y) - f(\sqrt{xx^* + yy^*}\ )$$

for all $x, y \in A$, where X is a normed linear space.

Theorem 3.1. *Let X be a linear space and let $f : A \to X$ be a function with $f(0) = 0$ for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{3.1}$$

for all $x, y \in A$. If there exists a constant $0 < L < 1$ such that

$$\varphi(2x, 2y) \leq 4L\varphi(x, y) \tag{3.2}$$

for all $x, y \in A$, then there exists a unique s-quadratic function $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \le \frac{1}{4-4L}\phi(x) \tag{3.3}$$

for all $x \in A$, where $\phi(x) := \varphi(\sqrt{2}x, \sqrt{2}x) + \varphi(2x, 0) + 2\varphi(\sqrt{2}x, 0) + 2\varphi(x, x)$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic, i.e., $Q(tx) = t^2Q(x)$ for all $x \in A$ and all $t \in \mathbb{R}$.

Proof. It follows from (3.1) and (3.2) that

$$\|f(\sqrt{2}x) + f(\sqrt{2}y) - f(\sqrt{2xx^* + 2yy^*}\ )\| \le \varphi(\sqrt{2}x, \sqrt{2}y), \tag{3.4}$$

$$\lim_{k\to\infty} \frac{1}{4^k}\varphi(2^kx, 2^ky) = 0 \tag{3.5}$$

for all $x, y \in A$. Replacing x and y by $x + y$ and $x - y$ in (3.1), respectively, we get

$$\|f(x + y) + f(x - y) - f(\sqrt{2xx^* + 2yy^*}\ )\| \le \varphi(x + y, x - y) \tag{3.6}$$

for all $x, y \in A$. It follows from (3.4) and (3.6) that

$$\begin{aligned} &\|f(x + y) + f(x - y) - f(\sqrt{2}x) - f(\sqrt{2}y)\| \\ &\le \varphi(\sqrt{2}x, \sqrt{2}y) + \varphi(x + y, x - y) \end{aligned} \tag{3.7}$$

for all $x, y \in A$. Letting $y = 0$ in (3.7), we get

$$\|2f(x) - f(\sqrt{2}x)\| \le \varphi(\sqrt{2}x, 0) + \varphi(x, x) \tag{3.8}$$

for all $x \in A$. Therefore we have from (3.7) and (3.8) that

$$\begin{aligned} &\|f(x + y) + f(x - y) - 2f(x) - 2f(y)\| \\ &\le \varphi(\sqrt{2}x, \sqrt{2}y) + \varphi(x + y, x - y) \\ &\quad + \varphi(\sqrt{2}x, 0) + \varphi(x, x) + \varphi(\sqrt{2}y, 0) + \varphi(y, y) \end{aligned} \tag{3.9}$$

for all $x, y \in A$. Setting $x = y$ in (3.9), we get

$$\|f(2x) - 4f(x)\| \le \phi(x) \tag{3.10}$$

for all $x \in A$. By (3.2) we have $\phi(2x) \le 4L\phi(x)$ for all $x \in A$. Let E be the set of all functions $g : A \to X$ with $g(0) = 0$ and introduce a generalized metric on E as follows:

$$d(g, h) := \inf\{\, C \in [0, \infty] : \|g(x) - h(x)\| \le C\phi(x) \quad \text{for all } x \in A\,\}.$$

It is easy to show that (E, d) is a generalized complete metric space [5].

Now we consider the function $\Lambda : E \to E$ defined by

$$(\Lambda g)(x) = \frac{1}{4}g(2x), \quad \text{for all } g \in E \text{ and } x \in A.$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \le C$. From the definition of d, we have $\|g(x) - h(x)\| \le C\phi(x)$, for all $x \in A$. By the assumption and the last inequality, we have

$$\|(\Lambda g)(x) - (\Lambda h)(x)\| = \frac{1}{4}\|g(2x) - h(2x)\| \le \frac{1}{4}C\phi(2x) \le CL\phi(x), \text{ for all } x \in A.$$

Thus $d(\Lambda g, \Lambda h) \leq Ld(g,h)$, for any $g,h \in E$. It follows from (3.10) that $d(\Lambda f, f) \leq \frac{1}{4}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point Q of Λ, i.e.,

$$Q : A \to X, \quad Q(x) = \lim_{k\to\infty} (\Lambda^k f)(x) = \lim_{k\to\infty} \frac{1}{4^k} f(2^k x)$$

and $Q(2x) = 4Q(x)$ for all $x \in A$. Also Q is the unique fixed point of Λ in the set $E^* = \{g \in E : d(f,g) < \infty\}$ and

$$d(Q,f) \leq \frac{1}{1-L} d(\Lambda f, f) \leq \frac{1}{4-4L},$$

i.e., inequality (3.3) holds true for all $x \in A$. It follows from the definition of Q, (3.1) and (3.5) that

$$\|DQ(x,y)\| = \lim_{k\to\infty} \frac{1}{4^k} \|Df(2^k x, 2^k y)\| \leq \lim_{k\to\infty} \frac{1}{4^k} \varphi(2^k x, 2^k y) = 0$$

for all $x,y \in A$. So Q is s-quadratic. By Theorem 2.1, the function $Q : A \to X$ is quadratic. Finally it remains to prove the uniqueness of Q. Let $T : A \to X$ be another s-quadratic function satisfying (3.3). Since $d(f,T) \leq \frac{1}{4-4L}$ and T is quadratic, we get $T \in E^*$ and $(\Lambda T)(x) = \frac{1}{4} T(2x) = T(x)$ for all $x \in A$, i.e., T is a fixed point of Λ. Since Q is the unique fixed point of Λ in E^*, then $T = Q$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then by the same reasoning as in the proof of [35] Q is $\mathbb{R}$-quadratic. □

Corollary 3.2. *Let $0 < r < 2$ and θ, δ be non-negative real numbers and let $f : A \to X$ be a function with $f(0) = 0$ such that*

$$\|Df(x,y)\| \leq \delta + \theta(\|x\|^r + \|y\|^r)$$

for all $x,y \in A$. Then there exists a unique s-quadratic function $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{6\delta}{4-2^r} + \frac{4 + 4(\sqrt{2})^r + 2^r}{4-2^r} \theta \|x\|^r$$

for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

The following theorem is an alternative result of Theorem 3.1 and we leave its proof to the reader.

Theorem 3.3. *Let $f : A \to X$ be a function for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying (3.1) for all $x,y \in A$. If there exists a constant $0 < L < 1$ such that*

$$4\varphi(x,y) \leq L\varphi(2x,2y)$$

for all $x,y \in A$, then there exists a unique s-quadratic function $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{4-4L} \phi(x)$$

for all $x \in A$, where $\phi(x)$ is defined as in Theorem 3.1. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

Corollary 3.4. *Let $r > 2$ and θ be non-negative real numbers and let $f : A \to X$ be an even function such that*

$$\|Df(x,y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in A$. Then there exists a unique s-quadratic function $Q : A \to X$ such that

$$\|f(x) - Q(x)\| \leq \frac{4 + 4(\sqrt{2})^r + 2^r}{2^r - 4}\theta\|x\|^r$$

for all $x \in A$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then Q is $\mathbb{R}$-quadratic.

For the case $r = 2$ we have the following counterexample which is a modification of the example of S. Czwerwik [7].

Example 3.1. Let $\phi : \mathbb{C} \to \mathbb{C}$ be defined by

$$\phi(x) := \begin{cases} |x|^2 & \text{for } |x| < 1; \\ 1 & \text{for } |x| \geq 1. \end{cases}$$

Consider the function $f : \mathbb{C} \to \mathbb{C}$ by the formula

$$f(x) := \sum_{n=0}^{\infty} \frac{1}{4^n}\phi(2^n x).$$

It is clear that f is continuous and bounded by $\frac{4}{3}$ on $\mathbb{C}$. We prove that

$$\left|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2}\ )\right| \leq 16(|x|^2 + |y|^2) \tag{3.11}$$

for all $x, y \in \mathbb{C}$. To see this, if $|x|^2 + |y|^2 = 0$ or $|x|^2 + |y|^2 \geq \frac{1}{4}$, then

$$\left|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2}\ )\right| \leq 4 \leq 16(|x|^2 + |y|^2).$$

Now suppose that $|x|^2 + |y|^2 < \frac{1}{4}$. Then there exists a positive integer k such that

$$\frac{1}{4^{k+1}} \leq |x|^2 + |y|^2 < \frac{1}{4^k}. \tag{3.12}$$

Then $2^k|x|, 2^k|y|, 2^k\sqrt{|x|^2 + |y|^2} \in (-1, 1)$ and $2^m|x|, 2^m|y|, 2^m\sqrt{|x|^2 + |y|^2} \in (-1, 1)$, for all $m = 0, 1, ..., k$.

From the definition of f and (3.12), we have

$$\begin{aligned} &\left|f(x) + f(y) - f(\sqrt{|x|^2 + |y|^2}\ )\right| \\ &= \Big| \sum_{n=k+1}^{\infty} \frac{1}{4^n}\big[\phi(2^n x) + \phi(2^n y) + \phi(2^n\sqrt{|x|^2 + |y|^2}\ )\Big| \\ &\leq 3\sum_{n=k+1}^{\infty} \frac{1}{4^n} = \frac{4}{4^{k+1}} \leq 4(|x|^2 + |y|^2). \end{aligned}$$

Therefore f satisfies (3.11). Let $Q : \mathbb{C} \to \mathbb{C}$ be a quadratic function such that $\|f(x) - Q(x)\| \leq \beta|x|^2$, for all $x \in \mathbb{C}$, where β is a positive constant. Then there exists a constant $c \in \mathbb{C}$ such that $Q(x) = cx^2$ for all $x \in \mathbb{Q}$. So we have

$$|f(x)| \leq (\beta + |c|)|x|^2 \tag{3.13}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m > \beta + |c|$. If $x_0 \in (0, 2^{-m}) \cap \mathbb{Q}$, then $2^n x_0 \in (0,1)$ for all $n = 0, 1, ..., m-1$. So

$$f(x_0) \geq \sum_{n=0}^{m-1} \frac{1}{4^n} \phi(2^n x_0) = m x_0^2 > (\beta + |c|) x_0^2$$

which contradicts (3.13).

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