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SHRINKING PROJECTION ALGORITHM FOR FIXED POINTS OF FIRMLY NONEXPANSIVE MAPPINGS AND ITS APPLICATIONS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. The purpose of this paper is to study the shrinking projection method for finding common fixed points of firmly nonexpansive mappings. Some strong convergence theorems are proved. The main convergence theorem is also applied to the equilibrium and optimization problems. The results of this paper improve and extend the results of Koji Aoyama, Fumiaki Kohsaka, Wataru Takahashi [Koji Aoyama, Fumiaki Kohsaka, Wataru Takahashi, Shrinking projection methods for firmly nonexpansive mappings, Nonlinear Analysis (2009), **71**(2009), 1626-1632] in the following respects: (1) the main convergence theorem has been proved by using the new method; (2) the condition of family of firmly nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ has been relaxed from the condition (Z) to uniformly closed; (3) the application has been given to find the solution of equilibrium and optimization problems.

Key Words and Phrases: Shrinking projection method, firmly nonexpansive mapping, uniformly closed, equilibrium problem.
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1. Introduction

We know of many attempts to obtain a strong convergence iteration to find a fixed point of a nonexpansive mapping [1-22]. A popular method is the hybrid projection method developed in [1-5]; see also [6-10] and references therein. Recently Takahashi, Takeuchi and Kubota [11] introduced an alternative projection method, which is called the shrinking projection method, and they showed several strong convergence theorems for a family of nonexpansive mappings; see also [12]. Therefore Koji Aoyama, Fumiaki Kohsaka, Wataru Takahashi [18] studied the shrinking projection

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method for a family of firmly nonexpansive mappings. Then they proved a strong convergence theorem and presented its applications.

The class of firmly nonexpansive mappings is one of the most important class in nonlinear mappings. In a Hilbert space, it is known that each metric projection onto a nonempty closed convex set is firmly nonexpansive; see [14]. It is also known that a firmly nonexpansive mapping is easily generated from a nonexpansive mapping. Let C be a nonempty subset of a Hilbert space H, $T : C \to H$ a nonexpansive mapping, and I the identity mapping. Then $S = (I + T)/2$ is a firmly nonexpansive mapping of C into H; see [14].

In this paper, motivated by the results mentioned above, we further study the shrinking projection method for a family of firmly nonexpansive mappings. Then we prove some strong convergence theorems and give their applications. The results of this paper improve and extend the results of Koji Aoyama, Fumiaki Kohsaka, Wataru Takahashi [18] in the following respects:

(1)the main convergence theorem has been proved by using the new method which is different with the method of [18];

(2) the condition of family of firmly nonexpansive mappings $\{T_n\}_{n=1}^{\infty}$ has been relaxed from the condition (Z) to uniformly closed;

(3) the application has been given to find the solution of equilibrium problems.

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C a nonempty subset of H. Strong convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. If C is closed and convex and $T : C \to H$ is nonexpansive, then it is known that the fixed points set $F(T)$ is closed and convex, and moreover, $I - T$ is demi-closed at zero, that is, $p \in F(T)$ whenever $x_n - Tx_n \to 0$ and $x_n \rightharpoonup p$.

A mapping $T : C \to H$ is said to be firmly nonexpansive if

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2$$

for all $x, y \in C$. It is obvious that every firmly nonexpansive mapping is nonexpansive. It is also obvious that if a mapping $T : C \to H$ is firmly nonexpansive and $F(T) \neq \emptyset$, then

$$\langle Tx - u, x - Tx \rangle \geq 0 \tag{1.1}$$

for all $x \in C$ and $u \in F(T)$. It is known that if a mapping T is nonexpansive, then $S = (I + T)/2$ is firmly nonexpansive, where I is the identity mapping; see, for example, [14].

Let C be a nonempty closed convex subset of H. The metric projection of H onto C is denoted by P_C, It is known that P_C is firmly nonexpansive.

Definition 1.1. ([18]) Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into H such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. We say that $\{T_n\}_{n=1}^{\infty}$ satisfies the condition (Z) if every weak cluster point of $\{x_n\}$ is in $\bigcap_{n=1}^{\infty} F(T_n)$ whenever $\{x_n\}$ is a bounded sequence in C and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

In this paper, we present the definition of uniformly closed for a sequence of mappings as follows.

Definition 1.2. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into H such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. We say that $\{T_n\}_{n=1}^{\infty}$ is uniformly closed, if $p \in \bigcap_{n=1}^{\infty} F(T_n)$ whenever $\{x_n\} \subset C$ converges strongly to p and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

It is easy to show that, if the sequence of mappings $\{T_n\}$ satisfies the condition (Z), then $\{T_n\}$ must be uniformly closed. In the next section, we shall give a example which is uniformly closed.

K. Aoyama, F. Kohsaka, W. Takahashi [18] proved the following strong convergence theorem.

Theorem 1.3. ([18]) *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{T_n\}$ a sequence of firmly nonexpansive mappings of C into H such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and $x \in H$. Let $\{x_n\}$ be a sequence in C defined by*

$$\begin{cases} x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle T_n x_n - z, x_n - T_n x_n \rangle \geq 0\} \end{cases}$$

for all $n \geq 1$. If $\{T_n\}$ satisfies the condition (Z), then $\{x_n\}$ converges strongly to $P_F(x)$.

2. Main results

Theorem 2.1. *Let H be a Hilbert space, C a nonempty closed convex subset of H, $\{T_n\}$ a sequence of firmly nonexpansive mappings of C into H such that $F = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty, and $x \in H$. Let $\{x_n\}$ be a sequence in C defined by*

$$\begin{cases} x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle T_n x_n - z, x_n - T_n x_n \rangle \geq 0\} \end{cases} \tag{2.1}$$

for all $n \geq 1$. If $\{T_n\}$ is uniformly closed, then $\{x_n\}$ converges strongly to $P_F(x)$.

Proof. It is obvious that each C_n is a closed convex subset of H for every $n \geq 1$. First let us show that $F \subset C_n$ for all $n \geq 1$. Clearly we have $F \subset C_1 = C$. Assume that $F \subset C_n$ for some $n \geq 1$. Since T_n is firmly nonexpansive, it follows from (1.1) that $F \subset C_{n+1}$. By induction on n, we see that $F \subset C_n$ for all $n \geq 1$ and hence $\{x_n\}$ is well defined.

Since $F \subset C_n$ and $C_{n+1} \subset C_n$ for all $n \geq 1$, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\| \leq \|P_F x - x\|,$$

and hence the limit $\lim_{n\to\infty} \|x_n - x\|$ exists.

Therefore, since P_{C_n} is firmly nonexpansive and $x_{n+m} \in C_n$, we have

$$\begin{aligned}\|x_{n+m} - x_n\|^2 &= \|x_{n+m} - P_{C_n}x\|^2 \\ &\le \|x_{n+m} - x\|^2 - \|x - P_{C_n}x\|^2 \\ &\le \|x_{n+m} - x\|^2 - \|x - x_n\|^2 \to 0,\end{aligned}$$

as $n \to \infty$. Then $\{x_n\}$ is Cauchy sequence, so that $\{x_n\}$ converges strongly to an element $p \in C$.

On the other hand, since $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned}0 &\le \|T_nx_n - x_{n+1}\|^2 + \|x_n - T_nx_n\|^2 \\ &\le \|T_nx_n - x_{n+1}\|^2 + \|x_n - T_nx_n\|^2 + 2\langle T_nx_n - x_{n+1}, x_n - T_nx_n\rangle \\ &= \|T_nx_n - x_{n+1} + x_n - T_nx_n\|^2 \\ &= \|x_{n+1} - x_n\|^2 \to 0,\end{aligned}$$

as $n \to \infty$. It follows that $\|T_nx_n - x_n\| \to 0$ as $n \to \infty$. Since $\{T_n\}$ is uniformly closed, we have $p \in F = \bigcap_{n=1}^{\infty} F(T_n)$.

Finally, we claim that $p = P_Fx$. If not, we have $\|x - p\| > \|x - P_Fx\|$. There must exist a positive integer N, if $n > N$, then $\|x - x_n\| > \|x - P_Fx\|$, which leads to

$$\begin{aligned}\|x - P_Fx\|^2 =&\|x - x_n + x_n - P_Fx\|^2 \\ =&\|x - x_n\|^2 + \|x_n - P_Fx\|^2 + 2\langle x_n - P_Fx, x - x_n\rangle.\end{aligned}$$

It follows that $\langle x_n - P_Fx, x - x_n\rangle < 0$, this together with $x_n = P_{C_n}x$ implies that $P_Fx \notin C_n$. Since we have proved that $F \subset C_n$ for all $n \ge 1$. This is a contradiction, hence $p = P_Fx$. This completes the proof. □

A direct consequence of Theorem 2.1 is Theorem 1.3 proved by K. Aoyama, F. Kohsaka, W. Takahashi [18].

Theorem 2.2. *Let H be a Hilbert space, C a nonempty closed convex subset of H, S_n a sequence of nonexpansive mappings of C into H such that $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0,1)$ such that $\sup_n \alpha_n < 1$, and $x \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H defined by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = \alpha_n x_n + (1-\alpha_n)S_nx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\} \end{cases} \tag{2.2}$$

for all $n \ge 1$. If $\{S_n\}$ is uniformly closed, then $\{x_n\}$ converges strongly to P_Fx.

Proof. Let $\{T_n\}$ be a sequence of mappings defined by

$$T_n = \frac{1}{2}I + \frac{1}{2}(\alpha_nI + (1-\alpha_n)S_n) = \frac{1+\alpha_n}{2}I + \frac{1-\alpha_n}{2}S_n$$

for all $n \ge 1$, where I is the identity operator. Then $F(T_n) = F(S_n)$ and hence $F = \bigcap_{n=1}^{\infty} F(T_n)$. It is obvious that, T_n is nonexpansive and firmly nonexpansive.

Since

$$\begin{aligned}\|z - x_n\|^2 - \|z - y_n\|^2 &= \|x_n\|^2 - 2\langle z, x_n\rangle - \|y_n\|^2 + 2\langle z, y_n\rangle \\ &= \langle x_n + y_n - 2z, x_n - y_n\rangle \\ &= 4\langle T_n x_n - z, x_n - T_n x_n\rangle.\end{aligned}$$

Then the iterative scheme (2.2) can be rewritten as follows

$$\begin{cases} x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle T_n x_n - z, x_n - T_n x_n\rangle \geq 0\}. \end{cases} \tag{2.3}$$

Since

$$\|T_n x_n - x_n\| = \frac{1 - \alpha_n}{2}\|S_n x_n - x_n\|,$$

and $\{S_n\}$ is uniformly closed, it follows that, $\{T_n\}$ is also uniformly closed. Therefore by using Theorem 2.1 we obtain the conclusion of Theorem 2.2.

A direct consequence of Theorem 2.2 is the following Theorem 2.3 (Theorem 3.4 in [18]) proved by K. Aoyama, F. Kohsaka, W. Takahashi [18].

Theorem 2.3. *Let H be a Hilbert space, C a nonempty closed convex subset of H, S_n a sequence of nonexpansive mappings of C into H such that $F = \bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. $\{\alpha_n\}$ a sequence in $[0, 1)$ such that $\sup_n \alpha_n < 1$, and $x \in H$. Let $\{x_n\}$ be a sequence in C and $\{C_n\}$ a sequence of closed convex subsets of H defined by $C_1 = C$ and*

$$\begin{cases} x_n = P_{C_n}(x), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\} \end{cases} \tag{2.2}$$

for all $n \geq 1$. If $\{S_n\}$ satisfies the condition (Z), then $\{x_n\}$ converges strongly to $P_F x$.

3. An application to equilibrium and optimization problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$f(x, y) \geq 0 \quad \text{for all } y \in C. \tag{3.1}$$

The set of solutions of (3.1) is denoted by $EP(f)$. Given a mapping $T : C \to H$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in EP(f)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (3.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [2, 3, 4, 11, 24].

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x,x)=0$ for all $x\in C$;
(A2) f is monotone, i.e., $f(x,y)+f(y,x)\le 0$ for all $x,y\in C$;
(A3) for each $x,y,z\in C$,

$$\lim_{t\downarrow 0} f(tz+(1-t)x,y)\le f(x,y);$$

(A4) for each $x\in C$, $y\mapsto f(x,y)$ is convex and lower semicontinuous.

We need the following lemmas for the proof of our main results.

Lemma 3.1. [2, 3] *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that*

$$f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C.$$

Lemma 3.2. [2, 3] *Assume that $f: C\times C\to\mathbb{R}$ satisfies $(A1)-(A4)$. For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\{z\in C:\ f(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\ge 0,\ \forall y\in C\}$$

for all $z\in H$. Then, the following hold:
(1) T_r is single-valued;
(2) T_r is firmly nonexpansive, i.e., for any $x,y\in H$,

$$\|T_rx-T_ry\|^2\le\langle T_rx-T_ry,x-y\rangle;$$

(3) $F(T_r)=EP(f)$;
(4) $EP(f)$ is closed and convex.

Remark. The T_r is also nonexpansive for all $r>0$.

Now, we prove the following Lemma which is very important for the main results of this section.

Lemma 3.3. *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying $(A1)-(A4)$. Let $\{r_n\}$ be a positive real sequence such that $\lim_{n\to\infty} r_n=r>0$. Then the sequence of mappings $\{T_{r_n}\}$ is uniformly closed.*

Proof. (1) Let $\{x_n\}$ be a convergent sequence in C. Let $z_n=T_{r_n}x_n$ for all n, then

$$f(z_n,y)+\frac{1}{r_n}\langle y-z_n,z_n-x_n\rangle\ge 0,\ \forall\ y\in C, \tag{3.2}$$

and

$$f(z_{n+m},y)+\frac{1}{r_{n+m}}\langle y-z_{n+m},z_{n+m}-x_{n+m}\rangle\ge 0,\ \forall\ y\in C. \tag{3.3}$$

Putting $y=z_{n+m}$ in (3.2) and $y=z_n$ in (3.3), we have

$$f(z_n,z_{n+m})+\frac{1}{r_n}\langle z_{n+m}-z_n,z_n-x_n\rangle\ge 0,\ \forall\ y\in C,$$

and

$$f(z_{n+m},z_n)+\frac{1}{r_{n+m}}\langle z_n-z_{n+m},z_{n+m}-x_{n+m}\rangle\ge 0,\ \forall\ y\in C.$$

So, from (A2) we have

$$\langle z_{n+m} - z_n, \frac{z_n - x_n}{r_n} - \frac{z_{n+m} - x_{n+m}}{r_{n+m}} \rangle \geq 0,$$

and hence

$$\langle z_{n+m} - z_n, z_n - x_n - \frac{r_n}{r_{n+m}}(z_{n+m} - x_{n+m}) \rangle \geq 0.$$

Thus, we have

$$\langle z_{n+m} - z_n, z_n - z_{n+m} + z_{n+m} - x_n - \frac{r_n}{r_{n+m}}(z_{n+m} - x_{n+m}) \rangle \geq 0,$$

which implies that

$$\begin{aligned} \|z_{n+m} - z_n\|^2 &\leq \langle z_{n+m} - z_n, z_{n+m} - x_n - \frac{r_n}{r_{n+m}}(z_{n+m} - x_{n+m}) \rangle \\ &= \langle z_{n+m} - z_n, (1 - \frac{r_n}{r_{n+m}}) z_{n+m} + \frac{r_n}{r_{n+m}} x_{n+m} - x_n) \rangle. \end{aligned}$$

Therefore, we get

$$\|z_{n+m} - z_n\| \leq |1 - \frac{r_n}{r_{n+m}}| \|z_{n+m}\| + \|\frac{r_n}{r_{n+m}} x_{n+m} - x_n\|. \tag{3.4}$$

On the other hand, for any $p \in EP(f)$, from $z_n = T_{r_n} x_n$, we have

$$\|z_n - p\| = \|T_{r_n} x_n - p\| \leq \|x_n - p\|,$$

so that $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} r_n = r > 0$, this together with (3.4) implies that $\{z_n\}$ is a Cauchy sequence. Hence $T_{r_n} x_n = z_n$ is convergent.

(2) By using the Lemma 3.2, we know that,

$$\bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f) \neq \emptyset.$$

(3) From (1) we know that, $\lim_{n\to\infty} T_{r_n} x$ exists for all $x \in C$. So, we can define a mapping T from C into itself by

$$Tx = \lim_{n\to\infty} T_{r_n} x, \ \forall \ x \in C.$$

It is obvious that, T is nonexpansive. It is easy to see that

$$EP(f) = \bigcap_{n=1}^{\infty} F(T_{r_n}) \subset F(T).$$

On the other hand, let $w \in F(T), \ w_n = T_{r_n} w$, we have

$$f(w_n, y) + \frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq 0, \ \forall y \in C.$$

By (A2) we know

$$\frac{1}{r_n} \langle y - w_n, w_n - w \rangle \geq f(y, w_n), \ \forall y \in C.$$

Since $w_n \to Tw = w$ and from (A4), we have $f(y, w) \leq 0$, for all $y \in C$. Then, for $t \in (0, 1]$ and $y \in C$,

$$\begin{aligned} 0 &= f(ty + (1-t)w, ty + (1-t)w) \\ &\leq tf(ty + (1-t)w, y) + (1-t)f(ty + (1-t)w, w) \\ &\leq tf(ty + (1-t)w, y). \end{aligned}$$

Therefore, we have

$$f(ty + (1-t)w, y) \geq 0.$$

Letting $t \downarrow 0$ and using (A3), we get

$$f(w, y) \geq 0, \ \forall\, y \in C.$$

and hence $w \in EP(f)$. From above two respects, we know that, $F(T) = \bigcap_{n=0}^{\infty} F(T_{r_n})$.

Next we show $\{T_{r_n}\}$ is uniformly closed. Assume $x_n \to x$ and $\|x_n - T_{r_n}x_n\| \to 0$, from above results we know that, $Tx = \lim_{n\to\infty} T_{r_n}x$. On the other hand, from $\|x_n - T_{r_n}x_n\| \to 0$, we also get $\lim_{n\to\infty} T_{r_n}x = x$, so that $x \in F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$. That is, the sequence of mappings $\{T_{r_n}\}$ is uniformly closed. This completes the proof. □

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ into $R = (-\infty, +\infty)$ satisfying (A1)-(A4) and $EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequences generated by $x \in H$ and*

$$\begin{cases} x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle T_{r_n}x_n - z, x_n - T_{r_n}x_n\rangle \geq 0\} \end{cases}$$

for all $n \geq 1$. Assume $\lim_{n\to\infty} r_n = r > 0$. Then $\{x_n\}$ converges strongly to $P_{EP(f)}x$, where $P_{EP(f)}$ is the metric projection from H onto $EP(f)$.

Proof. By using the Lemma 3.3, we know that, $\{T_{r_n}\}$ is uniformly closed and $\bigcap_{n=1}^{\infty} F(T_{r_n}) = EP(f)$. By using the Theorem 2.1, we obtain the conclusion of Theorem 3.4 □

Now, we study a kind of optimization problem by using the above results of this paper. That is, we will give an iterative algorithm of solution for the following optimization problem with nonempty set of solutions.

$$\begin{cases} \min h(x), \\ \ x \in C, \end{cases} \tag{3.5}$$

where $h(x)$ is a convex and lower semicontinuous functional defined on a closed convex subset C of a Hilbert space H. We denoted by A the set of solutions of (3.5). Let f be a bifunction from $C \times C$ to R defined by $f(x, y) = h(y) - h(x)$. We consider the following equilibrium problem, that is to find $x \in C$ such that

$$f(x, y) \geq 0, \quad \forall\, y \in C. \tag{3.6}$$

It is obvious that $EP(f) = A$, where $EP(f)$ denotes the set of solutions of equilibrium problem (3.6). In addition, it is easy to see that $f(x,y)$ satisfies the conditions (A1)-(A4) in the section 2. Therefore, from the Theorem 3.4, we can obtain the following Theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $h(x)$ be a convex and lower semicontinuous functional defined on C. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x \in C$ and*

$$\begin{cases} h(x_n) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall\ y \in C, \\ x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle u_n - z, x_n - u_n\rangle \geq 0\} \end{cases} \tag{3.7}$$

Assume $\lim_{n\to\infty} r_n = r > 0$. *Then $\{x_n\}$ converges strongly to $P_A x_0$.*

Proof. Let

$$T_{h,r}(x) = \{z \in C : h(x) - h(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall\ y \in C\},$$

for any $r > 0$. From the conditions of Theorem 3.5, we know $T_{h,r}(x)$ is single-valued firmly nonexpansive for all $r > 0$. Therefore iterative scheme (3.7) is equivalent to the following scheme

$$\begin{cases} x_n = P_{C_n}(x), \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \langle T_{h,r_n}x_n - z, x_n - T_{h,r_n}x_n\rangle \geq 0\}. \end{cases}$$

By using Theorem 3.4, $\{x_n\}$ converges strongly to $P_F x$, where $F = \bigcap_{n=1}^{\infty} F(T_{h,r_n})$. Furthermore, by using Lemma 3.2, we have $A = EP(f) = \bigcap_{n=1}^{\infty} F(T_{h,r_n}) = F$. Then $\{x_n\}$ converges strongly to $P_A x$. This completes the proof. □

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