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CONVERGENCE OF VISCOSITY ITERATIVE SCHEMES FOR NONEXPANSIVE SEMIGROUPS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with a gauge function φ, C a nonempty closed convex subset of E, $f : C \to C$ a contraction, and $\{T(t) : t \geq 0\}$ a nonexpansive semigroup on C with the fixed point set $F := \bigcap_{t\geq 0} F(T(t)) \neq \emptyset$. Strong convergence theorems of the following implicit and explicit viscosity iterative schemes are established:

$$x_t = \lambda_t f(x_t) + (1-\lambda_t)T(t)x_t, \quad t \in (0,\infty)$$

where $\{\lambda_t\}_{t\in(0,\infty)}$ is a net in $(0,1)$ such that $\lim_{t\to\infty} \lambda_t = 0$, and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset \mathbb{R}^+$. The limit point is the unique solution of a certain variational inequality.

Key Words and Phrases: Viscosity iterative scheme, nonexpansive semigroups, common fixed point, contraction, weakly sequentially continuous duality mapping, variational inequality.
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1. Introduction

Let E be a real Banach space and C be a nonempty closed convex subset of E. Recall that a mapping $f : C \to C$ is a *contraction* on C if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x-y\|, \forall x, y \in C$. We use Σ_C to denote the collection of mappings f verifying the above inequality. Let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x-y\|\ \forall x, y \in C$) and $F(T)$ denotes the set of fixed points of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that a family $\{T(t) : t \geq 0\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:
(1) $T(t_1 + t_2)x = T(t_1)T(t_2)x$ for any $t_1,\ t_2 \in \mathbb{R}^+$ and $x \in C$;

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(2) $T(0)x = x$ for each $x \in C$;
(3) for each $x \in C$, $t \mapsto T(t)x$ is continuous;
(4) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for each $t \in \mathbb{R}^+$ and $x,\ y \in C$.

Given a real number $t \in (0, 1)$, a contraction $f \in \Sigma_C$ and a nonexpasive mapping T, let a contraction $T_t := T_t^f : C \to C$ be defined by

$$T_t x = tf(x) + (1 - t)Tx, \quad x \in C.$$

Denote by $x_t := x_t^f \in C$ the unique fixed point of T_t. Then x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t. \tag{1.1}$$

A special case of (1.1) has been considered in a Hilbert space by Browder [4] as follows. Fix $u \in C$ and define a contraction G_t on C by

$$G_t x = tu + (1 - t)Tx, \quad x \in C.$$

Let $z_t \in C$ be the unique fixed point of G_t. Then z_t satisfies the equation

$$z_t = tu + (1 - t)Tz_t.$$

(Such a sequence $\{z_t\}$ is said to be an approximating fixed point of T since it possesses the property that if $\{z_t\}$ is bounded, then $\lim_{t \to 0} \|Tz_t - z_t\| = 0$.) In 1967, the strong convergence of $\{z_t\}$ as $t \to 0$ for a self-mapping T of a bounded C was proved in a Hilbert space by Browder [4] and Halpern [10]. In 1980, Reich [16] extended the result of Browder [4] to a uniformly smooth Banach space and showed that the limit defines the (unique) sunny nonexpansive retraction from C onto $F(T)$. Takahashi and Ueda [21] improved results of Reich [16] to a reflexive Banach space having a uniformly Gâteaux differentiable norm (see also Ha and Jung [9]).

In 1967, Halpern [10] firstly introduced the following explicit iterative scheme in Hilbert space:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)x_n. \tag{1.2}$$

He pointed out that the control conditions

$$\text{(C1)} \quad \lim_{n \to \infty} \alpha_n = 0 \quad \text{and} \quad \text{(C2)} \quad \sum_{n=0}^{\infty} \alpha_n = \infty$$

are necessary for the convergence of the iterative scheme (1.2) to a fixed point of T.

In 1992, Wittmann [22] obtained a strong convergence result in a Hilbert space for the iterative scheme (1.2) under the control conditions (C1), (C2) and (C3) $\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$.

Shioji and Takahashi [17] extended Wittmann's results to a reflexive Banach space having a uniformly Gâteaux differentiable norm and in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings. For other control conditions, we also refer Lions [13] and Reich [16].

On the other hand, the viscosity approximation method of selecting a particular fixed point of a given nonexpansive mapping was proposed by Moudafi [15]. In 2004, in order to extend Theorem 2.2 of Moudafi [15] to a Banach space setting, Xu [24] consider the the following explicit viscosity iterative scheme in a uniformly smooth Banach space: for $T : C \to C$ a nonexpansive mapping, $f \in \Sigma_C$ and $\alpha_n \in (0, 1)$,

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.3}$$

and under control conditions (C1), (C2) and (C3) or (C4) $\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$, he studied the strong convergence of the sequence $\{x_n\}$ generated by (1.3) to a fixed point of T which is the unique solution of a certain variational inequality. Moreover, in [24], he also studied the strong convergence of $\{x_t\}$ defined by (1.1) as $t \to 0$ in either a Hilbert space or a uniformly smooth Banach space. For the case of more general Banach spaces, see also Jung [11].

On the another hand, in order to extend Browder's and Reich's results to the nonexpansive semigroup case, Shioji and Takahashi [18] introduced in a Hilbert space the implicit iterative scheme

$$x_n = \alpha_n u + (1 - \alpha_n)\sigma_{t_n}(x_n), \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{t_n\} \subset \mathbb{R}^+$ such that $\lim_{n\to\infty} t_n = \infty$, and for each $t > 0$ and $x \in C$, $\sigma_t(x)$ is the average given by

$$\sigma_t(x) = \frac{1}{t}\int_0^t T(s)xds.$$

Under suitable conditions on the sequence $\{\alpha_n\}$, they proved the strong convergence of $\{x_n\}$ to a point in $F := \bigcap_{t\geq 0} F(T(t))$.

In 2007, Chen and Song [5] considered the following implicit and explicit viscosity iterative scheme:

$$x_n = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n} x_n, \quad n \geq 1,$$
$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)\sigma_{t_n} x_n, \quad n \geq 1,$$

and proved that the sequence $\{x_n\}$ converges to a same point of F in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm.

In 2003, Suzuki [20] introduced firstly in Hilbert space the following implicit iterative scheme for the nonexpansive semigroup case:

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 1, \tag{1.4}$$

and proved strong convergence of the sequence $\{x_n\}$ generated by (1.4) with appropriate conditions imposed upon the parameter sequence $\{\alpha_n\}$. In 2005, Xu [25] proved that Suzuki's result holds in a uniformly convex Banach space having a weakly sequentially continuous duality mapping.

In 2005, Aleyner and Reich [2] considered the following explicit iterative scheme:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 0 \tag{1.5}$$

in a reflexive Banach space having a uniformly Gâteaux differentiable such that each nonempty, bounded, closed and convex subset of E has the fixed point property for nonexpansive mappings (Note that all these assumptions are fulfilled whenever E is uniformly smooth). Under appropriate assumptions imposed upon the semigroup $\{T(t) : t \geq 0\}$ and the control conditions (C1), (C2) and (C3) on $\{\alpha_n\}$, they showed that the sequence $\{x_n\}$ generated by (1.5) converges strongly to Qu, where Q is the unique sunny nonexpansive retraction from C onto $F := \bigcap_{t\geq 0} F(T(t))$, $Qu = s - \lim_{t\to\infty} z_t$ and z_t is the unique solution of the following equation:

$$z_t = \lambda_t u + (1-\lambda_t)T(t)z_t, \quad t \in (0,\infty),$$

where $\{\lambda_t\}_{t\in(0,\infty)}$ is a net in (0,1) such that $\lim_{t\to\infty} \lambda_t = 0$. Benavides et al. [3] also studied in a uniformly smooth Banach space that the implicit iterative scheme (1.4) and the explicit iterative scheme (1.5) converges to a same point of F under the asymptotic regularity on the semigroup $\{T(t) : t \geq 0\}$ and the control conditions (C1), (C2) and (C4) on $\{\alpha_n\}$.

In 2008, Song and Xu [19] considered the following implicit and explicit viscosity iterative schemes in a reflexive and strictly convex Banach space having a uniformly Gâteaux differentiable norm:

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1,$$

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset \mathbb{R}^+$ such that $\lim_{n\to\infty} t_n = \infty$, and proved that the sequence $\{x_n\}$ converges to a same point of F under the control conditions (C1) and (C2) on $\{\alpha_n\}$ and uniform asymptotic regularity on $\{T(t) : t \geq 0\}$.

In this paper, motivated by above-mentioned results, we consider the following implicit and explicit viscosity iterative schemes for nonexpansive semigroup $\{T(t) : t \geq 0\}$ from $C \to C$; for $f \in \Sigma_C$,

$$x_t = \lambda_t f(x_t) + (1-\lambda_t)T(t)x_t, \quad t \in (0,\infty) \tag{1.6}$$

where $\{\lambda_t\}_{t\in(0,\infty)}$ is a net in (0,1) such that $\lim_{t\to\infty} \lambda_t = 0$, and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \quad n \geq 0. \tag{1.7}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset \mathbb{R}^+$. First, by using the uniform asymptotic regularity on $\{T(t) : t \geq 0\}$, we establish a strong convergence theorem for the sequence $\{x_t\}$ defined by (1.6) in a reflexive Banach space having a weakly sequentially continuous duality mapping. Then, under the control conditions (C1) and (C2) on $\{\alpha_n\}$ and $\lim_{n\to\infty} t_n = \infty$, and the uniform asymptotic regularity on $\{T(t) : t \geq 0\}$, we prove in the same Banach space that the sequence $\{x_n\}$ generated by (1.7) converges strongly to a common fixed point of $\{T(t) : t \geq 0\}$, which is the unique solution of a certain variational inequality. The main results develop and improve the corresponding results of Aleyner and Censor [1], Aleyner and Reich [2], Benavides et al. [3], Chen and Song [5], Shioji and Takahashi [18], Song and Xu [19], Suzuki [20] and Xu [25].

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by $\langle x, f\rangle$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x.

The norm of E is said to be *Gâteaux differentiable* (and E is said to be *smooth*) if

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t}$$

exists for each $x,\ y$ in its unit sphere $U=\{x\in E:\|x\|=1\}$.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0,\infty)$ such that $\varphi(0)=0$ and $\lim_{r\to\infty}\varphi(r)=\infty$. The mapping $J_\varphi : E\to 2^{E^*}$ defined by

$$J_\varphi(x)=\{f\in E^*:\langle x,f\rangle=\|x\|\|f\|,\|f\|=\varphi(\|x\|)\},\quad \text{for all } x\in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t)=t$ denoted by J, is referred to as the *normalized duality mapping*. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued. The following property of duality mapping is also well-known ([6]):

$$J_\varphi(\lambda x)=\operatorname{sign}\lambda\left(\frac{\varphi(|\lambda|\cdot\|x\|)}{\|x\|}\right)J(x)\ \text{ for all } x\in E\setminus\{0\},\quad \lambda\in\mathbb{R}, \tag{2.1}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $J(-x)=-J(x)$ for all $x\in E$.

We say that a Banach space E has a weakly sequentially continuous duality mapping if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\}\in E$ with $x_n\rightharpoonup x$, $J_\varphi(x_n)\overset{*}{\rightharpoonup} J_\varphi(x)$. For example, every l^p space $(1<p<\infty)$ has a weakly sequentially continuous duality mapping with gauge function $\varphi(t)=t^{p-1}$. Set

$$\Phi(t)=\int_0^t\varphi(\tau)d\tau,\quad \text{for all } t\in\mathbb{R}^+.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x. Thus it is easy to see ([6]) that the normalized duality mapping $J(x)$ can also be defined as the subdifferential of the convex functional $\Phi(\|x\|)=\|x\|^2/2$, that is, for all $x\in E$

$$J(x)=\partial\Phi(\|x\|)=\{f\in E^*:\Phi(\|y\|)-\Phi(\|x\|)\ge\langle y-x,f\rangle\ \text{ for all } y\in E\}.$$

Let D be a subset of C. Then a mapping $Q:C\to D$ is said to be a retraction from C onto D if $Qx=x$ for all $x\in D$. A retraction $Q:C\to D$ is said to be *sunny* if $Q(Qx+t(x-Qx))=Qx$ for all $x\in C$ and $t\ge 0$ with $Qx+t(x-Qx)\in C$. A subset D of C is said to be a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction of C onto D. In a smooth Banach space E, it is well-known [8, p. 48]) that Q is a sunny nonexpansive retraction from C onto D if and only if the following condition holds:

$$\langle x-Qx,J(z-Qx)\rangle\le 0,\quad x\in C,\quad z\in D. \tag{2.2}$$

(Note that this fact still holds if the normalized duality mapping J is replaced by a general duality mapping J_φ with gauge function φ.)

We need the following lemmas for the proof of our main results. (Lemma 2.1 was also given in [12]. Lemma 2.2 is essentially Lemma 2 in [14] (also see [23]). We refer also [6, 7, 8] for Lemmas 2.3, 2.4 and 2.5).

Lemma 2.1. *Let E be a real Banach space and φ a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \text{for all } t \in \mathbb{R}^+.$$

Then the following inequality holds:

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle, \quad \text{for all } x, y \in E,$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$. In particular, if E is smooth, then one has

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle, \quad \text{for all } x, y \in E.$$

Lemma 2.2. *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \leq (1 - \lambda_n)s_n + \lambda_n\beta_n + \gamma_n, \quad n \geq 0,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1 - \lambda_n) = 0$,

(ii) $\limsup_{n\to\infty} \beta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\beta_n < \infty$,

(iii) $\gamma_n \geq 0$ $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (Demiclosedness principle)**.** *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with a gauge function φ, C a nonempty closed convex subset of E and $T : C \to E$ a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I - T)x_n \to y$ imply that $x \in C$ and $(I - T)x = y$.*

Lemma 2.4. *If E is a Banach space such that E^* is strictly convex, then E is smooth and any duality mapping is norm-to-weak* continuous.*

Lemma 2.5. *Let E be a smooth Banach space, C a nonempty closed convex subset of E and $T : C \to C$ a nonexpansive mapping. If J is the normalized duality mapping on E, then*

$$\langle (I - T)(x) - (I - T)(y), J(x - y)\rangle \geq 0, \quad \text{for all } \ x, \ y \in C.$$

Finally, recall that a nonexpansive semigroup $\{T(t) : t \geq 0\}$ on C is said to be *uniformly asymptotically regular* (shortly, u.a.r) on bounded subsets of C if

$$T(s + t)x = T(s)T(t)x, \text{ for all } s, t \geq 0 \text{ and } x \in C$$

and for all bounded subset K of C there holds

$$\lim_{t\to\infty}\sup_{K}\|T(s)T(t)x-T(t)x\|=0, \tag{2.3}$$

uniformly for all $s\geq 0$ (also see [1, 2, 3]). Examples of u.a.r operator semigroup can be found in [1, 19].

3. Main results

First, we study the convergence of implicit viscosity iterative scheme to the unique solution of a certain variational inequality.

For any $t\geq 0$, $T(t):C\to C$ is nonexpansive and so, for any $\lambda_t\in(0,1)$ and $f\in\Sigma_C$, $\lambda_t f+(1-\lambda_t)T(t):C\to C$ defines a contraction. Thus, by the Banach contraction mapping principle, there exists a unique fixed point x_t^f satisfying

$$x_t^f=\lambda_t f(x_t^f)+(1-\lambda_t)T(t)x_t^f. \tag{3.1}$$

For simplicity we will write x_t for x_t^f provided no confusion occurs.

Now we show that the sequence $\{x_t\}$ defined by (3.1) converges strongly some common fixed point of $\{T(t):t\geq 0\}$.

Theorem 3.1. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $\{T(t):t\geq 0\}$ a u.a.r. nonexpansive semigroup from C into itself with $F:=\bigcap_{t\geq 0}F(T(t))\neq\emptyset$. Let $\{x_t\}$ be defined by (3.1) and $\lambda_t\in(0,1)$ such that $\lim\limits_{t\to\infty}\lambda_t=0$. Then as $t\to\infty$, $\{x_t\}$ converges strongly to a point in F. If we define $Q:\Sigma_C\to F$ by*

$$Q(f)=\overline{q}:=\lim_{t\to\infty}x_t,\quad f\in\Sigma_C,$$

then $\overline{q}$ is the unique solution in F of the variational inequality

$$\langle (I-f)(\overline{q}),J_\varphi(\overline{q}-p)\rangle\leq 0,\quad f\in\Sigma_C,\quad p\in F.$$

Proof. Note that the definition of the weak sequential continuity of duality mapping J_φ implies that E is smooth. Let $\{x_{t_n}\}$ be a subsequence of $\{x_t\}$ such that $\lim\limits_{n\to\infty}t_n=\infty$ and

$$x_{t_n}=\lambda_{t_n}f(x_{t_n})+(1-\lambda_{t_n})T(t_n)x_{t_n}.$$

Let $p\in F$. Then $x_{t_n}-p=\lambda_{t_n}(f(x_{t_n})-p)+(1-\lambda_{t_n})(T(t_n)x_{t_n}-T(t_n)p)$. Therefore

$$\begin{aligned}\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|)=&\ \langle x_{t_n}-p,J_\varphi(x_{t_n}-p)\rangle\\ \leq \lambda_{t_n}\langle f(x_{t_n})-p,J_\varphi(x_{t_n}-p)\rangle &+(1-\lambda_{t_n})\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|).\end{aligned}$$

It follows that for all $p\in F$,

$$\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|)\leq\langle f(x_{t_n})-p,J_\varphi(x_{t_n}-p)\rangle. \tag{3.2}$$

Hence

$$\begin{aligned}&\langle x_{t_n}-f(x_{t_n}),J_\varphi(x_{t_n}-p)\rangle\\ &=\langle x_{t_n}-p,J_\varphi(x_{t_n}-p)\rangle+\langle p-f(x_{t_n}),J_\varphi(x_{t_n}-p)\rangle\\ &\geq\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|)-\|x_{t_n}-p\|\varphi(\|x_{t_n}-p\|)=0.\end{aligned}$$

That is, $\langle z_{t_n} - f(x_{t_n}), J_\varphi(x_{t_n} - p)\rangle \geq 0$. Now

$$\begin{aligned}\|x_{t_n} - p\| \leq\ & \lambda_{t_n}\|f(x_{t_n}) - p\| + (1-\lambda_{t_n})\|T(t_n)x_{t_n} - T(t_n)p\| \\ \leq\ & \lambda_{t_n}\|f(x_{t_n}) - p\| + (1-\lambda_{t_n})\|x_{t_n} - p\|.\end{aligned}$$

This gives that

$$\begin{aligned}\|x_{t_n} - p\| \leq\ \|f(x_{t_n}) - p\| &\leq \|f(x_{t_n}) - f(p)\| + \|f(p) - p\| \\ &\leq\ k\|x_{t_n} - p\| + \|f(p) - p\|,\end{aligned}$$

and so $\|x_{t_n} - p\| \leq \frac{1}{1-k}\|f(p) - p\|$. In particular, $\{x_{t_n}\}$ is bounded, so are $\{f(x_{t_n})\}$ and $\{T(t_n)x_{t_n}\}$. Since E is reflexive, $\{x_{t_n}\}$ has a weakly convergence subsequence $\{x_{t_{n_k}}\}$, say, $x_{t_{n_k}} \rightharpoonup u \in E$. Since $\lambda_{t_n} \to 0$,

$$\|x_{t_n} - T(t_n)x_{t_n}\| = \lambda_{t_n}\|f(x_{t_n}) - T(t_n)x_{t_n}\| \to 0 \quad (\text{as } n \to \infty).$$

Also by (2.3), we have for each $t \in \mathbb{R}^+$,

$$\begin{aligned}&\|T(t)x_{t_n} - x_{t_n}\| \\ &\leq \|T(t)T(t_n)x_{t_n} - T(t)x_{t_n}\| + \|T(t)T(t_n)x_{t_n} - T(t_n)x_{t_n}\| \\ &\qquad + \|T(t_n)x_{t_n} - x_{t_n}\| \\ &\leq \|T(t)T(t_n)x_{t_n} - T(t)x_{t_n}\| + 2\|T(t_n)x_{t_n} - x_{t_n}\| \\ &\leq \sup_K \|T(t)T(t_n)x - T(t_n)x\| + 2\|T(t_n)x_{t_n} - x_{t_n}\| \to 0 \ \ (\text{as } n \to \infty),\end{aligned}$$

where K is a bounded subset containing $\{x_{t_n}\}$. So, by Lemma 2.3, $u \in F$. Therefore by (3.2) and the assumption that J_φ is weakly sequentially continuous, we obtain

$$\|x_{t_n} - u\|\varphi(\|x_{t_n} - u\|) \leq \langle f(x_{t_n}) - u, J_\varphi(x_{t_n} - u)\rangle \to 0 \quad (\text{as } n \to \infty).$$

Since φ is continuous and strictly increasing, we must have $x_{t_{n_k}} \to u \in F$.

We will now show that every weakly convergent subsequence of $\{x_{t_n}\}$ has the same limit. Suppose that $x_{t_{n_k}} \rightharpoonup u$ and $x_{t_{n_j}} \rightharpoonup v$. Then by the above proof, $u,\ v \in F$, $x_{t_{n_k}} \to u$ and $x_{t_{n_j}} \to v$. It follows from (3.2) that

$$\|u - v\|\varphi(\|u - v\|) \leq \langle f(u) - v, J_\varphi(u - v)\rangle \tag{3.3}$$

and

$$\|v - u\|\varphi(\|v - u\|) \leq \langle f(v) - u, J_\varphi(v - u)\rangle. \tag{3.4}$$

Adding (3.3) and (3.4) yields

$$\begin{aligned}2\|u - v\|\varphi(\|u - v\|) &\leq \|u - v\|\varphi(\|u - v\|) - \langle f(u) - f(v), J_\varphi(u - v)\rangle \\ &\leq (1-k)\|u - v\|\varphi(\|u - v\|).\end{aligned}$$

Since $k \in (0,1)$, this implies that $\|u - v\|\varphi(\|u - v\|) \leq 0$, and so $u = v$. Hence $\{x_{t_n}\}$ converges strongly to a point in F as $t_n \to \infty$.

The same argument shows that if $t_l \to \infty$, then the subsequence $\{x_{t_l}\}$ of $\{x_t\}$ converges strongly to the same limit. Thus, as $t \to \infty$, $\{x_t\}$ converges strongly to a point in F.

If we define $Q : \sum_C \to F$ by $Q(f) = \overline{q} = \lim_{t\to\infty} x_t,\ f \in \sum_C$, then $\overline{q}$ solves the variational inequality

$$\langle (I-f)(\overline{q}), J(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F.$$

In fact, since $(I-f)(x_t) = -\frac{1-\lambda_t}{\lambda_t}(I-T(t))x_t$, by Lemma 2.5, we have for $p \in F$,

$$\langle (I-f)(x_t), J(x_t-p)\rangle = -\frac{1-\lambda_t}{\lambda_t}\langle (I-T(t))x_t - (I-T(t))p, J(x_t-p)\rangle \le 0.$$

Since E is smooth, it follows that E^* is strictly convex for E reflexive (cf, [6, p. 43]). Noting that J is norm-to-weak* continuous by Lemma 2.4 and taking the limit as $t \to \infty$, we obtain $\langle (I-f)(\overline{q}), J(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F$. Since $J_\varphi(x) = (\varphi(\|x\|)/\|x\|)J(x)$ for $x \neq 0$ by (2.1), this implies that

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F. \tag{3.5}$$

Finally, we show the uniqueness of solution of the variational inequality (3.5) in F. In fact, suppose that $p, q \in F$ satisfy (3.5). Then we have

$$\langle (I-f)(p), J_\varphi(p-q)\rangle \le 0 \tag{3.6}$$

and

$$\langle (I-f)(q), J_\varphi(q-p)\rangle \le 0. \tag{3.7}$$

Combining (3.6) and (3.7), it follows that

$$(1-k)\|p-q\|\varphi(\|p-q\|) \le \langle (I-f)(p) - (I-f)(q), J_\varphi(p-q)\rangle \le 0.$$

So, we have $p = q$ and the uniqueness is proved. $\square$

Remark 3.2. (1) In Theorem 3.1, if $f(x) = u$, $x \in C$, is a constant, then

$$\langle Qu - u, J_\varphi(Qu-p)\rangle \le 0, \quad u \in C, \quad p \in F.$$

Hence by (2.2), Q reduces to the sunny nonexpansive retraction from C to F.

(2) Theorem 3.1 develops Theorem 3.2 in Song and Xu [19] to different Banach space.

(3) Theorem 3.1 generalizes the corresponding results of Suzuki [20] and Xu [25] to the viscosity method in more general Banach space.

(4) When $f(x) = u$ for all $x \in C$, Theorem 3.1 complements the corresponding results of Aleyner and Reich [2] and Benavides et al. [3] in different Banach space.

(5) Theorem 3.1 also improves the corresponding results of Chen and Song [5] and Shioji and Takahasi [18].

By using Theorem 3.1, we establish the strong convergence of the explicit viscosity iterative scheme.

Theorem 3.3. *Let E be a reflexive Banach space having a weakly sequentially continuous duality mapping J_φ with gauge function φ. Let C be a nonempty closed convex subset of E and $\{T(t) : t \ge 0\}$ a u.a.r. nonexpansive semigroup from C into itself with $F := \bigcap_{t\ge 0} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\} \subset (0,1)$ and $\{t_n\} \subset \mathbb{R}^+$ be sequences satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=0}^{\infty} \alpha_n = \infty$; (ii) $\lim_{n\to\infty} t_n = \infty$.

Let $f \in \sum_C$ and $x_0 \in C$ be chosen arbitrarily . Let $\{x_n\}$ be defined by

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(t_n)x_n, \end{cases} \tag{3.8}$$

Then $\{x_n\}$ converges strongly to $\overline{q} \in F$, which is the unique solution of the variational inequality

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F.$$

Proof. First we note that by Theorem 3.1, there exists the unique solution $\overline{q} \in F$ of the variational inequality

$$\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0, \quad f \in \Sigma_C, \quad p \in F, \tag{3.9}$$

where $\overline{q} = \lim_{t\to\infty} x_t$ and x_t is defined by (3.1). We will show that $x_n \to \overline{q}$.

We divide the proof into several steps.

Step 1. We show that $\|x_n - z\| \le \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z)-z\|\}$ for all $n \ge 0$ and all $z \in F$ and so $\{x_n\}$, $\{f(x_n)\}$ and $\{T(t_n)x_n\}$ are bounded. Indeed, let $z \in F$. Then we have

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n(f(x_n) - z) + (1-\alpha_n)(T(t_n)x_n - z)\| \\ &\le \alpha_n\|f(x_n) - z\| + (1-\alpha_n)\|x_n - z\| \\ &\le \alpha_n(\|f(x_n) - f(z)\| + \|f(z) - z\|) + (1-\alpha_n)\|x_n - z\| \\ &\le \alpha_n k\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\ &= (1-(1-k)\alpha_n)\|x_n - z\| + \alpha_n\|f(z) - z\| \\ &\le \max\left\{\|x_n - z\|, \frac{1}{1-k}\|f(z) - z\|\right\}. \end{aligned}$$

Using an induction, we obtain

$$\|x_n - z\| \le \max\left\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\right\}, \text{ for all } n \ge 0.$$

Hence $\{x_n\}$ is bounded, and so are $\{T(t_n)x_n\}$ and $\{f(x_n)\}$.

Step 2. We show that $\lim_{n\to\infty} \|T(r)x_n - x_n\| = 0$ uniformly in $r \in \mathbb{R}^+$. Indeed, it follows from condition (i) that

$$\|x_{n+1} - T(t_n)x_n\| = \alpha_n\|f(x_n) - T(t_n)x_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.10}$$

Since $\{T(t)\}$ is u.a.r. nonexpansive semigroup,

$$\lim_{n\to\infty} \|T(r)T(t_n)x_n - T(t_n)x_n\| \le \lim_{n\to\infty} \sup_{x\in D} \|T(r)T(t_n)x - T(t_n)x\| = 0, \tag{3.11}$$

uniformly $r \in \mathbb{R}^+$, where $D = \{x \in C : \|x - z\| \le \max\{\|x_0 - z\|, \frac{1}{1-k}\|f(z) - z\|\}$ for $z \in F$. Combining (3.10) and (3.11), we have that for all $r > 0$

$$\begin{aligned} \|T(r)x_{n+1} - x_{n+1}\| &\le \|T(r)x_{n+1} - T(r)T(t_n)x_n\| \\ &\quad + \|T(r)T(t_n)x_n - T(t_n)x_n\| + \|T(t_n)x_n - x_{n+1}\| \\ &\le 2\|x_{n+1} - T(t_n)x_n\| + \sup_{x\in D}\|T(r)T(t_n)x - T(t_n)x\| \to 0, n \to \infty. \end{aligned}$$

Step 3. We show that $\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_n)\rangle \le 0$. Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup p \in C$ and

$$\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_n)\rangle = \lim_{j\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_{n_j})\rangle$$

From Step 2. it follows that $\|T(r)x_{n_i} - x_{n_i}\| \to 0$ as $i\to\infty$. By Lemma 2.3, we have $p = T(r)p$ for each $r \in \mathbb{R}^+$ and so $p \in F$. Thus by the weakly sequentially continuity of the duality mapping J_φ and (3.9), we have

$$\begin{aligned}\limsup_{n\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_n)\rangle &= \lim_{j\to\infty}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_{n_j})\rangle \\ &= \langle (I-f)(\overline{q}), J_\varphi(\overline{q}-p)\rangle \le 0.\end{aligned}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n-\overline{q}\| = 0$. By using (3.8), we have

$$x_{n+1}-\overline{q} = \alpha_n(f(x_n)-f(\overline{q})) + (1-\alpha_n)(T(t_n)x_n-\overline{q}) + \alpha_n(f(\overline{q})-\overline{q}).$$

As a consequence, since Φ is an increasing convex function with $\Phi(0)=0$, by applying Lemma 2.1, we obtain

$$\begin{aligned}&\Phi(\|x_{n+1}-\overline{q}\|)\\ &\le \Phi(\|\alpha_n(f(x_n)-f(\overline{q})) + (1-\alpha_n)(T(t_n)x_n-\overline{q})\|)\\ &\quad + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle\\ &\le \Phi(k\alpha_n\|x_n-\overline{q}\| + (1-\alpha_n)\|x_n-\overline{q}\|)\\ &\quad + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle\\ &\le (1-(1-k)\alpha_n)\Phi(\|x_n-\overline{q}\|) + \alpha_n\langle f(\overline{q})-\overline{q}, J_\varphi(x_{n+1}-\overline{q})\rangle.\end{aligned}\tag{3.12}$$

Put $\lambda_n = (1-k)\alpha_n$ and $\delta_n = \frac{1}{1-k}\langle (I-f)(\overline{q}), J_\varphi(\overline{q}-x_{n+1})\rangle$. From (i) and Step 3, it follows that $\lambda_n \to 0$, $\sum_{n=0}^{\infty}\lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Since (3.12) reduces to

$$\Phi(\|x_{n+1}-\overline{q}\|) \le (1-\lambda_n)\Phi(\|x_n-\overline{q}\|) + \lambda_n\delta_n,$$

from Lemma 2.2, we conclude that $\lim_{n\to\infty}\Phi(\|x_n-\overline{q}\|)=0$, and thus $\lim_{n\to\infty}x_n = \overline{q}$. □

Remark 3.4. (1) Theorem 3.3 develops Theorem 4.2 in Song and Xu [19] in different Banach space.

(2) Theorem 3.3 complements the corresponding result in Aleyner and Censor [1], Aleyner and Reich [2], Benavides et al. [3] to the viscosity method in different Banach space. In particular, the conditions $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n+1}| < \infty$ in [1, 2] and $\lim_{n\to\infty}\frac{\alpha_n}{\alpha_{n+1}} = 1$ in [3] can be removed.

(3) Theorem 3.3 also improves the corresponding results of Suzuki [20] to the viscosity method in more general Banach space.

References

[1] A. Aleyner, Y. Censor, *Best approximation to common fixed points of a semigroup of nonexpansive operators*, J. Nonlinear Convex Anal., **6**(2005), 137-151.

[2] A. Aleyner, S. Reich, *An explicit construction of sunny nonexpansive retractions in Banach spaces*, Fixed Point Theory and Appl., **2005**(2005), 295-305.

[3] T.D. Benavides, G. Lopez Acedo, H.K. Xu, *Construction of sunny nonexpansive retractions in Banach spaces*, Bull. Austral. Math. Soc., **66**(2002), 9-16.

[4] F.E. Browder, *Convergence of approximations to fixed points of nonexpansive mappings in Banach spaces*, Arch. Ration. Mech. Anal. **24**(1967), 82-90.

[5] R. Chen, Y. Song, *Convergence to common fixed point of nonexpansive semigroup*, J. Comput. Appl. Math., **200**(2007) 566-575.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[7] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics,vol. 28, Cambridge Univ. Press, Cambridge, UK, 1990.

[8] K. Goebel, S. Reich, *Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.

[9] K.S. Ha, J.S. Jung, *Strong convergence theorems for accretive operators in Banach spaces*, J. Math. Anal. Appl., **147**(1990), 330-339.

[10] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc., **73**(1967), 957-961.

[11] J.S. Jung, *Viscosity approximation methods for a family of finite nonexpansive mappings in Banach spaces*, Nonlinear Anal., **64**(2006), 2536-2552.

[12] J.S. Jung, C. Morales, *The Mann process for perturbed m-accretive operators in Banach spaces*, Nonlinear Anal., **46**(2001), 231-243.

[13] P.L. Lions, *Approximation de points fixes de contractions*, C.R. Acad. Sci. Sér A-B, Paris, **284**(1977), 1357-1359.

[14] L.S. Liu, *Iterative processes with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., **194**(1995), 114-125.

[15] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl., **241**(2000), 46-55.

[16] S. Reich, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl., **75**(1980), 287-292.

[17] N. Shioji, W. Takahashi, *Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces*, Proc. Amer. Math. Soc., **125**(1997), 3641-3645.

[18] N. Shioji, W. Takahashi, *Strong convergence theorems for asymptotically nonexpansive mappings in Hilbert spaces*, Nonlinear Anal., **34**(1998), 87-99.

[19] Y. Song, S. Xu, *Strong convergence theorems for nonexpansive semigroup in Banach spaces*, J. Math. Anal. Appl., **338**(2008), 152-161.

[20] T. Suzuki, *On strong convergence to a common fixed poont of nonexpansive semigroup in Hilbert spaces*, Proc. Amer. Math. Soc., **131**(2003), 2133-2136.

[21] W. Takahashi, Y. Ueda, *On Reich's strong convergence for resolvents of accretive operators*, J. Math. Anal. Appl., **104**(1984), 546-553.

[22] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **59**(1992), 486-491.

[23] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.

[24] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl., **298**(2004), 279-291.

[25] H.K. Xu, *Strong convergence for contraction semigroups in Banach spaces*, Bull. Austral. Math. Soc., **72**(2005), 371-379.

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