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GLOBAL BIFURCATION FOR NEUMANN PROBLEMS INVOLVING NONHOMOGENEOUS OPERATORS

IN-SOOK KIM* AND YUN-HO KIM**

Dedicated to Wataru Takahashi on the occasion of his retirement

*Department of Mathematics, Sungkyunkwan University
Suwon 440-746, Republic of Korea
E-mail: iskim@skku.edu

**Department of Mathematics, University of Iowa
Iowa City, Iowa 52242, USA
E-mail: ykim16@math.uiowa.edu

Abstract. We consider a Neumann problem involving nonhomogeneous operators

$$-\mathrm{div}(\Psi(x,\nabla u)) + \Phi(x,u) = \mu\,|u|^{p-2}\,u + f(\lambda,x,u,\nabla u) \quad \text{in } \Omega$$

when Ψ, Φ, and f satisfy certain conditions and μ is not an eigenvalue in some sense. The aim of this paper is to study the structure of the set of solutions for the above equation, by applying a bifurcation result for nonlinear equations and a nonlinear spectral theory for homogeneous operators.

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1. Introduction

Some eigenvalue problems with Neumann boundary conditions have been investigated in [5, 9, 13, 14, 15]. From this background, a bifurcation theory for Neumann problems can be developed; see e.g. [6, 13, 14]. In fact, Khalil and Ouanan [6] obtained some bifurcation results for nonlinear Neumann problem of the form

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda m(x)|u|^{p-2}u + f(\lambda,x,u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$, and $\frac{\partial u}{\partial n}$ denotes the outer normal derivative of u with respect to $\partial\Omega$. It is based on the fact in [5] that the first eigenvalue of the p-Laplacian is simple and isolated, under suitable conditions on m.

Concerning Dirichlet boundary conditions, various bifurcation problems from the first eigenvalue of the p-Laplacian can be found in [1, 3, 10, 11, 12]; see [2] for the

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degenerated p-Laplacian. While one thus deals with bifurcation at the first eigenvalue, Väth [16] has attempted another approach in the case when μ is not an eigenvalue of the p-Laplacian; see [7]. In this direction, a global bifurcation result for Dirichlet problems involving nonhomogeneous operators

$$\begin{cases} -\mathrm{div}(\Psi(x, \nabla u)) = \mu\, |u|^{p-2}\, u + f(\lambda, x, u, \nabla u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is given in [8]. It is required that Ψ behaves asymptotically at infinity like the p-Laplacian.

In the present paper, we consider the following nonlinear Neumann problem

$$\begin{cases} -\mathrm{div}(\Psi(x, \nabla u)) + \Phi(x, u) = \mu\, |u|^{p-2}\, u + f(\lambda, x, u, \nabla u) & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{B}$$

when μ is not an eigenvalue of the form

$$\begin{cases} -\mathrm{div}(w(x)\, |\nabla u|^{p-2}\, \nabla u) + \nu(x)\, |u|^{p-2}\, u = \mu\, |u|^{p-2}\, u & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{E}$$

Here $\Psi(x, \cdot) : \mathbb{R}^N \to \mathbb{R}^N$ and $\Phi(x, \cdot) : \mathbb{R} \to \mathbb{R}$ are not necessarily positively homogeneous and $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies a Carathéodory condition. It is known in [4, 16] that the condition that μ is not an eigenvalue of the p-Laplacian is closely related to nonlinear spectral theory for homogeneous operators. The aim of this paper is to study the structure of the solution set for the above Neumann problem (B). A key tool is to use a bifurcation result for nonlinear equations given in [8], with the aid of nonlinear spectral theory.

This note is organized as follows: In Section 2, to solve our bifurcation problem (B) in the weak sense, we give some properties of the corresponding integral operators. In Section 3, the main idea in our approach is to observe the asymptotic behavior of the integral operator induced by Ψ and Φ at infinity, as the Dirichlet problem has shown in [8]. With this observation, we obtain a spectral result concerning nonhomogeneous operators provided that μ is not an eigenvalue of (E). In Section 4, we prove the main theorem on global bifurcation for the above Neumann problem (B).

2. Some properties of integral operators

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary. Let $1 < p < \infty$ and $p' := p/(p-1)$. Let $X = W^{1,p}(\Omega)$ be the Sobolev space, endowed with the norm

$$\|u\|_X = \Big(\int_\Omega |\nabla u|^p\, dx + \int_\Omega |u|^p\, dx \Big)^{\frac{1}{p}},$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$ or $\mathbb{R}^1$. We assume that

(J1) $\Psi\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $\Phi\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy a Carathéodory condition, respectively, that is, $\Psi(\cdot, v)$ is measurable on Ω for all $v \in \mathbb{R}^N$, $\Psi(x, \cdot)$ is continuous on $\mathbb{R}^N$ for almost all $x \in \Omega$, $\Phi(\cdot, u)$ is measurable on Ω for all $u \in \mathbb{R}$, and $\Phi(x, \cdot)$ is continuous on $\mathbb{R}$ for almost all $x \in \Omega$.

(J2) There are functions $a_i \in L^{p'}(\Omega)$ and nonnegative constants b_i $(i = 1, 2)$ such that for almost all $x \in \Omega$, the following growth conditions hold:

$$|\Psi(x, v)| \le a_1(x) + b_1 |v|^{p-1} \qquad \text{and} \qquad |\Phi(x, u)| \le a_2(x) + b_2 |u|^{p-1}$$

for all $v \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$.

(J3) There are positive constants c_i $(i = 1, 2)$ such that for almost all $x \in \Omega$, the estimates hold:

$$\langle \Psi(x, v_1) - \Psi(x, v_2), v_1 - v_2 \rangle \ge \begin{cases} c_1 \min\left\{1, (|v_1| + |v_2|)^{p-2}\right\} |v_1 - v_2|^2 & \text{if } 1 < p < 2 \\ c_1 |v_1 - v_2|^p & \text{if } 2 \le p < \infty \end{cases}$$

for all $v_1, v_2 \in \mathbb{R}^N$ and

$$\langle \Phi(x, u_1) - \Phi(x, u_2), u_1 - u_2 \rangle \ge \begin{cases} c_2 \min\left\{1, (|u_1| + |u_2|)^{p-2}\right\} |u_1 - u_2|^2 & \text{if } 1 < p < 2 \\ c_2 |u_1 - u_2|^p & \text{if } 2 \le p < \infty \end{cases}$$

for all $u_1, u_2 \in \mathbb{R}$.

Under (J1) and (J2), we define an operator $J \colon X \to X^*$ by

$$\langle J(u), v \rangle := \int_\Omega \Big(\langle \Psi(x, \nabla u(x)), \nabla v(x) \rangle + \langle \Phi(x, u(x)), v(x) \rangle \Big)\, dx, \tag{2.1}$$

where $\langle \cdot\,, \cdot \rangle$ denotes the pairing of X and its dual X^* and the Euclidean scalar product on $\mathbb{R}^N$ or $\mathbb{R}^1$, respectively.

The following examples are a particular form of Corollary 3.1 in [8].

Example 2.1. Suppose that $\psi \colon \Omega \times [0, \infty) \to [0, \infty)$ has the property that $\psi(\cdot, t)$ is measurable on Ω for all $t \in [0, \infty)$ and $\psi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$. Assume that there exists a positive constant c_3 such that the following conditions are satisfied for almost all $x \in \Omega$:

$$\psi(x, t) \ge c_3 t^{p-2} \qquad \text{and} \qquad t \frac{\partial \psi}{\partial t}(x, t) + \psi(x, t) \ge c_3 t^{p-2} \tag{2.2}$$

for almost all $t \in (0, 1)$ and in case $2 \le p < \infty$ the condition (2.2) holds for almost all $t \in (1, \infty)$ and in case $1 < p < 2$

$$\psi(x, t) \ge c_3 \qquad \text{and} \qquad t \frac{\partial \psi}{\partial t}(x, t) + \psi(x, t) \ge c_3 \tag{2.3}$$

holds for almost all $t \in (1, \infty)$. Then $\Psi \colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $\Psi(x, v) = \psi(x, |v|) v$ satisfies (J1) and (J3).

Example 2.2. Suppose that $\phi \colon \Omega \times [0, \infty) \to [0, \infty)$ has the property that $\phi(\cdot, u)$ is measurable on Ω for all $u \in [0, \infty)$ and $\phi(x, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for almost all $x \in \Omega$. Assume that there exists a positive constant c_4 such that the following conditions are satisfied for almost all $x \in \Omega$:

$$\phi(x, u) \ge c_4 u^{p-2} \qquad \text{and} \qquad u \frac{\partial \phi}{\partial u}(x, u) + \phi(x, u) \ge c_4 u^{p-2} \tag{2.4}$$

for almost all $u \in (0,1)$ and in case $2 \le p < \infty$ the condition (2.4) holds for almost all $u \in (1,\infty)$ and in case $1 < p < 2$

$$\phi(x,u) \ge c_4 \qquad \text{and} \qquad u\frac{\partial \phi}{\partial u}(x,u) + \phi(x,u) \ge c_4 \tag{2.5}$$

holds for almost all $u \in (1,\infty)$. Then $\Phi\colon \Omega\times\mathbb{R} \to \mathbb{R}$, $\Phi(x,u) = \phi(x,|u|)u$ satisfies (J1) and (J3).

To observe the above integral operator J in a more concrete situation, we assume that

(J4) w and ν belong to $L^\infty(\Omega)$ and there are positive constants d_i $(i = 1,2)$ such that

$$w(x) \ge d_1 \quad \text{and} \quad \nu(x) \ge d_2 \qquad \text{for almost all } \ x \in \Omega.$$

Setting $\Psi_p\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $\Phi_p\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\Psi_p(x,v) = w(x)\,|v|^{p-2}\,v \quad \text{and} \quad \Phi_p(x,u) = \nu(x)\,|u|^{p-2}\,u,$$

we define another operator $J_p : X \to X^*$ by

$$\langle J_p(u), v\rangle := \int_\Omega \Big(\langle \Psi_p(x,\nabla u(x)), \nabla v(x)\rangle + \langle \Phi_p(x,u(x)), v(x)\rangle \Big)\, dx. \tag{2.6}$$

The first goal of this section is to show that two operators J and J_p are bounded homeomorphisms, where the term $\Phi(x,u)$ or $\Phi_p(x,u)$ is essential for the case of Neumann conditions.

Lemma 2.3. *Under assumptions* (J1), (J2), *and* (J3), *the operator* $J\colon X \to X^*$ *is a bounded homeomorphism. Under assumption* (J4), *the operator* $J_p : X \to X^*$ *is a bounded homeomorphism.*

Proof. In view of (J1) and (J2), it is obvious that J is bounded and continuous on X. We will now show that J^{-1} is continuous on X^*; see the proof of Theorem 3.1 in [8], where the term $\Phi(x,u)$ is not necessary. Set $c = \min\{c_1, c_2\}$. Assume first that $2 \le p < \infty$. Let $u, v \in X$. Then for almost all $x \in \Omega$, we have by (J3) that

$$\langle \Psi(x,\nabla u(x)) - \Psi(x,\nabla v(x)), \nabla u(x) - \nabla v(x)\rangle \ge c_1 |\nabla u(x) - \nabla v(x)|^p$$

and

$$\langle \Phi(x,u(x)) - \Phi(x,v(x)), u(x) - v(x)\rangle \ge c_2 |u(x) - v(x)|^p.$$

taking the integral at both sides of these inequalities, we obtain

$$\begin{aligned}\langle J(u) - J(v), u - v\rangle &= \int_\Omega \langle \Psi(x,\nabla u) - \Psi(x,\nabla v), \nabla u - \nabla v\rangle\, dx \\ &\quad + \int_\Omega \langle \Phi(x,u) - \Phi(x,v), u - v\rangle\, dx \ge c\, \|u - v\|_X^p .\end{aligned}$$

Next, assume that $1 < p < 2$. Then for all $u, v \in X$ with $(u,v) \neq (0,0)$, we obtain by (J3) that

$$\langle J(u) - J(v), u - v\rangle \geq c \int_{\Omega_0} \Big(m_1^{p-2} \, |\nabla u - \nabla v|^2 + m_2^{p-2} \, |u - v|^2 \Big) \, dx,$$

where we put $\Omega_0 := \{x \in \Omega : (u(x), v(x)) \neq (0,0)\}$ and use the shortcuts

$$m_1(x) := \min\{1, |\nabla u(x)| + |\nabla v(x)|\} \quad \text{and} \quad m_2(x) := \min\{1, |u(x)| + |v(x)|\}.$$

From Hölder's and Minkowski's inequalities, and the inequality

$$a^{\frac{1}{q'}} r^{\frac{1}{q}} + b^{\frac{1}{q'}} s^{\frac{1}{q}} \leq (a+b)^{\frac{1}{q'}} (r+s)^{\frac{1}{q}}$$

for any positive numbers a, b, r, s, it follows that

$$\begin{aligned}
\|u-v\|_X^p &= \int_{\Omega_0} m_1^{-p(p-2)/2} \, m_1^{p(p-2)/2} |\nabla u - \nabla v|^p \, dx + \int_{\Omega_0} m_2^{-p(p-2)/2} \, m_2^{p(p-2)/2} |u-v|^p \, dx \\
&\leq \Big(\int_{\Omega_0} |m_1|^p \; dx \Big)^{(2-p)/2} \Big(\int_{\Omega_0} m_1^{p-2} |\nabla u - \nabla v|^2 \, dx \Big)^{p/2} \\
&\quad + \Big(\int_{\Omega_0} |m_2|^p \; dx \Big)^{(2-p)/2} \Big(\int_{\Omega_0} m_2^{p-2} |u - v|^2 \, dx \Big)^{p/2} \\
&\leq \Big(\int_{\Omega_0} \big(|m_1|^p + |m_2|^p \big) \, dx \Big)^{(2-p)/2} \Big(\int_{\Omega_0} \big(m_1^{p-2} |\nabla u - \nabla v|^2 + m_2^{p-2} |u-v|^2 \big) \, dx \Big)^{p/2}.
\end{aligned}$$

Applying Minkowski's inequality twice, we have

$$\begin{aligned}
\int_{\Omega_0} \big(|m_1|^p + |m_2|^p \big) \, dx &\leq \int_{\Omega_0} \big(|\nabla u| + |\nabla v| \big)^p \, dx + \int_{\Omega_0} \big(|u| + |v| \big)^p \, dx \\
&\leq \big(\|\nabla u\|_{L^p(\Omega, \mathbb{R}^N)} + \|\nabla v\|_{L^p(\Omega, \mathbb{R}^N)} \big)^p \\
&\quad + \big(\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)} \big)^p \\
&\leq (\|u\|_X + \|v\|_X)^p
\end{aligned}$$

and hence

$$\int_{\Omega_0} \big(m_1^{p-2} |\nabla u - \nabla v|^2 + m_2^{p-2} |u - v|^2 \big) \, dx \geq \; (\|u\|_X + \|v\|_X)^{p-2} \; \|u - v\|_X^2 \, .$$

Consequently, we obtain

$$\langle J(u) - J(v), u - v\rangle \geq \begin{cases} c \, (\|u\|_X + \|v\|_X)^{p-2} \, \|u - v\|_X^2 & \text{if} \quad 1 < p < 2 \text{ and} \\ & \qquad (u,v) \neq (0,0) \\ c \, \|u - v\|_X^p & \text{if} \quad 2 \leq p < \infty. \end{cases} \tag{2.7}$$

It is easily checked that J is strictly monotone and coercive. Since J is bounded and continuous on X, the Browder-Minty theorem hence implies that J is a bounded homeomorphism on X and J^{-1} is bounded on X^*; see e.g. [18, Theorem 26.A]. Moreover, we see from (J4) that Ψ_p and Φ_p satisfy (J1), (J2), and (J3), respectively. Using the first assertion gives that J_p is a bounded homeomorphism on X. This completes the proof. □

For the sake of convenience, we prove the following result which is formally Corollary 3.4 of [8]. The difference from the assumptions in [8] is the presence of the term $\Phi(x,u)$ in (2.1) which makes possible to work in the whole Sobolev space. We point out that the continuity of the map $(t,f)\mapsto J_t^{-1}(f)$ is crucial for obtaining our main result in Section 4.

Lemma 2.4. *Suppose that two operators $J, J_0\colon X\to X^*$ of the form* (2.1) *are generated by functions satisfying* (J1), (J2), *and* (J3), *respectively. Then for each $t\in[0,1]$, the operator $J_t\colon X\to X^*$ defined by $J_t := tJ+(1-t)J_0$ is a bounded homeomorphism onto X^*. Moreover, the map $h:[0,1]\times X^*\to X,\ (t,f)\mapsto J_t^{-1}(f)$ is continuous on $[0,1]\times X^*$.*

Proof. For all $t\in[0,1]$ and $u,v\in X$, we have

$$\langle J_t(u)-J_t(v),u-v\rangle \geq \min\{\langle J_0(u)-J_0(v),u-v\rangle,\langle J_1(u)-J_1(v),u-v\rangle\} \tag{2.8}$$

and by (2.7)

$$\langle J_i(u)-J_i(v),u-v\rangle \geq \begin{cases} c\,(\|u\|_X+\|v\|_X)^{p-2}\,\|u-v\|_X^2 & \text{if } 1<p<2 \text{ and } (u,v)\neq(0,0) \\ c\,\|u-v\|_X^p & \text{if } 2\leq p<\infty \end{cases} \tag{2.9}$$

for $i=0,1$. Since each bounded continuous operator J_t is thus strictly monotone and coercive, we know as before that it is a bounded homeomorphism. For all $s,t\in[0,1]$ and $u,v\in X$, it follows from the relation

$$\langle J_t(u)-J_s(v),u-v\rangle=(t-s)\,\langle J_1(v)-J_0(v),u-v\rangle+\langle J_t(u)-J_t(v),u-v\rangle$$

that

$$\big(\,|t-s|\,\|J_1(v)-J_0(v)\|_{X^*}+\|J_t(u)-J_s(v)\|_{X^*}\,\big)\,\|u-v\|_X\geq\langle J_t(u)-J_t(v),u-v\rangle$$

and hence by (2.8) and (2.9)

$$\begin{aligned} &|t-s|\,\|J_1(v)-J_0(v)\|_{X^*}+\|J_t(u)-J_s(v)\|_{X^*} \\ &\qquad\geq\begin{cases} c\,(\|u\|_X+\|v\|_X)^{p-2}\,\|u-v\|_X & \text{if } 1<p<2 \text{ and } (u,v)\neq(0,0) \\ c\,\|u-v\|_X^{p-1} & \text{if } 2\leq p<\infty. \end{cases} \end{aligned} \tag{2.10}$$

To show the continuity of h, there are two cases to consider. For $1<p<2$, let (t_n,f_n) be any sequence in $[0,1]\times X^*$ such that $t_n\to t$ in $[0,1]$ and $f_n\to f$ in X^* as $n\to\infty$. Set $u_n=J_{t_n}^{-1}(f_n)$ and $u=J_t^{-1}(f)$. We obtain from (2.10) that

$$\begin{aligned}\|u_n-u\|_X\leq c^{-1}\,(\|u_n\|_X+\|u\|_X)^{2-p}(&|t_n-t|\,\|J_1(u)-J_0(u)\|_{X^*} \\ &+\|J_{t_n}(u_n)-J_t(u)\|_{X^*}).\end{aligned}$$

Note that h maps bounded sets into bounded sets, as follows from (2.10) with $t=s$ and $v=0$. Since $\{u_n : n\in\mathbb{N}\}$ is bounded and $J_{t_n}(u_n)\to J_t(u)$ in X^* as $n\to\infty$, we conclude that

$$u_n\to u \quad\text{and equivalently}\quad h(t_n,f_n)\to h(t,f) \quad\text{as}\quad n\to\infty.$$

Thus, h is continuous on $[0,1]\times X^*$. For $2\leq p<\infty$, it is clear by (2.10) that h is continuous on $[0,1]\times X^*$. This completes the proof. $\square$

Let p^* denote the critical Sobolev exponent, that is, $p^* = Np/(N-p)$ if $p < N$ and $p^* = \infty$ if $p \geq N$. We assume that

(F1) $f : \mathbb{R} \times \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory condition in the sense that $f(\lambda, \cdot, u, v)$ is measurable for all $(\lambda, u, v) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$ and $f(\cdot, x, \cdot, \cdot)$ is continuous for almost all $x \in \Omega$.

(F2) For each bounded interval $I \subset \mathbb{R}$, there are a function $a_I \in L^q(\Omega)$ and a nonnegative constant b_I such that

$$|f(\lambda, x, u, v)| \leq a_I(x) + b_I(|u|^{\frac{p}{q}} + |v|^{\frac{p}{q}})$$

for almost all $x \in \Omega$ and for all $(\lambda, u, v) \in I \times \mathbb{R} \times \mathbb{R}^N$, where the conjugate exponent of $q > 1$ is strictly less than p^*.

(F3) There exist a function $a \in L^{p'}(\Omega)$ and a locally bounded function $b : [0, \infty) \to \mathbb{R}$ with $\lim_{r\to\infty} b(r)/r^{p-1} = 0$ such that

$$|f(0, x, u, v)| \leq a(x) + b(|u| + |v|),$$

for almost all $x \in \Omega$ and for all $(u, v) \in \mathbb{R} \times \mathbb{R}^N$.

Under (F1) and (F2), we can define an operator $F : \mathbb{R} \times X \to X^*$ by

$$\langle F(\lambda, u), v\rangle = \int_\Omega f(\lambda, x, u(x), \nabla u(x))v(x)\, dx \tag{2.11}$$

and an operator $G : X \to X^*$ by

$$\langle G(u), v\rangle = \int_\Omega |u(x)|^{p-2}u(x)v(x)\, dx. \tag{2.12}$$

For our aim, we need the following result. The argument follows the lines of the proof of Theorem 4.1 in [8], although we work in the whole Sobolev space, not only in the subspace with zero traces. Recall that F is *completely continuous* if F is continuous and maps bounded sets into relatively compact sets.

Lemma 2.5. *If* (F1) *and* (F2) *hold, then* $F : \mathbb{R} \times X \to X^*$ *is completely continuous. The operator* $G : X \to X^*$ *is completely continuous.*

Proof. A linear operator $I_1 : \mathbb{R} \times X \to \mathbb{R} \times L^p(\Omega) \times L^p(\Omega, \mathbb{R}^N)$ defined by

$$I_1(\lambda, u) := (\lambda, u, \nabla u) \qquad \text{for } (\lambda, u) \in \mathbb{R} \times X$$

is clearly bounded. Let $\Gamma : Y = \mathbb{R} \times L^p(\Omega) \times L^p(\Omega, \mathbb{R}^N) \to L^q(\Omega)$ be defined by

$$\Gamma(\lambda, u, v)(x) := f(\lambda, x, u(x), v(x)).$$

If I is a bounded interval in $\mathbb{R}$ and $a_I \in L^q(\Omega)$ and $b_I \in [0, \infty)$ are chosen from (F2), then we have

$$\begin{aligned}\|\Gamma(\lambda, u, v)\|^q_{L^q(\Omega)} &\leq \int_\Omega (3\max\{|a_I|, b_I|u|^{\frac{p}{q}}, b_I|v|^{\frac{p}{q}}\})^q dx \\ &\leq 3^q\big(\|a_I\|^q_{L^q(\Omega)} + (b_I)^q \, \|u\|^p_{L^p(\Omega)} + (b_I)^q \, \|v\|^p_{L^p(\Omega, \mathbb{R}^N)} \big).\end{aligned}$$

Thus, Γ is bounded. Since Y is a generalized ideal space and $L^q(\Omega)$ is a regular ideal space, Theorem 6.4 of [17] implies that Γ is continuous on Y. The embedding I_2 :

$X \hookrightarrow L^{q'}(\Omega)$ is completely continuous and so is the adjoint operator $I_2^* : L^q(\Omega) \to X^*$ given by

$$(I_2^* v)(u) = \int_\Omega vu\, dx.$$

From the relation $F = I_2^* \circ \Gamma \circ I_1$ it follows that F is completely continuous. In particular, if we set $f(\lambda, x, u, v) = |u|^{p-2}u$ (with $q = p'$ in the notation of (F2)), then G is completely continuous. This completes the proof. $\square$

We close this section by making observation on the behavior of $F(0, \cdot)$ at infinity and it was proved in [16].

Lemma 2.6. *Under assumptions* (F1) *and* (F3)*, the operator* $F(0, \cdot) : X \to X^*$ *has the following estimate:*

$$\lim_{\|u\|_X \to \infty} \frac{\|F(0, u)\|_{X^*}}{\|u\|_X^{p-1}} = 0.$$

3. A Spectral Result

In this section, we study the asymptotic behavior of the integral operator J induced by Ψ and Φ and then deduce a spectral result for operators that are not necessarily positively homogeneous.

To do this, we consider the functions Ψ_p, Φ_p which induce the integral operator J_p of the form (2.6) and the following asymptotic hypothesis is required as in [8]:

(A) For each $\varepsilon > 0$ there are functions $M_i \in L^p(\Omega)$ $(i = 1, 2)$ such that for almost all $x \in \Omega$ the following estimates hold:

$$\frac{|\Psi(x, v) - \Psi_p(x, v)|}{|v|^{p-1}} \le \varepsilon, \text{ for all } v \in \mathbb{R}^N \text{ with } |v| > |M_1(x)|$$

and

$$\frac{|\Phi(x, u) - \Phi_p(x, u)|}{|u|^{p-1}} \le \varepsilon, \text{ for all } u \in \mathbb{R} \text{ with } |u| > |M_2(x)|\,.$$

Now we can show that the operators J and J_p are asymptotic at infinity, as in Proposition 5.1 of [8].

Proposition 3.1. *If* (J1), (J2), *and* (A) *hold, then*

$$\lim_{\|u\|_X \to \infty} \frac{\|J(u) - J_p(u)\|_{X^*}}{\|u\|_X^{p-1}} = 0.$$

Proof. Given $\varepsilon > 0$, choose functions $M_i \in L^p(\Omega)$ $(i = 1, 2)$ such that for almost all $x \in \Omega$, the following

$$|\Psi(x, v) - \Psi_p(x, v)| \le \varepsilon |v|^{p-1} \quad \text{and} \quad |\Phi(x, u) - \Phi_p(x, u)| \le \varepsilon |u|^{p-1}$$

hold for all $v \in \mathbb{R}^N$ with $|v| > |M_1(x)|$ and for all $u \in \mathbb{R}$ with $|u| > |M_2(x)|$. In view of (J2), choose functions $a_i \in L^{p'}(\Omega)$ and nonnegative constants b_i $(i = 1, 2)$ such that for almost all $x \in \Omega$, the estimates

$$|\Psi(x, v)| \le a_1(x) + b_1 |M_1(x)|^{p-1} \quad \text{and} \quad |\Phi(x, u)| \le a_2(x) + b_2 |M_2(x)|^{p-1}$$

hold for all $v \in \mathbb{R}^N$ with $|v| \le |M_1(x)|$ and for all $u \in \mathbb{R}$ with $|u| \le |M_2(x)|$. Set

$$\alpha_{M_1}(x) = a_1(x)+(b_1+w(x)) |M_1(x)|^{p-1} \text{ and } \alpha_{M_2}(x) = a_2(x)+(b_2+\nu(x)) |M_2(x)|^{p-1}.$$

Then α_{M_1} and α_{M_2} belong to $L^{p'}(\Omega)$ and for almost all $x \in \Omega$, the estimates

$$|\Psi(x, v) - \Psi_p(x, v)| \le \alpha_{M_1}(x) \quad \text{and} \quad |\Phi(x, u) - \Phi_p(x, u)| \le \alpha_{M_2}(x)$$

hold for all $v \in \mathbb{R}^N$ with $|v| \le |M_1(x)|$ and for all $u \in \mathbb{R}$ with $|u| \le |M_2(x)|$. Thus, for almost all $x \in \Omega$, the following relations

$$|\Psi(x, v) - \Psi_p(x, v)| \le \max\left\{\alpha_{M_1}(x), \varepsilon |v|^{p-1}\right\}$$

and

$$|\Phi(x, u) - \Phi_p(x, u)| \le \max\left\{\alpha_{M_2}(x), \varepsilon |u|^{p-1}\right\}$$

hold for all $v \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$. For all $u \in X$, we obtain by Hölder's and Minkowski's inequalities that

$$\begin{aligned}
\|J(u) - J_p(u)\|_{X^*}^{p'} &\le \int_\Omega \Big(|\Psi(x, \nabla u(x)) - \Psi_p(x, \nabla u(x))|^{p'} + |\Phi(x, u(x)) - \Phi_p(x, u(x))|^{p'} \Big)\, dx \\
&\le \int_\Omega \Big(\max\Big\{|\alpha_{M_1}(x)|^{p'},\ \varepsilon^{p'} |\nabla u(x)|^p\Big\}\max\Big\{|\alpha_{M_2}(x)|^{p'},\ \varepsilon^{p'} |u(x)|^p\Big\}\Big)\, dx \\
&\le \int_\Omega \Big(|\alpha_{M_1}(x)|^{p'} + |\alpha_{M_2}(x)|^{p'} + \varepsilon^{p'} (|\nabla u(x)|^p + |u(x)|^p)\Big)\, dx \\
&= \|\alpha_{M_1}\|_{L^{p'}(\Omega)}^{p'} + \|\alpha_{M_2}\|_{L^{p'}(\Omega)}^{p'} + \varepsilon^{p'} \|u\|_X^p .
\end{aligned}$$

Hence, for all $u \in X$ with $u \ne 0$, we have

$$\frac{\|J(u) - J_p(u)\|_{X^*}}{\|u\|_X^{p-1}} \le \left(\frac{\|\alpha_{M_1}\|_{L^{p'}(\Omega)}^{p'} + \|\alpha_{M_2}\|_{L^{p'}(\Omega)}^{p'}}{\|u\|_X^p} + \varepsilon^{p'} \right)^{\frac{1}{p'}} .$$

This completes the proof. □

Definition 3.2. A real number μ is called an *eigenvalue of* (E) if the equation

$$J_p(u) = \mu G(u)$$

has a solution u_0 in X that is different from the origin.

The following analogue of Theorem 5.1 in [8] will be used in the next section to prove the main theorem. We give another direct proof, by applying nonlinear spectral theory for homogeneous operators given in [4].

Lemma 3.3. *Suppose that* (J1), (J2), (J4) *and* (A) *hold. If μ is not an eigenvalue of* (E), *we have*

$$\liminf_{\|u\|_X\to\infty}\min_{t\in[0,1]}\frac{\|J_t(u)-\mu G(u)\|_{X^*}}{\|u\|_X^{p-1}}>0, \tag{3.1}$$

where $J_t := tJ + (1-t)J_p$ is a convex combination of J and J_p.

Proof. Using a spectral result for homogeneous operators stated in [4], we can obtain in view of Lemma 2.3 and Lemma 2.5 that

$$\alpha := \liminf_{\|u\|_X\to\infty}\frac{\|J_p(u)-\mu G(u)\|_{X^*}}{\|u\|_X^{p-1}}>0.$$

Let ε be an arbitrary positive number. Choose a positive number R_1 such that

$$\|u\|_X \geq R_1 \quad \text{implies} \quad \|J_p(u)-\mu G(u)\|_{X^*} > (\alpha-\varepsilon)\|u\|_X^{p-1}.$$

By Proposition 3.1, there exists a positive number R_2 such that

$$\|u\|_X \geq R_2 \quad \text{implies} \quad \|J(u)-J_p(u)\|_{X^*} < \frac{\alpha}{2}\|u\|_X^{p-1}.$$

Set $R := \max\{R_1, R_2\}$. For all $u \in X$ with $\|u\|_X \geq R$, we have

$$\begin{aligned}\min_{t\in[0,1]}\|J_t(u)-\mu G(u)\|_{X^*} &\geq \|J_p(u)-\mu G(u)\|_{X^*} - \max_{t\in[0,1]}\|J_p(u)-J_t(u)\|_{X^*}\\ &= \|J_p(u)-\mu G(u)\|_{X^*} - \|J(u)-J_p(u)\|_{X^*}\\ &> (\frac{\alpha}{2}-\varepsilon)\|u\|_X^{p-1}.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\liminf_{\|u\|_X\to\infty}\min_{t\in[0,1]}\frac{\|J_t(u)-\mu G(u)\|_{X^*}}{\|u\|_X^{p-1}}\geq\frac{\alpha}{2}>0.$$

□

4. Main Result

The following bifurcation result is taken from Theorem 2.2 of [8], as a key tool in obtaining our bifurcation result.

Lemma 4.1. *Let X be a Banach space and Y a normed space. Suppose that $J : X \to Y$ is a homeomorphism and $G : X \to Y$ is a completely continuous operator such that the Leray-Schauder degree in X satisfies*

$$\deg_X(I_X - (J^{-1}\circ(-G)), B_r, 0) \neq 0$$

for all sufficiently large $r > 0$, where I_X denotes the identity operator on X and B_r the open ball in X of radius r centered at the origin, respectively. Let $F : \mathbb{R}\times X \to Y$ be a completely continuous operator. If the set

$$\bigcup_{t\in[0,1]}\{u\in X : J(u)+G(u)=tF(0,u)\}$$

is bounded, then the set

$$\{(\lambda,u)\in\mathbb{R}\times X : J(u)+G(u)=F(\lambda,u)\}$$

has an unbounded connected set $C \subseteq (\mathbb{R} \setminus \{0\}) \times X$ *such that* $\overline{C}$ *intersects* $\{0\} \times X$.

Definition 4.2. A *weak solution of* (B) is a pair (λ, u) in $\mathbb{R} \times X$ such that

$$J(u) - \mu G(u) = F(\lambda, u) \quad \text{in } X^*,$$

where J, F, and G are defined by (2.1), (2.11), and (2.12), respectively.

We are now prepared to prove the main result on global bifurcation for Neumann problems.

Theorem 4.3. *Suppose that conditions* (J1)-(J4), (A)*, and* (F1)-(F3) *are satisfied. If* μ *is not an eigenvalue of* (E)*, then there is an unbounded connected set* $C \subseteq (\mathbb{R}\setminus\{0\})\times X$ *such that every point* (λ, u) *in* C *is a weak solution of the above Neumann problem* (B) *and* $\overline{C}$ *intersects* $\{0\} \times X$.

Proof. Since μ is not an eigenvalue of (E), we have by Lemma 3.3 that

$$\liminf_{\|u\|_X \to \infty} \frac{\|J(u) - \mu G(u)\|_{X^*}}{\|u\|_X^{p-1}} > 0.$$

In view of Lemma 2.6, for some $\beta > 0$, there is a positive constant R such that

$$\|J(u) - \mu G(u)\|_{X^*} > \beta \|u\|_X^{p-1} > \|F(0, u)\|_{X^*} \geq \|tF(0, u)\|_{X^*}$$

for all $u \in X$ with $\|u\|_X \geq R$ and for all $t \in [0, 1]$. Hence, the set

$$\bigcup_{t\in[0,1]} \{u \in X : J(u) - \mu G(u) = tF(0, u)\}$$

is bounded. To apply Lemma 4.1, it thus remains to prove that

$$\deg_X(I_X - J^{-1} \circ (\mu G), B_r, 0) \neq 0 \tag{4.1}$$

holds for all sufficiently large $r > R$. By Lemma 2.4, $J_t := tJ + (1-t)J_p : X \to X^*$ is a homeomorphism on X and $h : [0, 1] \times X^* \to X$, $h(t, f) := J_t^{-1}(f)$ is continuous on $[0, 1] \times X^*$. Since $g : [0, 1] \times X \to X^*$, $g(t, u) := (t, \mu G(u))$ is completely continuous by Lemma 2.5, the homotopy $H\colon [0, 1] \times X \to X^*$, $H(t, u) = J_t^{-1}(\mu G(u))$, is also completely continuous, as the composition of h with g. Moreover, we obtain by Lemma 3.3 that for sufficiently large $r > R$ we have $H(t, u) \neq u$ for all $(t, u) \in [0, 1] \times \partial B_r$. Hence the homotopy invariance of the degree implies that

$$\begin{aligned}
\deg_X(I_X - J^{-1} \circ (\mu G), B_r, 0) &= \deg_X(I_X - H(1, \cdot), B_r, 0)\\
&= \deg_X(I_X - H(0, \cdot), B_r, 0)\\
&= \deg_X(I_X - J_p^{-1} \circ (\mu G), B_r, 0).
\end{aligned}$$

Since $\Psi_p(x, \cdot)$ and $\Phi_p(x, \cdot)$ are odd for almost all $x \in \Omega$, Borsuk's theorem implies that the last degree is odd and so (4.1) holds. This completes the proof. $\square$

As a consequence of Theorem 4.3, we can show for which particular problems the assertion holds; see e.g., Examples 2.1 and 2.2.

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References

[1] M.A. del Pino, R.F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian*, J. Differential Equations, **92**(1991), 226-251.

[2] P. Drábek, A. Kufner, F. Nicolosi, *Quasilinear Elliptic Equations with Degenerations and Singularities*, W. de Gruyter, Berlin, 1997.

[3] N. Fukagai, M. Ito, K. Narukawa, *A bifurcation problem of some nonlinear degenerate elliptic equations*, Adv. Differential Equations, **2**(1997), 895-926.

[4] E. Giorgieri, J. Appell, M. Väth, *Nonlinear spectral theory for homogeneous operators*, Nonlinear Funct. Anal. Appl., **7**(2002), 589-618.

[5] Y.X. Huang, *On eigenvalue problems for the p-Laplacian with Neumann boundary conditions*, Proc. Amer. Math. Soc., **109**(1990), 177-184.

[6] A.E. Khalil, M. Ouanan, *A global bifurcation result of a Neumann problem with indefinite weight*, Electron. J. Qual. Theory Differential Equations, (2004), 1-14.

[7] I.-S. Kim, Y.-H. Kim, *Global bifurcation for nonlinear equations*, Nonlinear Anal., **69**(2008), 2362-2368.

[8] Y.-H. Kim, M. Väth, *Global solution branches for equations involving nonhomogeneous operators of p-Laplace type*, (Submitted).

[9] A. Lê, *Eigenvalue problems for the p-Laplacian*, Nonlinear Anal., **64**(2006), 1057-1099.

[10] K. Narukawa, *Global bifurcation for quasilinear elliptic equations*, Nonlinear Anal., **30**(1997), 5241-5249.

[11] K. Narukawa, *Global bifurcation of a class of p-Laplacian like operators*, Nonlinear Anal., **47**(2001), 4873-4884.

[12] K. Schmitt, I. Sim, *Bifurcation problems associated with generalized Laplacians*, Adv. Differential Equations, **9**(2004), 797-828.

[13] S. Senn, *Ein lineares und ein nichtlineares elliptisches Eigenwertproblem mit Neumann-Randbedingung*, Doctoral thesis, Universität Zürich, 1982.

[14] S. Senn, *On a nonlinear elliptic eigenvalue problem with Neumann boundary conditions, with an application to population genetics*, Comm. Partial Differential Equations, **8**(1983), 1199-1228.

[15] S. Senn, P. Hess, *On positive solutions of a linear elliptic eigenvalue problem with Neumann boundary conditions*, Math. Ann., **258**(1982), 459-470.

[16] M. Väth, *Global bifurcation of the p-Laplacian and related operators*, J. Differential Equations, **213**(2005), 389-409.

[17] M. Väth, *Continuity of single- and multivalued superposition operators in generalized ideal spaces of measurable vector functions*, Nonlinear Funct. Anal. Appl., **11**(2006), 607-646.

[18] E. Zeidler, *Nonlinear Functional Analysis and its Applications II/B*, Springer, New York, 1990.

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