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EXISTENCE RESULTS FOR SYSTEMS OF NONLINEAR EVOLUTION INCLUSIONS

RADU PRECUP AND ADRIAN VIOREL

Department of Mathematics, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania
E-mails: r.precup@math.ubbcluj.ro adrian.viorel@math.ubbcluj.ro

Abstract. Existence results for semilinear systems of abstract evolution inclusions are established by means of Nadler, Bohnenblust-Karlin and Leray-Schauder fixed point theorems and a new technique for the treatment of systems based on vector-valued metrics and convergent to zero matrices.
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1. Introduction and preliminaries

In [8] the first author developed a technique for the investigation of systems of nonlinear operator equations which is based on vector-valued metrics and convergent to zero matrices together with fundamental principles of nonlinear functional analysis. It is shown in [8] that the use of vector-valued metrics is more appropriate when treating systems of equations. The technique was applied in [10] to obtain existence of solutions for the Cauchy problem associated to a semilinear system of abstract evolution equations:

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) = F_1(t, u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) = F_2(t, u_1(t), u_2(t)) \\ u_1(0) = u_1^0 \\ u_2(0) = u_2^0. \end{cases} \tag{1.1}$$

The aim of this paper is to extend the methods and results from [10] to inclusions. More exactly we are concerned with the Cauchy problem associated to the semilinear system of abstract evolution inclusions:

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) \in F_1(u_1(t), u_2(t)) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) \in F_2(u_1(t), u_2(t)) \\ u_1(0) = u_1^0 \\ u_2(0) = u_2^0. \end{cases} \tag{1.2}$$

Here $A_i : D(A_i) \subseteq X_i \to X_i$ is assumed to be a linear operator, densely defined on the real Banach space X_i which generates the strongly continuous semigroup of contractions $\{S_i(t), t \geq 0\}$, and $F_i : X_1 \times X_2 \to 2^{X_i}$ is a multivalued operator, for $i = 1, 2$.

We shall look for *global mild solutions* to (1.2) on the interval $[0, T]$, i.e., $u = (u_1, u_2) \in C([0, T], X_1 \times X_2) = C([0, T], X_1) \times C([0, T], X_2)$ such that

$$u_i(t) = S_i(t)u_i^0 + \int_0^t S_i(t - \tau)w_i(\tau)d\tau \quad t \in [0, T],$$

where $w_i \in L^1([0, T], X_i)$ is a selection for the multivalued function $t \longmapsto F_i(u(t))$, i.e.,

$$w_i(t) \in F_i(u(t)) \quad \text{a.e. } t \in [0, T] \quad i = 1, 2.$$

In the next section three different fixed point principles will be used in order to prove the existence of solutions for the semilinear problem, namely the multivalued versions of the fixed point theorems of Perov, Schauder and Leray-Schauder (see [9]). In all three cases a key role will be played by the so called convergent to zero matrices. A square matrix M with nonnegative elements is said to be *convergent to zero* if

$$M^k \to 0 \quad \text{as} \quad k \to \infty.$$

It is known that the property of being convergent to zero is equivalent to each of the following three conditions (for details see [6], [9], [11]):

(a) $I - M$ is nonsingular and $(I - M)^{-1} = I + M + M^2 + ...$ (where I stands for the unit matrix of the same order as M);

(b) the eigenvalues of M are located inside the unit disc of the complex plane;

(c) $I - M$ is nonsingular and $(I - M)^{-1}$ has nonnegative elements.

Let X be a nonempty set. By a *vector-valued metric* on X we mean a mapping $d : X \times X \to \mathbb{R}^n_+$ such that

(i) $d(u, v) \geq 0$ for all $u, v \in X$ and if $d(u, v) = 0$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \leq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Here, if $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, by $x \leq y$ we mean $x_i \leq y_i$ for $i = 1, 2, ..., n$. We call the pair (X, d) a *generalized metric space*. For such a space convergence and completeness are similar to those in usual metric spaces.

Let (X, d) be a metric space. For two nonempty sets $A, B \subset X$ and $x \in X$ we use the following notations:

$$\begin{aligned} d(x, A) &= \inf\{d(x, a) : a \in A\}; \\ H(A, B) &= \max\left\{\sup_{a \in A} d(a, B); \sup_{b \in B} d(b, A)\right\}; \\ \delta(A, B) &= \sup\{d(a, b) : a \in A, b \in B\}. \end{aligned}$$

Recall that H is a metric (the Hausdorff-Pompeiu metric) on the set of all nonempty closed bounded subsets of (X, d). Also note the following property of this metric:

Remark 1.1. *Let (X, d) be a metric space, $A, B \subset X$ nonempty closed bounded sets and $q > 1$. Then for each $a \in A$ there exists $b \in B$ such that $d(a, b) \leq qH(A, B)$.*

We first give a vector version of Nadler's fixed point theorem [5, p 28].

Theorem 1.2. *Let $(X_1, d_1), (X_2, d_2)$ be two complete metric spaces and $N : X_1 \times X_2 \to 2^{X_1 \times X_2}$ a multivalued operator with $N(x)$ nonempty closed bounded for each $x \in X_1 \times X_2$. Assume that there exists matrix M which is convergent to zero, such that*

$$\begin{pmatrix} H_1(N_1(u), N_1(v)) \\ H_2(N_2(u), N_2(v)) \end{pmatrix} \leq M \begin{pmatrix} d_1(u_1, v_1) \\ d_2(u_2, v_2) \end{pmatrix} \tag{1.3}$$

for all $u = (u_1, u_2), v = (v_1, v_2) \in X_1 \times X_2$, where $N_1 : X_1 \times X_2 \to 2^{X_1}$ and $N_2 : X_1 \times X_2 \to 2^{X_2}$ are the two components of N and H_1, H_2 stand for the Hausdorff-Pompeiu metrics associated to d_1 and d_2, respectively. Then N has a fixed point.

Proof. First we note that $X_1 \times X_2$ endowed with the vector-valued metric

$$d(u,v) = \begin{pmatrix} d_1(u_1, v_1) \\ d_2(u_2, v_2) \end{pmatrix}$$

where $u = (u_1, u_2), v = (v_1, v_2)$, is a complete generalized metric space.

Starting with any $x^0 \in X_1 \times X_2$ and $x^1 \in N(x^0)$ we shall build a sequence of successive approximations. Take any number $q > 1$ such that qM is still convergent to zero. From Remark 1.1, there exists $x_i^2 \in N_i(x^1)$ with

$$d_i(x_i^1, x_i^2) \leq qH_i(N_i(x^0), N_i(x^1)).$$

Let $x^2 = (x_1^2, x_2^2)$. Clearly $x^2 \in F(x^1)$. Now we use relationship (1.3) to obtain

$$\begin{aligned} d(x^1, x^2) &\leq q \begin{pmatrix} H_1(N_1(x^1), N_1(x^2)) \\ H_2(N_2(x^1), N_2(x^2)) \end{pmatrix} \\ &\leq qM \begin{pmatrix} d_1(x_1^0, x_1^1) \\ d_2(x_2^0, x_2^1) \end{pmatrix} \\ &= qMd(x^0, x^1). \end{aligned}$$

Recursively we generate a sequence (x^n) such that for all $n \in \mathbb{N}$

$$x^{n+1} \in N(x^n) \quad \text{and} \quad d(x^n, x^{n+1}) \leq q^n M^n d(x^0, x^1).$$

We show that this sequence is fundamental and thus it converges in the complete generalized metric space $(X_1 \times X_2, d)$. Indeed,

$$\begin{aligned} d(x^n, x^{n+p}) &\leq d(x^n, x^{n+1}) + \dots + d(x^{n+p-1}, x^{n+p}) \\ &\leq q^n M^n (I + qM + \dots + q^{p-1}M^{p-1}) d(x^0, x^1) \\ &\leq q^n M^n (I - qM)^{-1} d(x^0, x^1). \end{aligned}$$

Since qM is convergent to zero, we deduce that (x^n) is fundamental as claimed.

Finally we show that the limit x^* of (x^n) is a fixed point of N. Indeed, for each $i = 1, 2$ we have

$$\begin{aligned} d_i(x_i^*, N_i(x^*)) &\leq d_i(x_i^*, x_i^{n+1}) + d_i(x_i^{n+1}, N_i(x^*)) \\ &\leq d_i(x_i^*, x_i^{n+1}) + H_i(N_i(x^n), N_i(x^*)) \end{aligned}$$

as $x_i^{n+1} \in N_i(x^n)$. Writing these inequalities in a vector form

$$\begin{pmatrix} d_1(x_1^*, N_1(x^*)) \\ d_2(x_2^*, N_2(x^*)) \end{pmatrix} \leq \begin{pmatrix} d_1\left(x_1^*, x_1^{n+1}\right) \\ d_2\left(x_2^*, x_2^{n+1}\right) \end{pmatrix} + \begin{pmatrix} H_1(N_1(x^n), N_1(x^*)) \\ H_2(N_2(x^n), N_2(x^*)) \end{pmatrix}$$

and applying (1.3) to the second term of the right hand side we obtain

$$\begin{aligned} \begin{pmatrix} d_1(x_1^*, N_1(x^*)) \\ d_2(x_2^*, N_2(x^*)) \end{pmatrix} &\leq \begin{pmatrix} d_1\left(x_1^*, x_1^{n+1}\right) \\ d_2\left(x_2^*, x_2^{n+1}\right) \end{pmatrix} + M \begin{pmatrix} d_1(x_1^*, x_2^n) \\ d_2(x_2^*, x_2^n) \end{pmatrix} \\ &= d\left(x^*, x^{n+1}\right) + Md(x^n, x^*) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $x^* \in N(x^*)$. □

Finally we recall two basic topological fixed point theorems for set-valued maps (see e.g. [3])

Theorem 1.3 (Bohnenblust-Karlin). *Let X be a Banach space, $D \subset X$ nonempty closed convex bounded and $N : D \to 2^X$ upper semicontinuous with $N(x)$ nonempty closed convex for all $x \in D$. If $N(D) \subset D$ and $N(D)$ is relatively compact, then N has at least one fixed point.*

Theorem 1.4 (Leray-Schauder). *Let X be a Banach space, $U \subset X$ open bounded and $N : \overline{U} \to 2^X$ upper semicontinuous with $N(x)$ nonempty closed convex for all $x \in \overline{U}$. If $N\left(\overline{U}\right)$ is relatively compact and $x_0 + \lambda(x - x_0) \notin N(x)$ on ∂U for all $\lambda > 1$, then N has at least one fixed point.*

2. Existence results

Obviously a mild solution for (1.2) is a fixed point of the multivalued operator

$$N : C([0,T], X_1) \times C([0,T], X_2) \to 2^{C([0,T], X_1 \times X_2)},$$
$$N(u) = (N_1(u), N_2(u)),$$

where

$$\begin{aligned} N_i(u) = \Big\{ S_i(t)u_i^0 + \textstyle\int_0^t S_i(t-\tau)w_i(\tau)d\tau : \\ w_i \in L^1([0,T], X_i), w_i(t) \in F_i(u(t)) \text{ a.e. } t \in [0,T] \Big\}. \end{aligned} \tag{2.1}$$

The multivalued operator N can be written as a composition of a single-valued operator $\mathcal{N}$ with a multivalued operator W, i.e., $N = \mathcal{N} \circ W$, where

$$W : C([0,T], X_1 \times X_2) \to 2^{L^1([0,T], X_1 \times X_2)}, \quad W(u) = (W_1(u), W_2(u)),$$
$$W_i(u) = \left\{ w_i \in L^1([0,T], X_i) : w_i(t) \in F_i(u(t)) \text{ a.e. } t \in [0,T] \right\}$$

and

$$\mathcal{N} : L^1([0,T], X_1 \times X_2) \to C([0,T], X_1 \times X_2), \quad \mathcal{N}(f) = (\mathcal{N}_1(f)(t), \mathcal{N}_2(f)(t)),$$
$$\mathcal{N}_i(f)(t) := S_i(t)u_i^0 + \int_0^t S_i(t-\tau)f_i(\tau)d\tau$$

for any $f = (f_1, f_2) \in L^1([0,T], X_1 \times X_2) = L^1([0,T], X_1) \times L^1([0,T], X_2)$.

We list below some properties of N.

Lemma 2.1. *If F_i has bounded values for $i = 1, 2$ then the operator N has bounded values $N(u)$.*

Proof. Since $N(u) = \mathcal{N}(W(u))$ and $\mathcal{N}$ is continuous we only have to show that $W(u)$ has bounded values.

$F_i(u)$ are all bounded and thus there is $R_i \in \mathbb{R}_+$ such that

$$\|f_i\| \leq R_i \text{ for all } f_i \in F_i(u(t)), \ t \in [0, T].$$

Let $w = (w_1, w_2) \in W(u)$, because $w_i(t) \in F_i(u(t))$ a.e. $t \in [0, T]$ we have

$$\|w_i(t)\| \leq R_i \text{ a.e. } t \in [0, T]$$

and

$$\|w_i\|_{L^1([0,T],X_i)} \leq \int_0^T R_i dt = R_i T$$

for $i = 1, 2$. And thus $W(u)$ is bounded. □

Lemma 2.2. *If F_i has closed values for $i = 1, 2$ then the operator N has closed values $N(u)$.*

Proof. Since $N(u) = \mathcal{N}(W(u))$ and $\mathcal{N}$ is continuous we only have to show that W has closed values.

Let $\left(w_i^k\right)_{k \in \mathbb{N}} \subset W_i(u)$ a convergent sequence $w_i^k \to w_i$. We show that $w_i \in W_i(u)$, i.e., $w_i \in L^1([0, T], X_i)$ and $w_i(t) \in F_i(u(t))$ a.e. $t \in [0, T]$. Obviously

$$w_i \in L^1([0, T], X_i) \tag{2.2}$$

is satisfied.

Now, since $w_i^k \to w_i$, we have, at least for a subsequence, that

$$w_i^k(t) \to w_i(t) \text{ a.e. } t \in [0, T].$$

But for all $k \in \mathbb{N}$

$$w_i^k(t) \in F_i(u(t)) \text{ a.e. } t \in [0, T],$$

and thus

$$w_i(t) \in F_i(u(t)) \text{ a.e. } t \in [0, T] \tag{2.3}$$

which concludes our proof. □

Our goal now is to prove that the multivalued operator N also preserves the *upper semicontinuity* of F_i.

Definition 2.3. *A multifunction $F : X \to 2^Y$, X and Y being two metric spaces, is* upper semicontinuous *if the set*

$$F^-(U) := \{x \in X : F(x) \cap U \neq \varnothing\}$$

is closed for each closed set $U \subset Y$.

Remark 2.4. *Let (X, d) be a metric space, if $F : X \to 2^X$ has bounded closed values and is upper semicontinuous then, for any sequence (x_k) in X such that $x_k \to x$, we have*

$$\sup_{y \in F_i(x_k)} D(y, F_i(x)) \to 0 \text{ as } k \to \infty.$$

Remark 2.5. *The composition of two upper semicontinuous operators is also upper semicontinuous.*

Lemma 2.6. *If F_i has bounded values and is upper semicontinuous for $i = 1, 2$ then the operator N is also upper semicontinuous.*

Proof. Since N can be written as

$$N(u) = \mathcal{N}(W(u))$$

and $\mathcal{N}$ is continuous it enough to show that W is upper semicontinuous for N to be so.

Let U be a closed set in $L^1([0, T], X_1) \times L^1([0, T], X_2)$ we have to show that

$$W^-(U) := \{x \in C([0, T], X_1) \times C([0, T], X_2) : W(x) \cap U \neq \varnothing\}$$

is closed.

For any sequence $(x_k) \subset W^-(U)$ such that $x_k \to x$ we show that $x \in W^-(U)$.

Inded, if $x_k \to x$ then the sequence also converges poinwise $x_k(t) \to x(t)$ and since F_i is upper semicontinuous, we can apply Remark 2.4 to obtain

$$\sup_{y \in F_i(x_k(t))} D_{X_i}(y, F_i(x(t))) \to 0 \text{ as } k \to \infty. \tag{2.4}$$

Now, since $x_k \in W^-(U)$, i.e., $W(x_k) \cap U \neq \varnothing$,

$$\exists w^k \in U \text{ such that } w_i^k(t) \in F_i(x_k(t)) \text{ a.e. } t \in [0, T] \tag{2.5}$$

for all $k \in \mathbb{N}$.

If we apply (2.4) to each w^k in (2.5) we obtain that there exists a $w_i(t) \in F_i(x(t))$ such that

$$w_i^k(t) \to w_i(t) \text{ a.e. } t \in [0, T].$$

The fact that the sets $F_i(x_k(t))$ are bounded for all $t \in [0, T]$ and $k \in \mathbb{N}$ assures the bondedness of $w_k(t)$ a.e. on $[0, T]$, and thus we can apply Lesbegue's bounded convergence theorem to prove that $w \in W(x) \cap U$. □

Our first existence result is established by means of the vector version of Nadler's theorem.

Theorem 2.7. *Let $F_i : X_1 \times X_2 \to 2^{X_i}$ and assume that $F_i(x)$ is nonempty closed bounded for each $x \in X_1 \times X_2$. In addition assume that there are constants $a_{ij} \geq 0$ for $i, j = 1, 2$ such that*

$$\delta_{X_i}(F_i(u), F_i(v)) \leq a_{i1} \|u_1 - v_1\|_{X_1} + a_{i2} \|u_2 - v_2\|_{X_2} \tag{2.6}$$

for all $u = (u_1, u_2)$, $v = (v_1, v_2) \in X_1 \times X_2$ and $i = 1, 2$. Then problem (1.2) has a mild solution.

Proof. We apply Theorem 1.2 to the operator N defined in (2.1). We will use the following notation for the space of continuous X_i-valued functions on $[0,T]$, endowed with the Bielecki norm

$$E_i := C([0,T], X_i) \quad \|u\|_{E_i} := \max_{t\in[0,T]} e^{-kt}\|u(t)\|_{X_i}$$

for some constant $k>0$.Let $u,v \in C([0,T], X_1\times X_2)$. Then for every $w^u \in W(u)$ and $w^v \in W(v)$, we have

$$\begin{aligned}\|\mathcal{N}_i(w^u)(t) - \mathcal{N}_i(w^v)(t)\|_{X_i} &\le \left\|\int_0^t S_i(t-\tau)(w_i^u(\tau) - w_i^v(\tau))\,d\tau\right\|_{X_i}\\ &\le \int_0^t \|S_i(t-\tau)\|\,\|(w_i^u(\tau)-w_i^v(\tau))\|_{X_i}\,d\tau\\ &\le \int_0^t \|(w_i^u(\tau)-w_i^v(\tau))\|_{X_i}\,d\tau.\end{aligned}$$

but since $w_i^u(\tau)\in F_i(u(\tau))$ and $w_i^v(\tau)\in F_i(v(\tau))$, by the definition of the functional δ and using (2.6) we have

$$\begin{aligned}\|(w_i^u(\tau)-w_i^v(\tau))\|_{X_i} &\le \delta(F_i(u(\tau)), F_i(v(\tau)))\\ &\le a_{i1}\|u_1(\tau)-v_1(\tau)\|_{X_1} + a_{i2}\|u_2(\tau)-v_2(\tau)\|_{X_2}.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{N}_i(w^u)(t) - \mathcal{N}_i(w^v)(t)\|_{X_i} &\le \int_0^t \left(a_{i1}\|u_1(\tau)-v_1(\tau)\|_{X_1} + a_{i2}\|u_2(\tau)-v_2(\tau)\|_{X_2}\right)d\tau\\ &\le \|u_1-v_1\|_{E_1}\int_0^t a_{i1}e^{k\tau}d\tau + \|u_2-v_2\|_{E_2}\int_0^t a_{i2}e^{k\tau}d\tau\\ &\le \frac{a_{i1}}{k}\|u_1-v_1\|_{E_1}e^{kt} + \frac{a_{i2}}{k}\|u_2-v_2\|_{E_2}e^{kt}\end{aligned}$$

for any $t\in[0,T]$ and $i=1,2$.

It follows that

$$\|\mathcal{N}_i(w^u) - \mathcal{N}_i(w^v)\|_{E_i} \le \frac{a_{i1}}{k}||u_1-v_1||_{E_1} + \frac{a_{i2}}{k}||u_2-v_2||_{E_2}.$$

Then by the definition of the Hausdorff-Pompeiu metric H_{E_i}

$$H_{E_i}(N_i(u), N_i(v)) \le \frac{a_{i1}}{k}||u_1-v_1||_{E_1} + \frac{a_{i2}}{k}||u_2-v_2||_{E_2}$$

for $i=1,2$. This can be written in matrix form as

$$\begin{pmatrix} H_{E_1}(N_1(u), N_1(v)) \\ H_{E_2}(N_2(u), N_2(v)) \end{pmatrix} \le M_k \begin{pmatrix} ||u_1-v_1||_{E_1} \\ ||u_2-v_2||_{E_2} \end{pmatrix}$$

with

$$M_k = \begin{pmatrix} \frac{a_{11}}{k} & \frac{a_{12}}{k} \\ \frac{a_{21}}{k} & \frac{a_{22}}{k} \end{pmatrix}. \tag{2.7}$$

Finally note that M_k is convergent to zero provided that $k>0$ is chosen sufficiently large. Now the conclusion follows from Theorem 1.2. $\square$

The next existence result is an application of Theorem 1.3 and uses growth conditions on F_i which are more general than the Lipschitz condition (2.6).

Theorem 2.8. *Let $F_i : X_1 \times X_2 \to 2^{X_i}$ be upper semicontinuous with $F_i(x)$ nonempty closed bounded for each $x \in X_1 \times X_2$. Assume that there exist constants $a_{ij} \geq 0$ and $b_i \geq 0$ for $i, j = 1, 2,$ such that*

$$\|w\|_{X_i} \leq a_{i1} \|u_1\|_{X_1} + a_{i2} \|u_2\|_{X_2} + b_i \tag{2.8}$$

for all $u = (u_1, u_2) \in X_1 \times X_2$ and $w \in F_i(u)$ $(i = 1, 2)$. If in addition operator N is completely continuous, then problem (1.2) has at least one mild solution.

Proof. In the Lemmas 2.1, 2.2, 2.6 we have proven that the operator N has bounded closed values and is upper semicontinuous. In order to apply Theorem 1.3 we need to find a nonempty closed convex bounded set $D \subset E_1 \times E_2$ such that

$$N(D) \subseteq D \tag{2.9}$$

Let us consider the set $D := \overline{B}_{R_1}(0; E_1) \times \overline{B}_{R_2}(0; E_2)$, where $\overline{B}_{R_i}(0; E_i)$ is the closed ball centered in origin of E_i of radius R_i. We try to find $R_1, R_2 > 0$ such that (2.9) holds. For $u \in C([0,T], X_1 \times X_2)$ and any $w \in W(u)$ we have

$$\begin{aligned} \|\mathcal{N}_i(w)(t)\|_{X_i} &\leq \left\|S_i(t) u_i^0\right\|_{X_i} + \left\|\int_0^t S_i(t-\tau) w_i(\tau) d\tau\right\|_{X_i} \\ &\leq \left\|u_i^0\right\|_{X_i} + \int_0^t \|w_i(\tau)\|_{X_i} d\tau. \end{aligned}$$

But since $w_i(\tau) \in F_i(u(\tau))$, we then have also using (2.8)

$$\begin{aligned} \|\mathcal{N}_i(w)(t)\|_{X_i} &\leq \left\|u_i^0\right\|_{X_i} + \int_0^t \left(a_{i1} \|u_1(\tau)\|_{X_1} + a_{i2} \|u_2(\tau)\|_{X_2} + b_i\right) d\tau \\ &\leq \|u_1\|_{E_1} \int_0^t a_{i1} e^{k\tau} d\tau + \|u_2\|_{E_2} \int_0^t a_{i2} e^{k\tau} d\tau + \left\|u_i^0\right\|_{X_i} + b_i T \end{aligned}$$

for any $t \in [0,T]$ and $i = 1, 2$. Consequently

$$\|\mathcal{N}_i(w)\|_{E_i} \leq \frac{a_{i1}}{k} ||u_1||_{E_1} + \frac{a_{i2}}{k} ||u_2||_{E_2} + c_i,$$

where $c_i = \left\|u_i^0\right\|_{X_i} + b_i T$. Now if $u \in D$, i.e., $||u_i||_{E_i} \leq R_i$ for $i = 1, 2$, we have

$$\|\mathcal{N}_i(w)\|_{E_i} \leq \frac{a_{i1}}{k} R_1 + \frac{a_{i2}}{k} R_2 + c_i \quad (i = 1, 2)$$

or, equivalently

$$\begin{pmatrix} \|\mathcal{N}_1(w)\|_{E_1} \\ \|\mathcal{N}_2(w)\|_{E_2} \end{pmatrix} \leq M_k \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

where M_k is given by (2.7). Now coose $k > 0$ such that M_k is convergent to zero and take R_1, R_2 the solution of the algebraic system

$$M_k \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} R_1 \\ R_2 \end{pmatrix}.$$

Hence

$$\begin{pmatrix} R_1 \\ R_2 \end{pmatrix} = (I - M_k)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{2.10}$$

Notice R_1, R_2 are nonnegative according to property (c) of convergent to zero matrices. Thus, for $i \in \{1,2\}$ we have that $\|\mathcal{N}_i(w)\|_{E_i} \leq R_i,$, for every $w \in W(u)$ and $u \in D$. This shows that $N(D) \subset D$. Therefore Theorem 1.3 applies. □

In the case of Hilbert spaces and if all mild solutions are classical solutions (i.e., each component is in $C([0,T], D(A_i)) \cap C^1([0,T], X_i)$ and satisfies (1.2)) the existence can be also derived from Theorem 1.4.

Theorem 2.9. *Let $(X_i, \langle .,. \rangle_{X_i})$, $i = 1,2$ be real Hilbert spaces, assume that all mild solutions of the system*

$$\begin{cases} \dfrac{du_1}{dt}(t) + A_1 u_1(t) \in \lambda F_1(u) \\ \dfrac{du_2}{dt}(t) + A_2 u_2(t) \in \lambda F_2(u) \\ u_1(0) = \lambda u_1^0 \\ u_2(0) = \lambda u_2^0 \end{cases} \tag{2.11}$$

for $\lambda \in (0,1)$ are classical solutions and that the nonlinear operator N is completely continuous. In addition, assume that there exist constants $a_{ij} \geq 0$ and $b_i \geq 0$ for $i,j = 1,2$ such that

$$\sup_{w_i \in F_i(u)} \langle w_i, u_i \rangle_{X_i} \leq a_{i1} \|u_1\|_{X_1}^2 + a_{i2} \|u_2\|_{X_2}^2 + b_i \tag{2.12}$$

for all $u \in X_1 \times X_2$, $i = 1,2$. Then problem (1.2) has at least one solution.

Proof. Let $u = (u_1, u_2)$ be any solution of (2.11). Then there exists $w \in W(u)$ such that

$$\frac{du_i}{dt}(t) + A_i u_i(t) = \lambda w_i(t), \quad i = 1,2.$$

Taking in this equation the inner product in X_i with $u_i(t)$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_i(t)\|_{X_i}^2 + \langle A_i u_i(t), u_i \rangle_{X_i} = \lambda \langle w_i, u_i \rangle_{X_i}.$$

Then using $\langle A_i x, x \rangle_{X_i} \geq 0$ for all $x \in D(A_i)$ (for details about this property of generators of semigroups in Hilbert spaces see for example ([7]), and (2.12)) we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_i(t)\|_{X_i}^2 \leq \lambda \langle w_i(t), u_i(t) \rangle_{X_i} \leq a_{i1} \|u_1(t)\|_{X_1}^2 + a_{i2} \|u_2(t)\|_{X_2}^2 + b_i.$$

Integrating with respect to t, we deduce that

$$\|u_i(t)\|_{X_i}^2 \leq \left\|u_i^0\right\|_{X_i}^2 + 2\int_0^t \left(a_{i1} \|u_1(\tau)\|_{X_1}^2 + a_{i2} \|u_2(\tau)\|_{X_2}^2 + b_i\right) d\tau$$

for all $t \in [0,T]$ and $i = 1,2$.

Using the same kind of estimates as in the proof of Theorem 2.8 we then deduce that

$$\|u_i\|_{E_i}^2 \leq \frac{a_{i1}}{k}||u_1||_{E_1}^2 + \frac{a_{i2}}{k}||u_2||_{E_2}^2 + c_i, \quad i = 1,2$$

where $c_i = \left\| u_i^0 \right\|_{X_i}^2 + 2b_iT$. This can be rewritten using the matrix form as follows

$$(I - M_k) \begin{pmatrix} ||u_1||_{E_1}^2 \\ ||u_2||_{E_2}^2 \end{pmatrix} \leq \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{2.13}$$

where again M_k is given by (2.7). For a sufficiently large k, matrix M_k is convergent to zero. Hence, $I - M_k$ is nonsingular and $(I - M_k)^{-1}$ has nonnegative elements. Multiplication of both sides of (2.13) with $(I - M_k)^{-1}$ yields

$$\begin{pmatrix} ||u_1||_{E_1}^2 \\ ||u_2||_{E_2}^2 \end{pmatrix} \leq (I - M_k)^{-1} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

This guarantees the a priori boundedness of all solutions $u = (u_1, u_2)$ of the equations $u \in \lambda N(u)$, for $\lambda \in (0, 1)$. Thus we may apply Theorem 1.4. □

Notice that sufficient conditions for the complete continuity of operator N, as well as for that mild solutions be classical solutions are available in literature, see for example [2], [13] and [14]. Related topics in connection with Perov's method can be found in [1] and [12].

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