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STRONG CONVERGENCE THEOREMS OF FIXED POINT FOR QUASI-PSEUDO-CONTRACTIONS BY HYBRID PROJECTION ALGORITHMS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. The purpose of this paper is to propose a modified hybrid projection algorithm and prove strong convergence theorems of fixed points for a Lipschitz quasi-pseudo-contraction in the framework of Hilbert spaces. Our results improve and extend the corresponding ones announced by many others.
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1. Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Assume that C is a nonempty closed convex subset of H and $T : C \to C$ is a nonlinear mapping. We use $F(T)$ to denote the set of fixed points of T. $P_C(\cdot)$ denotes the metric projection from H onto C.

Recall that the mapping T is said to be non-expansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

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T is said to be strictly pseudo-contractive if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + k\|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

Note that the class of strict pseudo-contractions strictly includes the class of non-expansive mappings. That is, T is non-expansive if and only if the coefficient $k = 0$. It is also said to be pseudo-contractive if $k = 1$. That is,

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2, \quad \forall x, y \in C. \tag{1.3}$$

T is said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0,1)$ such that $T - \lambda I$ is pseudo-contractive. Clearly, the class of strict pseudo-contractions falls into the one between classes of non-expansive mappings and pseudo-contractions. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of strict pseudo-contractions (see, e.g., [2-4,12]).

It is very clear that, in a real Hilbert space H, (1.3) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x-y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

T is called a quasi-strict pseudo-contraction if $F(T) \neq \emptyset$ and (1.2) holds for all $x \in C$ but $y \in F(T)$. In particular, if $k = 1$, then T is said to be quasi-pseudo-contractive; if $k = 0$, T is said to be quasi-non-expansive. Clearly pseudo-contraction with a nonempty fixed point set is quasi-pseudo-contractive, however, the converse may be not true. The following examples can be found in Chidume [5] and Zhou [12], respectively.

Example 1.1. Let $H = R^1$ and define a mapping by $T : H \to H$ by

$$Tx = \begin{cases} \frac{x}{2}\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then T is quasi-pseudo-contractive but not pseudo-contractive.

Example 1.2. Take $C = (0, \infty)$ and define a mapping by $T : C \to C$ by

$$Tx = \frac{x^2}{1+x}, \text{ for each } x \in C.$$

Then T is strict pseudo-contractive but not strong pseudo-contractive.

Example 1.3. Take $C = R^1$ and define a mapping $T : C \to C$ by

$$Tx = \begin{cases} 1, & x \in (-\infty, -1) \\ \sqrt{1-(1+x)^2}, & x \in [-1, 0) \\ -\sqrt{1-(x-1)^2}, & x \in [0, 1] \\ -1, & x \in (1, \infty). \end{cases}$$

Then T is a strong pseudo-contraction but not a strict pseudo-contraction.

Example 1.4. Take $H = R^2$ and $B = \{x \in R^2 : \|x\| \leq 1\}$, $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. If $x = (a, b) \in H$, we define $x^{\perp}$ to be $(b, -a) \in H$.

Define $T : B \to B$ by

$$Tx = \begin{cases} x + x^{\perp}, & x \in B_1, \\ \frac{x}{\|x\|} - x + x^{\perp}, & x \in B_2. \end{cases}$$

Then T is a Lipschitz pseudo-contraction but not a strict pseudo-contraction.

Recall that the normal Mann's iterative process was introduced by Mann [7] in 1953. Since then, construction of fixed points for non-expansive mappings and pseudo-contractions via the normal Mann's iterative process has been extensively investigated by many authors. The normal Mann's iterative process generates a sequence $\{x_n\}$ in the following manner:

$$\forall x_1 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad \forall n \geq 1, \tag{1.5}$$

where the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval (0,1).

If T is a non-expansive mapping with a fixed point and the control sequence $\{\alpha_n\}$ is chosen so that $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by normal Mann's iterative process (1.5) converges weakly to a fixed point of T (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [11]).

Attempts to modify the normal Mann iteration method (1.5) for non-expansive mappings, strict pseudo-contractions and pseudo-contractions so that strong convergence is guaranteed have recently been made; see, e.g., [1,6,8,9,10,12,13] and the references therein.

Nakajo and Takahashi [10] proposed the following modification of the Mann iteration for a single non-expansive mapping T in a Hilbert space. To be more precise, They proved the following theorem:

Theorem NT. *Let C be a closed convex subset of a Hilbert space H and let $T : C \to C$ be a non-expansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0, 1] such that $\alpha_n \leq 1 - \delta$ for some $\delta \in (0, 1]$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \forall\ n \geq 1. \end{cases} \tag{1.6}$$

Then $\{x_n\}$ converges in norm to $P_{F(T)}x_0$.

Recently, Kim and Xu [6] adapted the iteration (1.5) in Hilbert spaces. They extended the result of Nakajo an Takahashi [10] from non-expansive mappings to asymptotically non-expansive mappings. To be more precise, they proved the following result.

Theorem KX. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let $T : C \to C$ be an asymptotically non-expansive mapping with a sequence $\{k_n\}$ such that $k_n \to 1$ as $n \to \infty$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $[0, 1]$ such* *that* $\limsup_{n\to\infty} \alpha_n < 1$. *Define a sequence* $\{x_n\}$ *in* C *by the following algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \forall\ n \ge 1, \end{cases} \tag{1.7}$$

where

$$\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0, \quad \textit{as } n \to \infty.$$

Then $\{x_n\}$ *defined by* (1.7) *converges strongly to* $P_{F(T)}x_0$.

Subsequently, Marino and Xu [8] extended the result of Nakajo an Takahashi [10] from non-expansive mappings to strict pseudo-contractions. They proved

Theorem MX. *Let* C *be a closed convex subset of a Hilbert space* H *and* $T : C \to C$ *be a* k*-strict pseudo-contraction for some* $0 \le k < 1$ *and assume that the fixed point set* $F(T)$ *of* T *is nonempty. Define a sequence* $\{x_n\}$ *in* C *by the algorithm:*

$$\begin{cases} x_0 \in C \quad \textit{chosen arbitrarily}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + (k-\alpha_n)(1-\alpha_n)\|x_n - Tx_n\|^2\}, \\ Q_n = \{z \in C : \langle x_0 - x_n, x_n - z\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \ge 0. \end{cases} \tag{1.8}$$

Assume that the control sequence $\{\alpha_n\}$ *is such that* $0 \le \alpha_n < 1$ *for all* $n \ge 0$. *Then* $\{x_n\}$ *converges in norm to* $P_{F(T)}x_0$.

In this paper, motivated by Kim and Xu [6], Marino and Xu [8], Martinez-Yanes and Xu [9], Nakajo and Takahashi [10], and Zhou [12], we introduce a new hybrid projection algorithm to modify the normal Mann iterative scheme to obtain the strong convergence for Lipschitz quasi-pseudo-contractions in the framework of Hilbert spaces without any compact assumption.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([8]). *Let* H *be a real Hilbert space. Then the following equations hold:*

(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all* $x, y \in H$.

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H$.

Lemma 1.2. *Let* C *be a closed convex subset of real Hilbert space* H *and let* P_C *be the metric projection from* H *onto* C *(i.e., for* $x \in H$, $P_C x$ *is the only point in* C *such that* $\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}$*). Let* $x \in H$ *and* $z \in C$ *be given. Then* $z = P_C x$ *if and only if there holds the relations:*

$$\langle x - z, y - z\rangle \le 0 \quad \forall\ y \in C.$$

Lemma 1.3. *Let C be a nonempty closed convex subset of H and T a Lipschitz quasi-pseudo-contraction with the Lipschitz constant $L \geq 1$. Then $F(T)$ is nonempty closed convex subset of C.*

Proof. From the continuity of T, one has that $F(T)$ is closed. Next, we show $F(T)$ is convex. Let $p_1, p_2 \in F(T)$. We prove $p \in F(T)$, where $p = tp_1 + (1-t)p_2$, for $t \in (0,1)$. Put $y_\alpha = (1-\alpha)p + \alpha Tp$, where $\alpha \in (0, \frac{1}{1+L})$. For all $w \in F(T)$, one sees

$$\begin{aligned}
&\|p - Tp\|^2 \\
&= \langle p - Tp, p - Tp\rangle \\
&= \frac{1}{\alpha}\langle p - y_\alpha, p - Tp\rangle \\
&= \frac{1}{\alpha}\langle p - y_\alpha, p - Tp - (y_\alpha - Ty_\alpha)\rangle + \frac{1}{\alpha}\langle p - y_\alpha, y_\alpha - Ty_\alpha\rangle \\
&= \frac{1}{\alpha}\langle p - y_\alpha, p - Tp - (y_\alpha - Ty_\alpha)\rangle + \frac{1}{\alpha}\langle p - w + w - y_\alpha, y_\alpha - Ty_\alpha\rangle \\
&\leq \frac{1+L}{\alpha}\|p - y_\alpha\|^2 + \frac{1}{\alpha}\langle p - w, y_\alpha - Ty_\alpha\rangle + \frac{1}{\alpha}\langle w - y_\alpha, y_\alpha - Ty_\alpha\rangle \\
&\leq (1+L)\alpha\|p - Tp\|^2 + \frac{1}{\alpha}\langle p - w, y_\alpha - Ty_\alpha\rangle,
\end{aligned}$$

which yields that

$$\alpha[1 - (1+L)\alpha]\|p - Tp\|^2 \leq \langle p - w, y_\alpha - Ty_\alpha\rangle, \quad \forall w \in F(T). \tag{1.9}$$

Taking $w = p_i$ $i = 1, 2$ in (1.9), multiplying t and $(1-t)$ on the both sides of (1.9), respectively, and adding up, one has

$$\alpha[1 - (1+L)\alpha]\|p - Tp\|^2 \leq \langle p - p, y_\alpha - Ty_\alpha\rangle = 0.$$

This shows that $p \in F(T)$. This completes the proof.

2. Main results

Theorem 2.1. *Let C be a nonempty closed and convex subset of a Hilbert space H and T a Lipschitz quasi-pseudo-contraction from C into itself with the Lipschitz constant $L \geq 1$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases}
x_0 \in H \quad \text{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = P_{C_1}x_0, \\
y_n = (1-\alpha_n)x_n + \alpha_n Tx_n, \\
C_{n+1} = \{z \in C_n : \alpha_n[1 - (1+L)\alpha_n]\|x_n - Tx_n\|^2 \leq \langle x_n - z, y_n - Ty_n\rangle\}, \\
x_{n+1} = P_{C_{n+1}}x_0, \ \forall\, n \geq 1.
\end{cases} \tag{2.1}$$

Assume that the control sequence satisfies the restriction: $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n \leq a < 1$, *where* $a \in (0, \frac{1}{1+L})$. *Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Proof. First, we show that C_n is closed and convex for all $n \geq 1$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For any $z \in C_k$, one can easily see that

$$\alpha_k[1 - (1+L)\alpha_k]\|x_k - Tx_k\|^2 \leq \langle x_k - z, y_k - Ty_k\rangle$$

is closed and convex. So, C_{k+1} is closed and convex. Then, for all $n \geq 1$, C_n is closed and convex. This shows that $P_{C_{n+1}}x_0$ is well defined.

Next, we prove $F(T) \subset C_n$ for all $n \geq 1$. $F(T) \subset C_1 = C$ is obvious. Suppose $F(T) \subset C_k$ for some $k \in \mathbb{N}$. Then, for all $w \in F(T) \subset C_k$, one has

$$\begin{aligned}
&\|x_k - Tx_k\|^2 \\
&= \langle x_k - Tx_k, x_k - Tx_k\rangle \\
&= \frac{1}{\alpha_k}\langle x_k - y_k, x_k - Tx_k\rangle \\
&= \frac{1}{\alpha_k}\langle x_k - y_k, x_k - Tx_k - (y_k - Ty_k)\rangle + \frac{1}{\alpha_k}\langle x_k - y_k, y_k - Ty_k\rangle \\
&= \frac{1}{\alpha_k}\langle x_k - y_k, x_k - Tx_k - (y_k - Ty_k)\rangle + \frac{1}{\alpha_k}\langle x_k - w + w - y_k, y_k - Ty_k\rangle \\
&\leq \frac{1+L}{\alpha_k}\|x_k - y_k\|^2 + \frac{1}{\alpha_k}\langle x_k - w, y_k - Ty_k\rangle + \frac{1}{\alpha_k}\langle w - y_k, y_k - Ty_k\rangle \\
&\leq (1+L)\alpha_k\|x_k - Tx_k\|^2 + \frac{1}{\alpha_k}\langle x_k - w, y_k - Ty_k\rangle.
\end{aligned}$$

It follows that

$$\alpha_k[1 - (1+L)\alpha_k]\|x_k - Tx_k\|^2 \leq \langle x_k - w, y_k - Ty_k\rangle,$$

which shows $w \in C_{k+1}$. This implies that $F(T) \subset C_n$ for all $n \geq 1$. From $x_n = P_{C_n}x_0$, one sees that

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \quad \forall y \in C_n. \tag{2.2}$$

Since $F(T) \subset C_n$ for all $n \geq 1$, we arrive at

$$\langle x_0 - x_n, x_n - w\rangle \geq 0, \quad \forall w \in F(T). \tag{2.3}$$

From Lemma 1.3, we have that $P_{F(T)}x_0$ is well defined. There exists a unique p such that $p = P_{F(T)}x_0$. It follows that

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - p\|,
\end{aligned}$$

which yields that

$$\|x_0 - x_n\| \leq \|x_0 - p\|. \tag{2.4}$$

It follows from (2.4) that the sequence $\{x_n\}$ is bounded. Noticing that $x_{n+1} \in C_{n+1} \subset C_n$ and (2.2), one has

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0.$$

It follows that

$$\begin{aligned}0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\ &\leq -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|,\end{aligned}$$

that is, $\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|$. This together with the boundedness of $\{x_n\}$ implies that $\lim_{n\to\infty} \|x_0 - x_n\|$ exists. By the construction of C_n, one has that $C_m \subset C_n$ and $x_m = P_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. From (2.2), we have

$$\langle x_0 - x_n, x_n - x_{n+m}\rangle \geq 0. \tag{2.5}$$

It follows that

$$\begin{aligned}\|x_n - x_{n+m}\|^2 &= \|x_n - x_0 + x_0 - x_{n+m}\|^2 \\ &= \|x_n - x_0\|^2 + \|x_0 - x_{n+m}\|^2 - 2\langle x_0 - x_n, x_0 - x_{n+m}\rangle \\ &\leq \|x_0 - x_{n+m}\|^2 - \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_n - x_{n+m}\rangle \\ &\leq \|x_0 - x_{n+m}\|^2 - \|x_n - x_0\|^2.\end{aligned} \tag{2.6}$$

Letting $n \to \infty$ in (2.6), one has $\lim_{n\to\infty} \|x_n - x_{n+m}\| = 0$, for each $m > n$. Hence, $\{x_n\}$ is a Cauchy sequence. Since H is a Hilbert space and C is closed and convex, one can assume that

$$x_n \to q \in C \quad \text{as } n \to \infty.$$

Finally, we show that $q = P_{F(T)} x_0$. To show this, we first show $q \in F(T)$. By taking $m = 1$ in (2.6), one arrives at $\|x_n - x_{n+1}\| \to 0$, as $n \to \infty$. Noticing that $x_{n+1} \in C_{n+1}$, we obtain

$$\alpha_n[1 - (1 + L)\alpha_n]\|x_n - Tx_n\|^2 \leq \|x_n - x_{n+1}\|\|y_n - Ty_n\|.$$

It follows from the assumptions on $\{\alpha_n\}$ and boundedness of $\{y_n - Ty_n\}$ that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Thus, $Tx_n \to q$ as $n \to \infty$. Now the closed-ness property of T gives that q is a fixed point of T. From (2.3), one has

$$\langle x_0 - q, q - w\rangle \geq 0, \quad \forall w \in F(T),$$

which implies that $q = P_{F(T)} x_0$. This completes the proof.

If T is a Lipschitz pseudo-contraction in Theorem 2.1, then the following result can be obtained immediately.

Corollary 2.2. *Let C be a nonempty and closed convex subset of a Hilbert space H and T a Lipschitz pseudo-contraction from C into itself with the Lipschitz constant $L \geq 1$ with a nonempty fixed point set. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$\begin{cases} x_0 \in H \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n = (1 - \alpha_n)x_n + \alpha_n T x_n, \\ C_{n+1} = \{z \in C_n : \alpha_n[1 - (1 + L)\alpha_n]\|x_n - Tx_n\|^2 \leq \langle x_n - z, y_n - Ty_n\rangle\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \ \forall\, n \geq 1. \end{cases}$$

Assume that the control sequence satisfies the restriction:

$$0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n \le a < 1,$$

where $a \in (0, \frac{1}{1+L})$*, Then* $\{x_n\}$ *converges strongly to* $P_{F(T)}x_0$.

Remark 2.3. Corollary 2.2 improves the result of Marino and Xu [8] from strict pseudo-contractions to Lipschitz pseudo-contraction. It also includes the results of Nakajo and Takahashi [10] as a special case. The hybrid projection algorithm is also simpler than those studied by [8-10].

Remark 2.4. It is of interest to entitle the hybrid projection algorithm (2.1) "C method" regarding to the "CQ method" introduced by Martinez-Yanes and Xu [9].

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