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# AN ITERATIVE METHOD FOR A FUNCTIONAL-DIFFERENTIAL EQUATION WITH MIXED TYPE ARGUMENT

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**Abstract.** In this paper we shall study a functional-differential equations with mixed type argument. For this problem we give an algorithm based on the step method and the successive approximations method.

**Key Words and Phrases**: Functional-differential equations, mixed type argument, step method, successive approximation method, Newton's method.
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## 1. Introduction

In this paper we study the following problem

$$x'(t) = f(t, x(t), x(t-h), x(t+h)),\ t \in [-T, T], \tag{1.1}$$

$$x(t) = \varphi(t),\ t \in [-h, h]. \tag{1.2}$$

The problem (1.1)-(1.2) has been studied in the papers R. Driver [4], I.A. Rus and C. Iancu [9] and V.A. Darzu [2]. This problem is known in literature as Wheeler-Feynman problem. For this problem the above authors studied the existence and uniqueness of the solution using the step method.

The purpose of this paper is to elaborate an algorithm based on the step method and the successive approximation method and to apply it on the problem (1.1)-(1.2).

**Algorithm.** *At each step we have a problem like this*

$$\begin{cases} x'(t) = f(t, x(t), x(t-h), x(t+h)),\ t \in [a+h, a+2h] \\ x(t) = \theta(t),\ t \in [a, a+2h], \end{cases} \tag{1.3}$$

*where* $f \in C([a+h, a+2h] \times \mathbb{R}^3, \mathbb{R}), \theta \in C([a, a+2h], \mathbb{R})$ *and* $x : [a, a+3h] \to \mathbb{R}$.

*It follows that*

$$\theta'(t) = f(t, \theta(t), \theta(t-h), x(t+h)),\ t \in [a+h, a+2h].$$

*We denote* $\xi := t + h,\ \xi \in [a+2h, a+3h]$.

*Then*

$$\theta'(\xi - h) = f(\xi - h, \theta(\xi - h), \theta(\xi - 2h), x(\xi)),\ \xi \in [a+2h, a+3h].$$

*We denote* $F(\xi, x(\xi)) := f(\xi - h, \theta(\xi - h), \theta(\xi - 2h), x(\xi)) - \theta'(\xi - h)$. *So*

$$F(\xi, x(\xi)) = 0. \tag{1.4}$$

*The purpose is to impose conditions on* $f$ *such that equation (1.4) has a unique solution who can be approximated by Newton's method.*

In order to study the problem (1.1)-(1.2) we need the following well known results.

**Implicit function theorem.** *(*[1]*) We suppose that* $F : [a, b] \times \mathbb{R} \to \mathbb{R}$ *satisfy the following conditions*

(i) $F \in C^1([a, b] \times \mathbb{R})$;
(ii) *there exists* $\frac{\partial F(t,u)}{\partial u} \in \mathbb{R}^*$ *and* $\left|\frac{\partial F(t,u)}{\partial u}\right| \le M_1,\ \forall t \in [a, b], u \in \mathbb{R}$;
(iii) *for each* $t_0 \in [a, b]$, *there exists* $u_0 \in \mathbb{R}$ *such that* $F(t_0, u_0) = 0$.

*Then, there exists a unique solution* $x \in C^1[a, b]$ *such that* $F(t, x(t)) = 0$, $\forall (t, x(t)) \in [a, b] \times \mathbb{R}$, *solution that can be obtained using the successive approximations method.*

In terms of $f$, for the problem (1.1)-(1.2), the conditions from the above theorem are:

($C_1$) $f \in C^\infty([-T, T] \times \mathbb{R}^3, \mathbb{R}), \varphi \in C([-h, h], \mathbb{R})$;
($C_2$) $\frac{\partial f(t,u,v,w)}{\partial w} \in \mathbb{R}^*$, $\forall t \in [-T, T]$, $\forall u, v, w \in \mathbb{R}$;
($C_3$) $\left|\frac{\partial f(t,u,v,w)}{\partial w}\right| \le M_1, \forall t \in [-T, T],\ \forall u, v, w \in \mathbb{R}$;
($C_4$) $\forall t \in [-T, T], u, v, \eta \in \mathbb{R}$, the equation $f(t, u, v, w) - \eta = 0$ has a unique solution.

We shall use the notations, the terminology and some results given by I.A. Rus in the paper [6], and the following result, that is a generalization of the fibre contraction theorem (see [6]).

**Fibre contraction theorem.** *(Theorem 9.1.,* [7]*) Let* $(X_i, d_i),\ i = \overline{0, m},\ m \ge 1$, *be some metric spaces. Let*

$$A_i : X_0 \times \cdots \times X_i \to X_i,\ i = \overline{0, m}$$

*be some operators. We suppose that:*

(i) $(X_i, d_i),\ i = \overline{1, m}$, *are complete metric spaces;*
(ii) *the operator* $A_0$ *is WPO;*
(iii) *there exists* $\alpha_i \in (0; 1)$ *such that:*

$$A_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i,\ i = \overline{1, m}$$

*are* $\alpha_i$*-contractions;*

(iv) *the operators* $A_i,\ i=\overline{1,m}$, *are continuous.*

*The operator* $A : X_0 \times \cdots \times X_m \to X_0 \times \cdots \times X_m$,

$$A(x_0,\ldots,x_m) = (A_0(x_0), A_1(x_0,x_1),\ldots,A_m(x_0,\ldots,x_m))$$

*is WPO. If* $A_0$ *is PO, then* $A$ *is PO.*

## 2. Main result

In this section we apply the algorithm from section 1 for the problem (1.1)-(1.2).

Let $n \in \mathbb{N}^*$ be such that $nh \le T,\ (n+1)h > T$. In the conditions $(C_1)-(C_3)$, the step method consists in the following:

For $t \in [h,2h]$ we have

$$\begin{aligned} x'(t) &= f(t,x(t),x(t-h),x(t+h)),\\ x'(t-h) &= f(t-h,x(t-h),x(t-2h),x(t)),\\ x_0'(t-h) &= f(t-h,x_0(t-h),x_{-1}(t-2h),x(t)), \end{aligned}$$

where $x(t)=\varphi(t)=\begin{cases} x_{-1}(t), t\in[-h,0],\\ x_0(t), t\in[0,h].\end{cases}$

We denote $x(t) := x_1(t),\ t\in[h,2h]$.

Let

$F(t,x_1(t)) := f(t-h,x_0(t-h),x_{-1}(t-2h),x_1(t)) - x_0'(t-h) = 0,\ t\in[h,2h]$.

$F(t,x_1(t)) = 0,\ t\in[h,2h]$.

From the implicit function theorem there exists a solution $x_1^* \in C^1[h,2h]$ such that

$$F(t,x_1^*(t)) = 0, \forall t\in[h,2h].$$

The key of each step is to approximate the solution $x_1^* \in [h,2h]$ with the method of Newton:

$$x_{1m}(t) = x_{1,m-1}(t) - G(t,x_1^*(t))F(t,x_{1,m-1}(t)),$$

where $G(t,x_1^*(t)) \neq 0$ and $x_{1,m-1}(t) - G(t,x_1^*(t))F(t,x_{1,m-1}(t))$ is a contraction.

We choose the function $G : [h,2h]\times\mathbb{R}\to\mathbb{R}$ with $G(t,x_1^*(t)) := M\left(\frac{\partial F(h,x_1(h))}{\partial x_1}\right)^{-1}$, where $M\in(0,1)$ is a constant. It is obvious that $G(t,x_1^*(t))\neq 0$.

Now we consider the operator $A_1 : C[h,2h]\to C[h,2h]$, defined by

$$A_1(x_{1,m-1})(t) := x_{1,m-1}(t) - G(t,x_1^*(t))F(t,x_{1,m-1}(t)).$$

Proving that $A_1$ is a contraction we have the uniqueness of the solution $x_{1m}$ on $[h,2h]$. For all $t\in[h,2h]$ we have the inequality

$$\|A_1(x_{1,m-1})(t) - A_1(y_{1,m-1})(t)\| \le (1-M)\|x_{1,m-1}-y_{1,m-1}\|.$$

We have that $x_{1m} \overset{unif}{\to} x_1^*$ on $[h,2h]$, so in the next step we shall use $x_{1m}$ instead of $x_1^*$.

For $t \in [2h, 3h]$ we have

$$x'(t) = f(t, x(t), x(t-h), x(t+h)),$$
$$x'(t-h) = f(t-h, x(t-h), x(t-2h), x(t)),$$
$$x'_{1m}(t-h) = f(t-h, x_{1m}(t-h), x_0(t-2h), x(t)).$$

We denote $x(t) := x_2(t),\ t \in [2h, 3h]$.

Let

$$F(t, x_2(t)) := f(t-h, x_{1m}(t-h), x_0(t-2h), x_2(t)) - x'_{1m}(t-h) = 0,\ t \in [2h, 3h].$$
$$F(t, x_2(t)) = 0,\ t \in [2h, 3h].$$

Applying the implicit function theorem, we have that there exists the solution $x_2^* \in C^1[2h, 3h]$ such that

$$F(t, x_2^*(t)) = 0, \forall t \in [2h, 3h].$$

Now we approximate the solution $x_2^* \in [2h, 3h]$ with the method of Newton:

$$x_{2m}(t) = x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)),$$

where $G(t, x_2^*(t)) \neq 0$ and $x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t))$ is a contraction.

We choose $G : [2h, 3h] \times \mathbb{R} \to \mathbb{R}$ with $G(t, x_2^*(t)) := M \left(\frac{\partial F(2h, x_2(2h))}{\partial x_2}\right)^{-1}$, where $M \in (0, 1)$ is a constant. Then we have $G(t, x_2^*(t)) \neq 0$.

Let us consider the operator $A_2 : C[2h, 3h] \to C[2h, 3h]$, defined by

$$A_2(x_{2,m-1})(t) := x_{2,m-1}(t) - G(t, x_2^*(t))F(t, x_{2,m-1}(t)).$$

In the same way as in the previous step we prove that $A_2$ is a contraction. Follows that $x_{2m} \overset{unif}{\to} x_2^*$ on $[2h, 3h]$, so in the next step we shall use $x_{2m}$ instead of $x_2^*$.

By induction, for $t \in [nh, T]$ we have

$$x'(t) = f(t, x(t), x(t-h), x(t+h)),$$
$$x'(t-h) = f(t-h, x(t-h), x(t-2h), x(t)),$$
$$x'_{n-1,m}(t-h) = f(t-h, x_{n-1,m}(t-h), x_{n-2,m}(t-2h), x(t)).$$

We denote $x(t) := x_n(t),\ t \in [nh, T]$.

Let

$$F(t, x_n(t)) := f(t-h, x_{n-1,m}(t-h), x_{n-2,m}(t-2h), x(t)) - x'_{n-1,m}(t-h) = 0,$$
$$F(t, x_n(t)) = 0,\ t \in [nh, T].$$

Applying implicit function theorem, there exists the solution $x_n^* \in C^1[nh, T]$ such that $F(t, x_n^*(t)) = 0, \forall t \in [nh, T]$. We approximate the solution $x_n^* \in [nh, T]$ with the method of Newton, by $x_{nm}(t) = x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t))$, where $G(t, x_n^*(t)) \neq 0$ and $x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t))$ is a contraction.

The function chosen here is $G : [nh, T] \times \mathbb{R} \to \mathbb{R},\ G(t, x_n^*(t)) := M \left(\frac{\partial F(nh, x_n(nh))}{\partial x_n}\right)^{-1}$, where $M \in (0, 1)$ is a constant. Then $G(t, x_n^*(t)) \neq 0$.

Let the operator $A_n : C[nh, T] \to C[nh, T]$ defined by

$$A_n(x_{n,m-1})(t) := x_{n,m-1}(t) - G(t, x_n^*(t))F(t, x_{n,m-1}(t)).$$

Notice that $A_n$ is a contraction. Then we have that $x_{nm} \overset{unif}{\to} x_n^*$ on $[nh,T]$.

So, the following convergence takes place

$$\widetilde{x} = \begin{cases} x_{-1}, & t \in [-h,0] \\ x_0, & t \in [0,h] \\ x_{1m}, & t \in [h,2h] \\ \vdots & \\ x_{nm}, & t \in [nh,T] \end{cases} \to x^* = \begin{cases} x_{-1}, & t \in [-h,0] \\ x_0, & t \in [0,h] \\ x_1^*, & t \in [h,2h] \\ \vdots & \\ x_n^*, & t \in [nh,T]. \end{cases}$$

In what follows we present the step method for the solution determined with the above algorithm.

$(p_0)$ $x(t) = \varphi(t) = \begin{cases} x_{-1}(t), t \in [-h,0], \\ x_0(t), t \in [0,h]; \end{cases}$

$(p_1)$ $x_0^*(t-h) = x_0^*(0) + \displaystyle\int_h^t f(s-h, x_0^*(s-h), x_{-1}^*(s-2h), x_1(s))ds, t \in [h,2h];$

$(p_2)$ $x_1^*(t-h) = x_1^*(h) + \displaystyle\int_{2h}^t f(s-h, x_1^*(s-h), x_0(s-2h), x_2(s))ds, t \in [2h,3h];$

$(p_3)$ $x_2^*(t-h) = x_2^*(2h) + \displaystyle\int_{3h}^t f(s-h, x_2^*(s-h), x_1^*(s-2h), x_3(s))ds, t \in [3h,4h];$

$\vdots$

$(p_n)$ $x_{n-1}^*(t-h) = x_{n-1}^*((n-1)h) + \displaystyle\int_{nh}^t f(s-h, x_{n-1}^*(s-h), x_{n-2}^*(s-2h), x_n(s))ds,$
$t \in [nh,T].$

Thus we have the following theorem

**Theorem 2.1.** *In the conditions $(C_1)-(C_3)$ we have:*

a) *the problem (1.1)-(1.2) has in $C[-T,T]$ a unique solution*

$$x^*(t) = \begin{cases} \varphi(t), & t \in [-h,h] \\ x_1^*(t), & t \in [h,2h] \\ \vdots & \\ x_n^*(t), & t \in [nh,T]. \end{cases}$$

b) *the sequence define by*

$$x_0(t-h) = x_0(0) + \int_h^t f(s-h, x_0(s-h), x_{-1}(s-2h), x_{1m}(s))ds, t \in [h, 2h];$$

$$x_1^*(t-h) = x_1^*(h) + \int_{2h}^t f(s-h, x_1^*(s-h), x_0(s-2h), x_{2m}(s))ds, t \in [2h, 3h];$$

$$x_2^*(t-h) = x_2^*(2h) + \int_{3h}^t f(s-h, x_2^*(s-h), x_1^*(s-2h), x_{3m}(s))ds, t \in [3h, 4h];$$

$$\vdots$$

$$x_{n-1}^*(t-h) = x_{n-1}^*((n-1)h) + \int_{nh}^t f(s-h, x_{n-1}^*(s-h), x_{n-2}^*(s-2h), x_{nm}(s))ds,$$
$$t \in [nh, T];$$

*is convergent and* $\lim_{m\to\infty} x_{km} = x_k^*, \ k = \overline{1,n}$.

Theorem 2.1 gives a uniqueness result for the solution of the problem (1.1)-(1.2) by successive approximation method and now we want to improve the convergence of this solution. So, here cames the question: can we put $x_{i-1,m}(t)$ instead of $x_{i-1}^*(t), \ i = \overline{2,n}$ in the conclusion b) of Theorem 2.1? The answer of this question is given by the following theorem.

**Theorem 2.2.** *We suppose that the conditions* $(C_1)-(C_3)$ *and*

($C_4$) *there exists* $L_f > 0$ *such that*

$$|f(t,u,v,w_1) - f(t,u,v,w_2)| \le L_f |w_1 - w_2|, \forall t \in [-T,T], u,v,w_1,w_2 \in \mathbb{R};$$

*are satisfied. Then the sequence defined by*

$$x_0(t-h) = \varphi(0) + \int_h^t f(s-h, x_0(s-h), x_{-1}(s-2h), x_{1m}(s))ds, t \in [h, 2h]; \quad (2.1)$$

$$x_{1m}(t-h) = x_{1m}(h) + \int_{2h}^t f(s-h, x_{1m}(s-h), x_0(s-2h), x_{2m}(s))ds, t \in [2h, 3h];$$

$$x_{2m}(t-h) = x_{2m}(2h) + \int_{3h}^t f(s-h, x_{2m}(s-h), x_{1m}(s-2h), x_{3m}(s))ds, t \in [3h, 4h];$$

$$\ldots$$

$$x_{n-1,m}(t-h) = x_{n-1,m}((n-1)h) +$$
$$+ \int_{nh}^t f(s-h, x_{n-1,m}(s-h), x_{n-2,m}(s-2h), x_{nm}(s))ds, t \in [nh, T];$$

*is convergent and* $\lim_{m\to\infty} x_{km} = x_k^*, \ k = \overline{1,n}$.

*Proof.* We consider the Banach spaces

$$X_0 = (C[-h,h], \|\cdot\|_0) \text{ with } \|\cdot\|_0 = \max_{t\in[-h,h]}\{\|x(t)\|\, e^{-\lambda(t+h)}\},\ \lambda > 0,$$
$$X_i = (C[ih,(i+1)h], \|\cdot\|_i) \text{ with } \|\cdot\|_i = \max_{t\in[ih,(i+1)h]}\{\|x(t)\|\, e^{-\lambda(t-ih)}\},\ \lambda > 0,$$
$$X_n = (C[nh,T], \|\cdot\|_n) \text{ with } \|\cdot\|_n = \max_{t\in[nh,T]}\{\|x(t)\|\, e^{-\lambda(t-nh)}\},\ \lambda > 0,$$

and the operators

$$\begin{aligned}
A_0 &: X_0 \to X_0,\ A(x_0)(t) = \varphi(t),\ t\in[-h,h],\\
A_i &: X_{i-2}\times X_{i-1}\times X_i \to X_i,\ i=\overline{1,n-1}\\
A_i(x_{i-2},x_{i-1},x_i)(t) &= x_{i-1}((i-1)h)+\\
&+\int_{ih}^{t} f(s-h, x_{i-1}(s-h), x_{i-2}(s-2h), x_i(s))ds,\ t\in[ih,(i+1)h]\\
A_n &: X_{n-2}\times X_{n-1}\times X_n \to X_n,\\
A_n(x_{n-2},x_{n-1},x_n)(t) &= x_{n-1}((n-1)h+\\
&+\int_{nh}^{t} f(s-h, x_{n-1}(s-h), x_{n-2}(s-2h), x_n(s))ds,\ t\in[nh,T]
\end{aligned}$$

and

$$\begin{aligned}
A &: X_0\times\cdots\times X_n \to X_0\times\cdots\times X_n\\
A(x_0,\ldots,x_n) &= (A_0(x_0), A_1(x_{-1},x_0,x_1),\ldots,A_n(x_{n-2},x_{n-1},x_n)).
\end{aligned}$$

For fixed $(x_0,\ldots,x_n)\in X_0\times\cdots\times X_n$, the sequence (2.1) means

$$(x_{0m},\ldots,x_{nm}) = A^m(x_0,\ldots,x_n).$$

We need to prove that the operator $A$ is PO and for this we apply the fibre contraction theorem.

Since $A_0 : X_0 \to X_0$ is a constant operator then $A_0$ is $\alpha_0$-contraction with $\alpha_0 = 0$, so $A_0$ is PO and $F_{A_0} = \{x_0^*\}$, where $x_0^* = \varphi$. For $i = \overline{1,n}$ we have:

$$\|A_i(x_{i-2},x_{i-1},x_i) - A_i(x_{i-2},x_{i-1},y_i)\|_i \le \frac{L_f}{\lambda}\|x_i - y_i,\|$$

for all $x_{i-2}\in X_{i-2}, x_{i-1}\in X_{i-1}, x_i\in X_i$. Choosing $\lambda = L_f+1$, we get that $A_i(x_{i-2},x_{i-1},\cdot) : X_i\to X_i$ are $\alpha_i$-contractions with $\alpha_i = \frac{L_f}{L_f+1}$, so we are in the conditions of the fibre contraction theorem, therefore $A$ is PO and $F_A = \{(x_0^*,\ldots,x_n^*)\}$. Thus

$$(x_{0m},\ldots,x_{nm}) = A^m(x_0,\ldots,x_n) \to (x_0^*,\ldots,x_n^*),$$

where $x_0^m = \varphi$, for all $m\in\mathbb{N}$, and $x_{1m},\ldots,x_{nm}$ are defined by (2.1). From condition $(C_3)$ and from the definitions of $A_i, i=\overline{1,n}$, we have

$$x_{i-1}^*((i-1)h) = x_i^*((i-1)h), i=\overline{1,n},$$

therefore

$$x^*(t) = \begin{cases} \varphi(t), & t \in [-h, h] \\ x_1^*(t), & t \in [h, 2h] \\ \vdots & \\ x_n^*(t), & t \in [nh, T] \end{cases}$$

is the unique solution in $C[-T, T]$. □

## 3. Numerical example

In this section we give an example to test the numerical method presented above. We consider the following functional-differential problem with mixed type argument:

$$x'(t) = -4x(t) + x(t-h) + 3x(t+h) + (1-2h)/12, \ t \in [-7; 7], \ h = 1, \tag{3.1}$$
$$x(t) = (t-1)/12, \ t \in [-1; 1].$$

We divide the working interval $[-1; 7]$ by the points $P_n = nh$, $n = \overline{-1, 7}$. We develop the solution for the step of time $s = 0.1$, thus we obtain $N = 10$ points on each subinterval $I_n = [P_{n-1}, P_n]$. From implicit function theorem, on each $I_n$, there exists a solution $x_n(t)$ and this solution is approximated by Newton's method. Applying the algorithm explained in the previous section we get:

$$x_{nm}(t) = x_{nm-1}(t) - F(t, x_{nm-1}(t))/3 \tag{3.2}$$

with $F(t, x_{nm-1}(t)) = -4x(t-nh) + x(t-2nh) + 3x(t) + \frac{1-2nh}{12} - x'(t-nh)$.

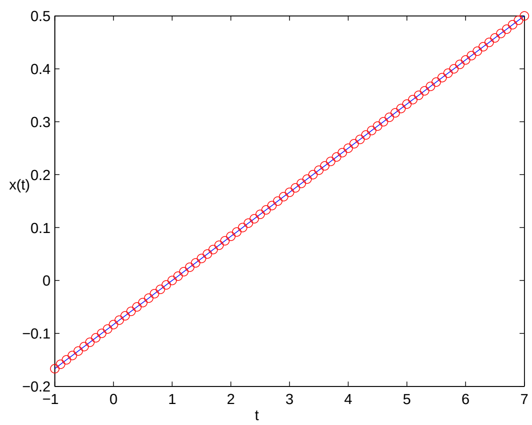


Figure 1. Exact and numerical solution for equation (3.1)

The algorithm from Section 2 is implemented using Matlab in the following way:

*Step 0:* We construct the vector $t$ formed by $2N+1$ points of the interval $[-h; h]$ at each step $s$. Further, we initialize the known solution for this interval with $\varphi(t) = (t-1)/12$ and its derivative with $\varphi'(t) = 1/12$.

*Step k:* We concatenate to the initial vector $t$ the rest of the points till $T$, constructing the interval $[nh, T],\ n = \overline{2, 7}$. For this interval we get the solution applying Newton's method. For starting this method, we initialize the value of the first solution with that computed to the last knot at the previous step.

*Stoping test:* We evaluate the difference in norm between two consecutive computed values $x_n^{(k)}$ and $x_n^{(k+1)}$ and the iterations stop when it is less than a chosen value (in our case $10^{-6}$). The last values of the solution are retained in the solution vector and are plotted along to the exact solution of the equation (3.1). These solutions are presented in Fig. 1.

We can see from Fig. 1 that for the equation (3.1), our algorithm work perfectly. The exact solution $x(t) = (t-1)/12$ are designed graphically by circles and the numerical solution $x = x_n(t)$ by line. We observe that the numerical solution is overlapping the exact solution.

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