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SOME RESULTS ON ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Let K be a nonempty closed convex subset of a real Banach space E, $T: K \to K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \in [0,1]$ be such that $\sum_{n\geq 0} \alpha_n^2 = \infty$ and $\lim\limits_{n\to\infty} \alpha_n = 0 = \lim\limits_{n\to\infty} \beta_n$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\, x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)\, x_n + \beta_n T^n x_n, \ n \geq 0. \end{aligned}$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.
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1. Introduction

Let E be a real Banach space and K be a nonempty convex subset of E. Let J denote the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2 \ \text{ and } \ ||f^*|| = ||x||\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. We shall denote the single-valued duality map by j.

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ in E.

Definition 1.1. *The mapping T is said to be uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$*

$$\|T^n x - T^n y\| \leq L \|x - y\|.$$

Definition 1.2. *T is said to be nonexpansive if for all x, $y \in D(T)$, the following inequality holds:*

$$\|Tx - Ty\| \leq \|x - y\|.$$

Definition 1.3. *T is said to be asymptotically nonexpansive [2], if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ such that*

$$\|T^n x - T^n y\| \leq k_n \|x - y\| \text{ for all } x,\ y \in D(T),\ n \geq 1.$$

Definition 1.4. *T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$ with $\lim\limits_{n\to\infty} k_n = 1$ and there exists $j(x-y) \in J(x-y)$ such that $\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x - y\|^2$ for all $x,\ y \in D(T),\ n \geq 1$.*

Remark 1.5. *1. It is easy to see that every asymptotically nonexpansive mapping is uniformly L-Lipschitzian.*

2. If T is asymptotically nonexpansive mapping then for all $x,\ y \in D(T)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq \|T^n x - T^n y\| \|x - y\| \leq k_n \|x - y\|^2,\ n \geq 1.$$

Hence every asymptotically nonexpansive mapping is asymptotically pseudocontractive.

3. Rhoades in [7] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The asymptotically pseudocontractive mappings were introduced by Schu [8] who proved the following theorem:

Theorem 1.6. *Let K be a nonempty bounded closed convex subset of a Hilbert space H and let $T : K \to K$ be a completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive with sequence $\{k_n\} \subset [1,\infty)$; $q_n = 2k_n - 1$, $\forall n \in N$; $\sum(q_n^2 - 1) < \infty$; $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$; $\epsilon < \alpha_n < \beta_n \leq b$, $\forall n \in N$, $\epsilon > 0$ and $b \in (0, L^{-2}[(1+L^2)^{\frac{1}{2}} - 1])$; $x_1 \in K$ for all $n \in N$, define*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n.$$

Then $\{x_n\}$ converges to some fixed point of T.

The recursion formula of Theorem 1.6 is a modification of the well-known Mann iteration process (see [5]).

Also among the most recent results about the same topic, following are due to Ofoedu [6].

Theorem 1.7. *[6] Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0,1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n (k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0,\infty) \to [0,\infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n ||x - x^*||^2 - \psi(||x - x^*||),\ \forall x \in K. \tag{O}$$

Then $\{x_n\}_{n\geq 0}$ is bounded.

Theorem 1.8. *[6] Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ such that $x^* \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ be such that $\sum_{n\geq 0} \alpha_n = \infty$, $\sum_{n\geq 0} \alpha_n^2 < \infty$ and $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T^n x_n,\ n \geq 0.$$

Suppose there exists a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$ such that

$$\langle T^n x - x^*, j(x - x^*)\rangle \leq k_n||x - x^*||^2 - \psi(||x - x^*||),\ \forall x \in K.$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $x^* \in F(T)$.

Remark 1.9. *One can easily see that if we take in Theorem 1.7, Theorem 1.8, $\alpha_n = \frac{1}{n^\sigma}$; $0 < \sigma < \frac{1}{2}$, then $\sum \alpha_n = \infty$, but also $\sum \alpha_n^2 = \infty$. Hence the conclusions of Theorems 1.7, 1.8 can be improved. The same argument can be applied on the results of Chidume and Chidume in [1].*

The purpose of this paper is to generalize the results of Schu [8] from Hilbert spaces to more general Banach spaces and improve the results of Ofoedu [6] in a significantly more general context by removing the conditions like $\sum_{n\geq 0} \alpha_n^2 < \infty$, $\sum_{n\geq 0} \alpha_n(k_n - 1) < \infty$ and (O) from the Theorems 1.7, 1.8.

2. Main Results

The following results are well known.

Lemma 2.1. *([10]) Let $J : E \to 2^E$ be the normalized duality mapping. Then for any $x, y \in E$, we have*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, j(x + y)\rangle, \quad \forall j(x + y) \in J(x + y).$$

Lemma 2.2. *([9]) If there exists a positive integer N such that for all $n \geq N, n \in \mathbb{N}$, we have $\rho_{n+1} \leq (1-\theta_n)\rho_n+b_n$, then $\lim_{n\to\infty} \rho_n = 0$, where $\theta_n \in [0, 1)$, $\sum_{n=0}^{\infty} \theta_n = \infty$, and $b_n = o(\theta_n)$.*

Lemma 2.3. *([4]) Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnrgative real sequences satisfying, for each $n \in \mathbb{N}$ the relation $a_{n+1} \leq (1-t_n)a_n+b_n+c_n$. Then $\lim_{n\to\infty} a_n = 0$, where $\{t_n\} \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$.*

We now prove our main results.

Lemma 2.4. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three nonnrgative real sequences satisfying, for each $n \in \mathbb{N}$ the relation $a_{n+1} \leq (1-t_n^l)a_n+b_n+c_n$; $l \geq 1$. Then, $\lim_{n\to\infty} a_n = 0$, where $\{t_n\} \in [0, 1]$, $\sum_{n=0}^{\infty} t_n^l = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$.*

Proof. Since $b_n = o(t_n)$, let $b_n = d_n t_n$, and $d_n \to 0$. By a straightforward induction, one obtains

$$0 \leq a_{n+1} \leq \prod_{j=k}^{n}(1 - t_j^l)a_k + \sum_{j=k}^{n}\left[t_j \prod_{i=j+1}^{n}(1 - t_i^l)\right] d_j + \sum_{j=k}^{n} c_j \prod_{i=j+1}^{n}(1 - t_i^l). \qquad \text{(L)}$$

We have

$$\prod_{j=k}^{n}(1-t_j^l) \leq e^{-\sum\limits_{j=k}^{n} t_j^l} \to 0,$$

and

$$\sum_{j=k}^{n} t_j \prod_{i=j+1}^{n}(1-t_i^l) \leq 1, \text{ for all } n,\ k.$$

Since $d_n \to 0$ and $\sum_{n=0}^{\infty} c_n < \infty$, for arbitrary $\varepsilon > 0$, there exists a natural number k such that $d_j < \varepsilon$ for all $j \geq k$, and $\sum_{j=k}^{\infty} c_j < \varepsilon$, we have from (L)

$$0 \leq \lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n \leq 2\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain $\lim\limits_{n\to\infty} a_n = 0$. This completes the proof. □

Theorem 2.5. *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly continuous asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that $p \in F(T) = \{x \in K : Tx = x\}$. Let $\{\alpha_n\}_{n\geq 0}, \{\beta_n\}_{n\geq 0} \in [0,1]$ be such that $\sum_{n\geq 0} \alpha_n^2 = \infty$ and $\lim\limits_{n\to\infty} \alpha_n = 0 = \lim\limits_{n\to\infty} \beta_n$. For arbitrary $x_0 \in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by:*

$$x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n T^n y_n,$$

$$y_n = (1-\beta_n)\, x_n + \beta_n T^n x_n,\ n \geq 0. \tag{2.1}$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p \in F(T)$.

Proof. Because p is a fixed point of T, then the set of fixed points $F(T)$ of T is nonempty.

Since T has bounded range, we set

$$M_1 = ||x_0 - p|| + \sup_{n\geq 0} ||T^n y_n - p||.$$

Obviously $M_1 < \infty$.

It is clear that $||x_0 - p|| \leq M_1$. Let $||x_n - p|| \leq M_1$. Next we will prove that $||x_{n+1} - p|| \leq M_1$.

Consider

$$\begin{aligned} ||x_{n+1} - p|| &= ||(1-\alpha_n)x_n + \alpha_n T^n y_n - p|| \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p)|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||T^n y_n - p|| \\ &\leq (1-\alpha_n)M_1 + M_1\alpha_n = M_1. \end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - p\}_{n\geq 0}$ is bounded. Let $M_2 = \sup\limits_{n\geq 0} ||x_n - p||$.

Denote $M = M_1 + M_2 + \sup\limits_{n\geq 0} ||T^n x_n - p||$. Obviously $M < \infty$.

Now from Lemma 2.1 for all $n \geq 0$, we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= ||(1-\alpha_n)x_n+\alpha_n T^n y_n - p||^2 \\
&= ||(1-\alpha_n)(x_n-p)+\alpha_n(T^n y_n-p)||^2 \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n\langle T^n y_n-p, j(x_{n+1}-p)\rangle \\
&= (1-\alpha_n)^2||x_n-p||^2+2\alpha_n\langle T^n x_{n+1}-p,\ j(x_{n+1}-p)\rangle \\
&\quad +2\alpha_n\langle T^n y_n-T^n x_{n+1}, j(x_{n+1}-p)\rangle \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n k_n||x_{n+1}-p||^2 \\
&\quad +2\alpha_n\left\|T^n y_n-T^n x_{n+1}\right\|\left\|x_{n+1}-p\right\| \\
&\leq (1-\alpha_n)^2||x_n-p||^2+2\alpha_n k_n||x_{n+1}-p||^2+2\alpha_n\lambda_n, \qquad (2.2)
\end{aligned}$$

where

$$\lambda_n = M\left\|T^n y_n - T^n x_{n+1}\right\|. \qquad (2.3)$$

Using (2.1) we have

$$\begin{aligned}
\|y_n-x_{n+1}\| &\leq \|y_n-x_n\|+\|x_n-x_{n+1}\| \\
&= \beta_n\left\|x_n-T^n x_n\right\|+\alpha_n\left\|x_n-T^n y_n\right\| \\
&\leq 2M\left(\alpha_n+\beta_n\right). \qquad (2.4)
\end{aligned}$$

From the conditions $\lim_{n\to\infty}\alpha_n = 0 = \lim_{n\to\infty}\beta_n$ and (2.4), we obtain $\lim_{n\to\infty}\|y_n-x_{n+1}\| = 0$, and the uniform continuity of T leads to $\lim_{n\to\infty}\|T^n y_n - T^n x_{n+1}\| = 0$, Thus, we have:

$$\lim_{n\to\infty}\lambda_n = 0. \qquad (2.5)$$

The real function $f:[0,\infty)\to[0,\infty)$, defined by $f(t)=t^2$ is increasing and convex. For all $\lambda\in[0,1]$ and $t_1,\ t_2>0$, we have

$$((1-\lambda)t_1+\lambda t_2)^2 \leq (1-\lambda)t_1^2+\lambda t_2^2. \qquad (2.6)$$

Consider

$$\begin{aligned}
||x_{n+1}-p||^2 &= ||(1-\alpha_n)x_n+\alpha_n T^n y_n-p||^2 \\
&= ||(1-\alpha_n)(x_n-p)+\alpha_n(T^n y_n-p)||^2 \\
&\leq \left[(1-\alpha_n)\left\|x_n-p\right\|+\alpha_n\left\|T^n y_n-p\right\|\right]^2 \\
&\leq (1-\alpha_n)\left\|x_n-p\right\|^2+\alpha_n\left\|T^n y_n-p\right\|^2 \\
&\leq (1-\alpha_n)\left\|x_n-p\right\|^2+M^2\alpha_n. \qquad (2.7)
\end{aligned}$$

Substituting (2.7) in (2.2), we get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq [(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)k_n]||x_n-p||^2 \\
&\quad +2\alpha_n\left(M^2k_n\alpha_n+\lambda_n\right). \qquad (2.8)
\end{aligned}$$

Consider

$$\begin{aligned}
(1-\alpha_n)^2+2\alpha_n(1-\alpha_n)k_n &= (1-\alpha_n)^2+2\alpha_n(1-\alpha_n) \\
&\quad +2\alpha_n(1-\alpha_n)(k_n-1)\leq 1-\alpha_n^2+2\alpha_n(k_n-1).
\end{aligned}$$

Consequently from (2.8), we obtain

$$\begin{aligned} \|x_{n+1}-p\|^2 &\leq \left[1-\alpha_n^2+2\alpha_n(k_n-1)\right]\|x_n-p\|^2+2\alpha_n\left(M^2k_n\alpha_n+\lambda_n\right) \\ &\leq \left(1-\alpha_n^2\right)\|x_n-p\|^2+2[M^2k_n\alpha_n+\lambda_n+M^2(k_n-1)]\alpha_n \\ &= \left(1-\alpha_n^2\right)\|x_n-p\|^2+\varepsilon_n\alpha_n, \qquad (2.9) \end{aligned}$$

where $\varepsilon_n = 2\left[M^2k_n\alpha_n+\lambda_n+M^2(k_n-1)\right]$. Now with the help of $\sum_{n\geq 0}\alpha_n^2=\infty$, $\lim\limits_{n\to\infty}\alpha_n=0$, (2.5) and Lemma 2.4, we obtain, from (2.9), that $\lim_{n\to\infty}\|x_n-p\|=0$, which completes the proof. □

Corollary 2.6. *Let K be a nonempty closed convex subset of a real Banach space E, $T: K\to K$ a uniformly L-Lipschitzian asymptotically pseudocontractive mapping having $T(K)$ bounded with sequence $\{k_n\}_{n\geq 0}\subset[1,\infty)$, $\lim\limits_{n\to\infty}k_n=1$ such that $p\in F(T)=\{x\in K: Tx=x\}$. Let $\{\alpha_n\}_{n\geq 0},\{\beta_n\}_{n\geq 0}\in[0,1]$ be such that $\sum_{n\geq 0}\alpha_n^2=\infty$ and $\lim\limits_{n\to\infty}\alpha_n=0=\lim\limits_{n\to\infty}\beta_n$. For arbitrary $x_0\in K$ let $\{x_n\}_{n\geq 0}$ be iteratively defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)\,x_n+\alpha_nT^ny_n, \\ y_n &= (1-\beta_n)\,x_n+\beta_nT^nx_n,\ n\geq 0. \end{aligned}$$

Then $\{x_n\}_{n\geq 0}$ converges strongly to $p\in F(T)$.

Remark 2.7. *We will try to remove conditions like (O) form the existing literature.*

References

[1] C.E. Chidume, C.O. Chidume, *Convergence theorem for fixed points of uniformly continuous generalized phihemicontractive mappings*, J. Math. Anal. Appl., **303**(2005), 545-554.

[2] K. Goebel, W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., **35**(1972), 171-174.

[3] S. Ishikawa, *Fixed point by a new iteration method*, Proc. Amer. Math. Soc., **44**(1974), 147-150.

[4] L.S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194**(1995), 114-125.

[5] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506-510.

[6] E.U. Ofoedu, *Strong convergence theorem for uniformly L-Lipschitzian asymptotically pseudocontractive mapping in real Banach space*, J. Math. Anal. Appl., **321**(2006), No. 2, 722-728.

[7] B.E. Rhoades, *A comparison of various definition of contractive mappings*, Trans. Amer. Math. Soc., **226**(1977), 257-290.

[8] J. Schu, *Iterative construction of fixed point of asymptotically nonexpansive mappings*, J. Math. Anal. Appl., **158**(1991), 407-413.

[9] X. Weng, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc., **113**(1991), No. 3, 727-731.

[10] H.K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., **16**(1991), No. 12, 1127-1138.

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