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EQUIDIFFERENTIABILITY OF SUBSETS OF INFINITELY DIFFERENTIABLE FUNCTION SPACES

SATOSHI KODAMA AND SHIGEO AKASHI

Dedicated to Wataru Takahashi on the occasion of his retirement

Department of Information Sciences, Faculty of Science and Technology
Tokyo University of Science, 2641, Yamazaki, Noda City, Chiba Prefecture, 278-8510 JAPAN
E-mails: j6309701@ed.noda.tus.ac.jp akashi@is.noda.tus.ac.jp

Abstract. In this paper, the concept of equidifferentiability, which is analogous to the concept of equicontinuity, is introduced and this concept is applied to some relations between the sets consisting of finitely equidifferentiable functions defined on $[0, 1]$ and the sets consisting of infinitely equidifferentiable functions defined on $[0, 1]$. Moreover, fixed point theorems for the finitely differentiable function spaces and those for the infinitely differentiable function spaces are remarked.
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1. Introduction

The concept of compactness plays an important role in the function space theory. For example, the solution to Hilbert's 13th problem (cf. [1],[2],[3],[8]), which is related to the superposition representability of the functions of several variables, has required a compactness criterion represented in terms of the theory of function spaces. Moreover, Ascoli's theorem (cf. [6],[7]) shows that equicontinuity can be regarded as a characterization of compactness of subsets of function spaces. By the way, as for the several researches investigating the existence of fixed points under the conditions such as a family of nonexpansive mappings and a family of several metrics, we can refer to the results obtained by Petruşel and Rus (cf. [5]) and the results obtained by Nakajo, Shimoji and Takahashi (cf. [4]), respectively.
In this paper, the concept of equidifferentiability, which is analogeous to the concept of equicontinuity, is introduced and this concept is applied to some relations between the sets consisting of finitely equidifferentiable functions defined on $[0, 1]$ and the sets consisting of infinetely equidifferentiable functions defined on $[0, 1]$. Moreover, the fixed point theorems for the finitely differentiable function spaces and those for the infinitely differentiable function spaces are remarked.

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2. A criterion for compactness

Let $\mathbb{N}$ be the set of all positive integers and let k be a positive integer. Moreover, $\mathcal{D}^k([0,1])$ denotes the set of all k-time continuously differentiable real-valued functions defined on $[0,1]$ and $\|\cdot\|_k$ denotes the norm defined as

$$\|f\|_k = \max_{0\leq i\leq k} \sup_{x\in[0,1]} |f^{(i)}(x)|, \quad f\in \mathcal{D}^k([0,1]).$$

By the same way as above, $\|\cdot\|_\infty$ denotes the function defined as

$$\|f\|_\infty = \max_{0\leq i\leq k} \sup_{x\in[0,1]} |f^{(i)}(x)|, \quad f\in \mathcal{D}^k([0,1]),$$

where f is an infinitely differentiable function. It can be easily proved that $(\mathcal{D}^k([0,1]), \|\cdot\|_k)$ is a Banach space and that there exists an infinitely differentiable function f satisfying $\|f\|_\infty = \infty$.

Let $\mathcal{D}^\infty([0,1])$ be the set of all infinity differentiable functions defined on $[0,1]$ and $d_\infty(\cdot,\cdot)$ denotes the metric defined as

$$d_\infty(f,g) = \sum_{k=1}^{\infty} \frac{\|f-g\|_k}{2^k(1+\|f-g\|_k)}, \quad f,g\in\mathcal{D}^\infty([0,1]).$$

Then,we can prove the follows. Let $\mathcal{F}$ be a non-empty subset of $\mathcal{D}^k([0,1])$. Then, $\mathcal{F}$ is said to be k-time equidifferentiable, if, for any positive number ε, there exists a positive number δ satisfying

$$\max_{0\leq i\leq k} \sup_{|x-y|<\delta} |f^{(i)}(x)-f^{(i)}(y)| \leq \varepsilon, \quad f\in\mathcal{F}.$$

Now, we can obtain the following:

Theorem 2.1. *Let $\mathcal{F}$ be a non-empty closed subset of $(\mathcal{D}^k([0,1]), \|\cdot\|_k))$. Then, $\mathcal{F}$ is compact if and only if F is bounded and k-time equidifferentiable.*

Proof. We first assume that $\mathcal{F}$ is compact. Since the compactness of $\mathcal{F}$ implies the boundedness of $\mathcal{F}$, it is sufficient to prove that $\mathcal{F}$ is k-time equidifferentiable. For any arbitrary positive number ε, we can find a finite $(\varepsilon/3)$-net which is denoted by $\mathcal{N}(\mathcal{F},\varepsilon/3)$. Therefore, there exists a positive number δ satisfying

$$\max_{0\leq j\leq k} \sup_{|x-y|<\delta} |f^{(j)}(x)-f^{(j)}(y)| \leq \frac{\varepsilon}{3}, \quad f\in\mathcal{N}\left(\mathcal{F},\frac{\varepsilon}{3}\right),$$

because all the k-time continuously differentiable functions and their derivative functions are uniformly continuous on $[0,1]$. Let g be an element of $\mathcal{D}^k([0,1])$. Then, there exists an element f_g belonging to $\mathcal{N}(\mathcal{F},\varepsilon/3)$ satisfying $\|g-f_g\|_k < \varepsilon/3$. Let x and y be any two elements belonging to $[0,1]$ satisfying $|x-y|<\delta$. Then, for any integer i satisrying $0\leq i\leq k$, we have

$$\begin{aligned} |g^{(i)}(x)-g^{(i)}(y)| &\leq |g^{(i)}(x)-f_g^{(i)}(x)| + |f_g^{(i)}(x)-f_g^{(i)}(y)| + |f_g^{(i)}(y)-g^{(i)}(y)| \\ &\leq \varepsilon. \end{aligned}$$

Therefore, the former half has been proved. Finally, we assume that $\mathcal{F}$ is bounded and k-time equidifferentiable. Since $\mathcal{F}$ is closed, it is sufficient to prove that every sequence cosisting of elements belonging to $\mathcal{F}$ has a Cauchy subsequence. Here,

$Q([0,1])$ denotes the set of all non-negative rational numbers belonging to $[0,1]$ and q denotes a bijective mapping defined on $\mathbb{N}$ with values in $Q([0,1])$. Let $\{g_j\}_{j=1}^{\infty}$ be a sequence consisting of elements belonging to $\mathcal{F}$. Then, we can find a subsequence $\{g_j^1\}_{j=1}^{\infty}$ satisfying the condition that $\{g_j^1(q(1))\}_{j=1}^{\infty}$ converges. Moreover, we can find a subsequence of $\{g_j^1\}_{j=1}^{\infty}$ which is denoted by $\{g_j^2\}_{j=1}^{\infty}$ satisfying the condition that $\{g_j^1(q(2))\}_{j=1}^{\infty}$ also converges. If we continue this process iteratively, we can construct the array of sequence $\{\{g_j^i\}_{j=1}^{\infty}; i \in \mathbb{N}\}$. It is clear that the following inequality holds:

$$\max_{0\le j\le k} \sup_{|x-y|<\delta} \left| \left(g_i^i\right)^{(j)}(x)(x) - \left(g_i^i\right)^{(j)}(y) \right| \le \frac{\varepsilon}{3}, \quad i \in \mathbb{N},,$$

because $\{g_i^i\}_{i=1}^{\infty}$ is also k-time equidifferentiable. Let L be a positive integer and let $\{q(j_\ell); 1 \le \ell \le L\}$ be a finite subset of $\{q(j); j \in \mathbb{N}\}$ which satisfies the following:

$$[0,1] \subset \cup_{\ell=1}^{L} (q(j_\ell) - \delta, q(j_\ell) + \delta).$$

Therefore, we can find such a positive integer $n(\varepsilon)$ that, for any two positive integers m and n satisfying $m > n(\varepsilon)$ and $n > n(\varepsilon)$, the following inequality holds:

$$\left| (g_m^m)^{(j)}(q(j_\ell)) - (g_n^n)^{(j)}(q(j_\ell)) \right| \le \frac{\varepsilon}{3}, \quad 1 \le j \le k, \quad 1 \le \ell \le L.$$

These results lead us to the following inequalities:

$$\begin{aligned} |(g_m^m)^{(i)}(x) - (g_n^n)^{(i)}(x)| &\le |(g_m^m)^{(i)}(x) - (g_m^m)^{(i)}(q(j_\ell))| \\ &+ |(g_m^m)^{(i)}(q(j_\ell)) - (g_n^n)^{(i)}(q(j_\ell))| \\ &+ |(g_n^n)^{(i)}(q(j_\ell)) - (g_n^n)^{(i)}(x)| \\ &\le \varepsilon \end{aligned}$$

Therefore, we can conclude the proof of the latter half. □

Let k and ℓ be two positive integers satisfying $k < \ell$. Then, $\mathcal{D}^{\ell}([0,1])$ can be regarded as a closed subspace of $\mathcal{D}^{k}([0,1])$. Here, for any positive number M, $\mathcal{U}_k(M)$ and $\mathcal{U}_\infty(M)$ denote he subset of $\mathcal{D}^{k}([0,1])$ and the subset of $\mathcal{D}^{\infty}([0,1])$ defined as

$$\mathcal{U}_k(M) = \{f \in \mathcal{D}^k([0,1]); \|f\|_k \le M\}$$

and

$$\mathcal{U}_\infty(M) = \{f \in \mathcal{D}^\infty([0,1]); \|f\|_\infty \le M\},$$

respectively. Then, we can obtain the following:

Proposition 2.2. *Let m be a positive integer and let $S_\infty(m)$ denote the closed sphere of the metric space $(\mathcal{D}^\infty([0,1]), d(\cdot,\cdot))$ with its radius $1/2^m$. Then, $S_\infty(m)$ is a bounded subset of $(\mathcal{D}^m([0,1]), \|\cdot\|_m)$.*

Proof. Assume that we have

$$\sup_{f\in S_\infty(m)} \|f\|_m = \infty.$$

Then, for any positive integer ℓ which is greater than m, we have

$$\sup_{f\in S_\infty(m)} \|f\|_\ell = \infty.$$

This equality implies that the following inequalities hold:

$$\sup_{f\in S_\infty(m)} d_\infty(0,f) \geq \sum_{k=m}^{\infty} \frac{\|f\|_k}{2^k(1+\|f\|_k)} = \frac{1}{2^{m-1}}.$$

Therefore, we have the conclusion. □

Remark 2.3. For any non-negative integer k and for any positive number M,

$$\{f \in \mathcal{D}^k([0,1]); \|f\|_{k+1} \leq M\}$$

is a compact subset of $(\mathcal{D}^k([0,1]), \|\cdot\|_k))$, because a bounded subset included by $(\mathcal{D}^k([0,1]), \|\cdot\|_{k+1}))$ assures that all the elements belonging to the subset are equidifferentiable.

Remark 2.4. Let T_∞ be a continuous mapping on $\mathcal{D}^\infty([0,1])$ and, for any positive integer k, let T_k be the unique extension of T_∞ whose extended domain is $\mathcal{D}^k([0,1])$. Moreover, let $\mathcal{F}$ be a non-empty compact and convex subset of $\mathcal{D}^\infty([0,1])$. Then, we can apply Schauder's fixed point theorem to F, because F can be regarded as a compact and convex subset of $\mathcal{D}^k([0,1])$, and eventually, we can prove that T_k has a fixed point, which is denoted as p_k. Since $\mathcal{D}^\infty([0,1])$ can be regarded as a locally convex topological vector space, we can apply Tychonoff's fixed point theorm to F, andeventually, we can prove that T_∞ has a fixed point, which is denoted as p_∞. Here we can easily prove that p_k is exactly equal to p_∞, because $\mathcal{D}^\infty([0,1])$ is a dense subset of $\mathcal{D}^k([0,1])$.

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References

[1] S. Akashi, *A version of Hilbert's 13th problem for analytic functions*, Bull. London Math. Soc., **35**(2003), 8-14.

[2] K.I. Babenko, *On the best approximation of a class of analytic functions*, Izv., **22**(1958), 631-640.

[3] G.G. Lorentz, *Approximation of Functions*, Holt, Rinehart and Winston Inc., New York, 1966.

[4] K. Nakajo, K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces*, Journal of Nonlinear and Convex Analysis, **8**(2007), No. 1, 11-34.

[5] A. Petruşel, I.A. Rus, *Fixed point theory for multivalued operators on a set with two metrics*, Fixed Point Theory, **8**(2007), No. 1, 97-104.

[6] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Company, Third Edition, New York, 1987.

[7] G.F. Simons, *Introduction to Topology and Modern Analysis*, McGraw-Hill Kogakusha, Ltd, Tokyo, 1963.

[8] A.G. V̌itushkin, *Some properties of linear superpositions of smooth functions*, Dokl., **156**(1964), 1003-1006.

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