SOME RESULTS ON ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T : K \to K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n \geq 0} \subseteq [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \in [0, 1]$ be such that $\beta_n \geq 0$, $\alpha_n^2 = \infty$ and $\lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n \geq 0}$ be iteratively defined by

$$
x_{n+1} = (1-\alpha_n) x_n + \alpha_n T^n y_n, 
y_n = (1-\beta_n) x_n + \beta_n T^n x_n, \quad n \geq 0.
$$

Then $\{x_n\}_{n \geq 0}$ converges strongly to $p \in F(T)$.

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1. Introduction

Let $E$ be a real Banach space and $K$ be a nonempty convex subset of $E$. Let $J$ denote the normalized duality mapping from $E$ to $2^{E^*}$ defined by

$$
J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \text{ and } ||f^*|| = ||x||\},
$$

where $E^*$ denotes the dual space of $E$ and $\langle , \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by $j$.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in $E$.

Definition 1.1. The mapping $T$ is said to be uniformly $L$-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$

$$
\|T^n x - T^n y\| \leq L \|x - y\|.
$$

Definition 1.2. $T$ is said to be nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$
\|Tx - Ty\| \leq \|x - y\|.
$$
Definition 1.3. $T$ is said to be asymptotically nonexpansive [2], if there exists a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
$$
\|T^nx - T^ny\| \leq k_n \|x - y\| \quad \text{for all } x, y \in D(T), \ n \geq 1.
$$

Definition 1.4. $T$ is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ and there exists $j(x - y) \in J(x - y)$ such that
$$
(T^nx - T^ny, j(x - y)) \leq k_n \|x - y\|^2 \quad \text{for all } x, y \in D(T), \ n \geq 1.
$$

Remark 1.5. 1. It is easy to see that every asymptotically nonexpansive mapping is uniformly $L$-Lipschitzian.

2. If $T$ is asymptotically nonexpansive mapping then for all $x, y \in D(T)$ there exists $j(x - y) \in J(x - y)$ such that
$$
(T^nx - T^ny, j(x - y)) \leq \|T^nx - T^ny\| \|x - y\| \leq k_n \|x - y\|^2, \ n \geq 1.
$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [7] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [8] who proved the following theorem:

Theorem 1.6. Let $K$ be a nonempty bounded closed convex subset of a Hilbert space $H$ and let $T : K \to K$ be a completely continuous, uniformly $L$-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1, \infty)$; $q_n = 2k_n - 1$, $\forall n \in \mathbb{N}$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in \mathbb{N}$, $\epsilon > 0$ and $b \in (0, L^{-2}[(1 + L^2)^{1/2} - 1])$; $x_1 \in K$ for all $n \in \mathbb{N}$, define
$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n x_n.
$$

Then $\{x_n\}$ converges to some fixed point of $T$.

The recursion formula of Theorem 1.6 is a modification of the well-known Mann iteration process (see [5]).

Also among the most recent results about the same topic, following are due to Ofoedu [6].

Theorem 1.7. [6] Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T : K \to K$ a uniformly $L$-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ such that $\|x^*\| \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n \geq 0} \subset [0, 1]$ be such that $\sum_{n \geq 0} \alpha_n = \infty$, $\sum_{n \geq 0} \alpha_n^2 < \infty$, and $\sum_{n \geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n \geq 0}$ be iteratively defined by
$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_nT^n x_n, \ n \geq 0.
$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that
$$
(T^nx - x^*, j(x - x^*)) \leq k_n\|x - x^*\| - \psi(\|x - x^*\|), \ \forall x \in K.
$$

Then $\{x_n\}_{n \geq 0}$ is bounded.
Theorem 1.8. [6] Let $K$ be a nonempty closed convex subset of a real Banach space $E$, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$, $\lim_{n \to \infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n \geq 0} \subset [0, 1]$ be such that $\sum_{n \geq 0} \alpha_n = \infty$, $\sum_{n \geq 0} \sigma_n^2 < \infty$ and $\sum_{n \geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n \geq 0}$ be iteratively defined by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, \quad n \geq 0.$$ 

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*) \rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||), \quad \forall x \in K.$$ 

Then $\{x_n\}_{n \geq 0}$ converges strongly to $x^* \in F(T)$.

Remark 1.9. One can easily see that if we take in Theorem 1.7, Theorem 1.8, $\alpha_n = \frac{1}{n^2}$; $0 < \sigma < \frac{1}{2}$, then $\sum \alpha_n = \infty$, but also $\sum \sigma_n^2 = \infty$. Hence the conclusions of Theorems 1.7, 1.8 can be improved. The same argument can be applied on the results of Chidume and Chidume in [1].

The purpose of this paper is to generalize the results of Schu [8] from Hilbert spaces to more general Banach spaces and improve the results of Ofoedu [6] in a significantly more general context by removing the conditions like $\sum_{n \geq 0} \alpha_n^2 < \infty$, $\sum_{n \geq 0} \alpha_n(k_n - 1) < \infty$ and (O) from the Theorems 1.7, 1.8.

2. Main Results

The following results are well known.

Lemma 2.1. ([10]) Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y) \rangle, \quad \forall x, y \in J(x + y).$$

Lemma 2.2. ([9]) If there exists a positive integer $N$ such that for all $n \geq N$, $n \in \mathbb{N}$, we have $\rho_{n+1} \leq (1 - \theta_n) \rho_n + b_n$, then $\lim_{n \to \infty} \rho_n = 0$, where $\theta_n \in [0, 1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, and $b_n = o(\theta_n)$.

Lemma 2.3. ([4]) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying, for each $n \in \mathbb{N}$ the relation $a_{n+1} \leq (1 - t_n) a_n + b_n + c_n$. Then $\lim_{n \to \infty} a_n = 0$, where $\{t_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$.

We now prove our main results.

Lemma 2.4. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnegative real sequences satisfying, for each $n \in \mathbb{N}$ the relation $a_{n+1} \leq (1 - t_n^l) a_n + b_n + c_n$; $l \geq 1$. Then, $\lim_{n \to \infty} a_n = 0$, where $\{t_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} t_n^l = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$.

Proof. Since $b_n = o(t_n)$, let $b_n = d_n t_n$, and $d_n \to 0$. By a straightforward induction, one obtains

$$0 \leq a_{n+1} \leq \prod_{j=k}^{n} (1 - t_j^l) a_k + \sum_{j=k}^{n} \left[ t_j \prod_{i=j+1}^{n} (1 - t_i^l) \right] d_j + \sum_{j=k}^{n} c_j \prod_{i=j+1}^{n} (1 - t_i^l). \quad \text{(L)}$$
We have
\[ \prod_{j=k}^{n} (1 - t_j^i) \leq e^{-\sum_{j=k}^{n} t_j^i} \to 0, \]
and
\[ \sum_{j=k}^{n} t_j \prod_{i=j+1}^{n} (1 - t_i^j) \leq 1, \text{ for all } n, k. \]

Since \( d_n \to 0 \) and \( \sum_{n=0}^{\infty} c_n < \infty \), for arbitrary \( \varepsilon > 0 \), there exists a natural number \( k \) such that \( d_j < \varepsilon \) for all \( j \geq k \), and \( \sum_{j=k}^{\infty} c_j < \varepsilon \), we have from (L)
\[ 0 \leq \liminf_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n \leq 2\varepsilon. \]

Letting \( \varepsilon \to 0 \), we obtain \( \lim_{n \to \infty} a_n = 0 \). This completes the proof. \( \square \)

**Theorem 2.5.** Let \( K \) be a nonempty closed convex subset of a real Banach space \( E \), \( T : K \to K \) a uniformly continuous asymptotically pseudocontractive mapping having \( T(K) \) bounded with sequence \( \{k_n\}_{n \geq 0} \subset [1, \infty) \), \( \lim k_n = 1 \) such that \( p \in F(T) = \{x \in K : Tx = x\} \). Let \( \{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \in [0, 1] \) be such that \( \sum_{n \geq 0} \alpha_n^2 = \infty \) and \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \). For arbitrary \( x_0 \in K \) let \( \{x_n\}_{n \geq 0} \) be iteratively defined by:
\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, \]
\[ y_n = (1 - \beta_n) x_n + \beta_n T^n x_n, \quad n \geq 0. \quad (2.1) \]

Then \( \{x_n\}_{n \geq 0} \) converges strongly to \( p \in F(T) \).

**Proof.** Because \( p \) is a fixed point of \( T \), then the set of fixed points \( F(T) \) of \( T \) is nonempty.

Since \( T \) has bounded range, we set
\[ M_1 = ||x_0 - p|| + \sup_{n \geq 0} ||T^n y_n - p||. \]

Obviously \( M_1 < \infty \).

It is clear that \( ||x_0 - p|| \leq M_1 \). Let \( ||x_n - p|| \leq M_1 \). Next we will prove that \( ||x_{n+1} - p|| \leq M_1 \).

Consider
\[ ||x_{n+1} - p|| = ||(1 - \alpha_n)x_n + \alpha_n T^n y_n - p|| \]
\[ \leq ||(1 - \alpha_n)(x_n - p) + \alpha_n (T^n y_n - p)|| \]
\[ \leq (1 - \alpha_n)||x_n - p|| + \alpha_n ||T^n y_n - p|| \]
\[ \leq (1 - \alpha_n)M_1 + M_1 \alpha_n = M_1. \]

So, from the above discussion, we can conclude that the sequence \( \{x_n - p\}_{n \geq 0} \) is bounded. Let \( M_2 = \sup_{n \geq 0} ||x_n - p|| \).

Denote \( M = M_1 + M_2 + \sup_{n \geq 0} ||T^n y_n - p|| \). Obviously \( M < \infty \).
Now from Lemma 2.1 for all \( n \geq 0 \), we obtain
\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n)x_n + \alpha_n T^n y_n - p\|^2
\]
\[
= \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\|^2
\]
\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|T^n y_n - p, j(x_{n+1} - p)\|
\]
\[
= (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|T^n x_{n+1} - p, j(x_{n+1} - p)\|
\]
\[
+ 2\alpha_n \|T^n y_n - T^n x_{n+1}\| \|x_{n+1} - p\|
\]
\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n k_n \|x_{n+1} - p\|^2
\]
\[
+ 2\alpha_n \|T^n y_n - T^n x_{n+1}\| \|x_{n+1} - p\|
\]
\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n k_n \|x_{n+1} - p\|^2 + 2\alpha_n \lambda_n, \tag{2.2}
\]
where
\[
\lambda_n = M \|T^n y_n - T^n x_{n+1}\|. \tag{2.3}
\]
Using (2.1) we have
\[
\|y_n - x_{n+1}\| \leq \|y_n - x_n\| + \|x_n - x_{n+1}\|
\]
\[
= \beta_n \|x_n - T^n x_n\| + \alpha_n \|x_n - T^n y_n\|
\]
\[
\leq 2M (\alpha_n + \beta_n). \tag{2.4}
\]
From the conditions \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \) and (2.4), we obtain \( \lim_{n \to \infty} \|y_n - x_{n+1}\| = 0 \), and the uniform continuity of \( T \) leads to \( \lim_{n \to \infty} \|T^n y_n - T^n x_{n+1}\| = 0 \). Thus, we have:
\[
\lim_{n \to \infty} \lambda_n = 0. \tag{2.5}
\]
The real function \( f : [0, \infty) \to [0, \infty) \), defined by \( f(t) = t^2 \) is increasing and convex. For all \( \lambda \in [0, 1] \) and \( t_1, t_2 > 0 \), we have
\[
((1 - \lambda)t_1 + \lambda t_2)^2 \leq (1 - \lambda)t_1^2 + \lambda t_2^2. \tag{2.6}
\]
Consider
\[
\|x_{n+1} - p\|^2 = \|(1 - \alpha_n)x_n + \alpha_n T^n y_n - p\|^2
\]
\[
= \|(1 - \alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)\|^2
\]
\[
\leq (1 - \alpha_n)^2 \|x_n - p\|^2 + 2\alpha_n \|T^n y_n - p\|^2
\]
\[
\leq (1 - \alpha_n) \|x_n - p\|^2 + \alpha_n \|T^n y_n - p\|^2
\]
\[
\leq (1 - \alpha_n) \|x_n - p\|^2 + M^2\alpha_n. \tag{2.7}
\]
Substituting (2.7) in (2.2), we get
\[
\|x_{n+1} - p\|^2 \leq \|(1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n) k_n\| \|x_n - p\|^2
\]
\[
+ 2\alpha_n \left(M^2 k_n \alpha_n + \lambda_n\right). \tag{2.8}
\]
Consider
\[
(1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n) k_n = (1 - \alpha_n)^2 + 2\alpha_n (1 - \alpha_n)
\]
\[
+ 2\alpha_n (1 - \alpha_n)(k_n - 1) \leq 1 - \alpha_n^2 + 2\alpha_n (k_n - 1). \]
Consequently from (2.8), we obtain

\[ \|x_{n+1} - p\|^2 \leq (1 - \alpha_n^2) \|x_n - p\|^2 + 2\alpha_n (M^2 k_n \alpha_n + \lambda_n) \]

\[ \leq (1 - \alpha_n^2) \|x_n - p\|^2 + 2[M^2 k_n \alpha_n + \lambda_n + M^2 (k_n - 1)\alpha_n] \]

\[ = (1 - \alpha_n^2) \|x_n - p\|^2 + \varepsilon_n \alpha_n, \]  \hspace{1cm} (2.9)

where \( \varepsilon_n = 2[M^2 k_n \alpha_n + \lambda_n + M^2 (k_n - 1)] \). Now with the help of \( \sum_{n=0}^{\infty} \alpha_n^2 = \infty \), \( \lim_{n \to \infty} \alpha_n = 0 \), (2.5) and Lemma 2.4, we obtain, from (2.9), that \( \lim_{n \to \infty} \|x_n - p\| = 0 \), which completes the proof.

\[ \Box \]

**Corollary 2.6.** Let \( K \) be a nonempty closed convex subset of a real Banach space \( E, T : K \to K \) a uniformly \( L \)-Lipschitzian asymptotically pseudocontractive mapping having \( T(K) \) bounded with sequence \( \{k_n\}_{n \geq 0} \subset [1, \infty) \), \( \lim_{n \to \infty} k_n = 1 \) such that \( p \in F(T) = \{ x \in K : Tx = x \} \). Let \( \{\alpha_n\}_{n \geq 0}, \{\beta_n\}_{n \geq 0} \in [0, 1] \) be such that \( \sum_{n=0}^{\infty} \alpha_n^2 = \infty \) and \( \lim_{n \to \infty} \alpha_n = 0 = \lim_{n \to \infty} \beta_n \). For arbitrary \( x_0 \in K \) let \( \{x_n\}_{n \geq 0} \) be iteratively defined by

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n y_n, \]

\[ y_n = (1 - \beta_n) x_n + \beta_n T^n x_n, \quad n \geq 0. \]

Then \( \{x_n\}_{n \geq 0} \) converges strongly to \( p \in F(T) \).

**Remark 2.7.** We will try to remove conditions like (O) form the existing literature.

**References**


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