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CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY MONOTONE MAPPINGS

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Abstract. In this paper, we introduce a general iterative scheme for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings and the set of solutions of variational inequalities for an inverse-strongly monotone mapping.
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1. Introduction and preliminaries

Throughout this paper, we assume that H is a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, C is a closed convex subset of H and $A : C \to H$ is a nonlinear mapping. We denote by P_C be the projection of H onto the closed convex subset C. The classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

We denoted by $VI(C, A)$ the set of solutions of the variational inequality. For a given $z \in H, u \in C$ satisfies the inequality

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that $P_C x$ is characterized by the property: $P_C x \in C$ and $\langle x - P_C x, P_C x - y \rangle \geq 0$ for all $y \in C$.

Remark 1.1. One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem, that is, an element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where $\lambda > 0$ is a constant and I is the identity mapping.

Recall the following definitions.

(1) A mapping $A : C \to H$ is said to be inverse-strongly monotone if there exists a positive real number μ such that

$$\langle Ax - Ay, x - y\rangle \geq \mu\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.2}$$

For such a case, A is also said to be μ-inverse-strongly monotone.

(2) A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Next, we denote by $F(T)$ the set of fixed points of T.

(3) A mapping $f : C \to C$ is said to be contractive if there exists $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

(4) A linear bounded operator $B : C \to C$ is said to be strongly positive if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in C.$$

(5) A set-valued mapping $T : H \to 2^H$ is said to be monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$.

(6) A monotone mapping $T : H \to 2^H$ is said to be maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$.

Let A be a monotone mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [26]).

Regarding to the class of nonexpansive mappings, we have the following results.

Let C be a nonempty bounded closed and convex subset of a real Hilbert space H and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point in C.

Remark 1.2. The above result is still valid if the framework of the space is uniformly convex Banach spaces; see [1]. In 1965, Kirk [15] proved that the existence of fixed points of a single nonexpansive mapping in the framework of reflexive Banach spaces which enjoy the normal structure. We also remark that the existence of common fixed point for a nonexpansive semigroup was given by Browder, see [1] for more details.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see [11,16,31-33] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in \Omega} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle, \tag{1.3}$$

where B is a linear bounded operator on H, Ω is the fixed point set of a nonexpansive mapping S and b is a given point in H.

In [32], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n b, \quad \forall n \geq 0,$$

converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions.

Recently, Marino and Xu [16] introduced a general iterative scheme by the viscosity approximation method:

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad \forall n \geq 0, \tag{1.4}$$

where S is a nonexpansive mapping on H, f is a contraction on H with the coefficient α, B is a bounded linear strongly positive operator on H with the coefficient $\bar{\gamma}$ and γ is a constant such that $0 < \gamma < \bar{\gamma}/\alpha$. They proved that the sequence $\{x_n\}$ generated by the iterative scheme (1.4) converges strongly to the unique solution of the variational inequality:

$$\langle (B - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2}\langle Bx, x \rangle - h(x),$$

where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in H$.)

Recently, variational inequalities and fixed point problems have been considered by many authors. See, e.g., [8,12,13,17-20,25,28] and the references therein. For finding a common element of the sets of fixed points of nonexpansive mappings and solutions of variational inequalities for μ-inverse-strongly monotone mapping, Iiduka and Takahashi [12] proposed the following iterative scheme:

$$x_1 = x \in C, \quad x_{n+1} = \alpha_n x + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1, \tag{1.5}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\mu)$. They proved that the sequence $\{x_n\}$ defined by (1.5) converges strongly to some $z \in F(S) \cap VI(C, A)$.

Very recently, Chen et al. [8] studied the following iterative process:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \quad \forall n \geq 1,$$

and also obtained a strong convergence theorem by so-called viscosity approximation method discussed by Moudafi [17] in the framework of Hilbert spaces.

Concerning a family of nonexpansive mappings has been considered by many authors. See, e.g., [4,6,7,9,19-24,27,29,32] and the references therein. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings. See, e.g., [3,5,29] and the references therein. The problem of finding an optimal point that minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance. See e.g., [4,10,34] and the references therein. A simple algorithmic solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation. See, e.g., [14,34].

In this paper, we consider the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\cdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\cdots\\
&U_{n,2} = \gamma_2 T_2 U_{n,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned} \tag{1.6}$$

where $\gamma_1, \gamma_2, \ldots$ are real numbers such that $0 \le \gamma_n \le 1$ and $T_1, T_2, \cdots$ be an infinite family of mappings of C into itself. Nonexpansivity of each T_i ensures the nonexpansivity of W_n.

Concerning W_n, we have the following lemmas which are important to prove our main results.

Lemma 1.1. (Shimoji and Takahashi [27]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for all $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 1.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{1.7}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \cdots$ and $\gamma_1, \gamma_2, \cdots$.

Remark 1.3. Throughout this paper, we shall always assume that $0 < \gamma_i \le b < 1$ for all $i \ge 1$.

Lemma 1.2 (Shimoji and Takahashi [27]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\gamma_1, \gamma_2, \cdots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 1.3 (Chang et al. [6]; Ceng and Yao [7]). *Let C be a nonempty closed convex subset of a Hilbert space H. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself* *such that* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ *and* $\gamma_1, \gamma_2, \cdots$ *be a real sequence such that* $0 < \gamma_n \leq b < 1$ *for all* $n \geq 1$*. If* K *is any bounded subset of* C*, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

In this paper, motivated by Chen et al. [8], Cho et al. [9], Iiduka and Takahashi [12], Marino and Xu [16], Maingé [18] and Takahashi and Toyoda [28], we introduce a general iterative process as follows:

$$x_1 \in C, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B) W_n P_C (I - \lambda_n A) x_n, \quad \forall n \geq 1, \tag{1.8}$$

where A is a μ-inverse-strongly monotone mapping from C into H, B is a linear bounded strongly positive self-adjoint operator with the coefficient $\bar{\gamma}$, $f : C \to C$ is a contraction with the coefficient α $(0 < \alpha < 1)$ and W_n is a mapping defined by (1.6), and prove that the sequence $\{x_n\}$ generated by the iterative scheme (1.8) converges strongly to a common element of the sets of common fixed points of an infinite nonexpansive mappings and solutions of variational inequalities for the μ-inverse-strongly monotone mapping, which solves another variational inequality:

$$\langle \gamma f(q) - Bq, p - q \rangle \leq 0, \quad p \in \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A),$$

and is also the optimality condition for the minimization problem:

$$\min_{x\in F} \frac{1}{2} \langle Bx, x \rangle - h(x),$$

where F is the intersection of the common fixed point set of the infinite family of nonexpansive mappings $T_1, T_2, \cdots$ and the set of solutions of variational inequalities for μ-inverse-strongly monotone mappings, h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for all $x \in C$.)

The results obtained in this paper improve and extend the recent ones announced by Chen et al. [8], Iiduka and Takahashi [12], Marino and Xu [16] and many others.

In order to prove our main results, we also need the following lemmas.

Lemma 1.4 *In a real Hilbert space* H*, the the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in H.$$

Lemma 1.5. (Marino and Xu [16]) *Assume that* B *is a strong positive linear bounded operator on a Hilbert space* H *with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \leq \|B\|^{-1}$*. Then* $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$*.*

Lemma 1.6 (Marino and Xu [16]). *Let* H *be a Hilbert space,* B *be a strongly positive linear bounded self-adjoint operator on* H *with the coefficient* $\bar{\gamma} > 0$*. Assume that* $0 < \gamma < \bar{\gamma}/\alpha$*. Let* $T : H \to H$ *be a nonexpansive mapping with a fixed point* x_t *of the contraction* $x \mapsto t\gamma f(x) + (I - tB)Tx$*. Then* $\{x_t\}$ *converges strongly as* $t \to 0$ *to a fixed point* $\bar{x}$ *of* T*, which solves the variational inequality:*

$$\langle (B - \gamma f)\bar{x}, \bar{x} - z \rangle \leq 0, \quad \forall z \in F(T).$$

Equivalently, $\bar{x} = P_F(\gamma f + I - B)\bar{x}$.

Lemma 1.7 (Xu [31]). *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.8. *Let* H *be a Hilbert space,* C *a closed convex subset of* H*,* $f : C \to C$ *a contraction with the coefficient* $\alpha \in (0,1)$ *and* B *a strongly positive linear bounded operator with the coefficient* $\bar{\gamma} > 0$*. Then, for any* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$,

$$\langle x-y, (B-\gamma f)x - (B-\gamma f)y\rangle \geq (\bar{\gamma}-\gamma\alpha)\|x-y\|^2, \quad \forall x, y \in C.$$

That is, $B - \gamma f$ *is strongly monotone with the coefficient* $\bar{\gamma} - \alpha\gamma$.

Proof. From the definition of strongly positive linear bounded operator, we have

$$\langle x-y, B(x-y)\rangle \geq \bar{\gamma}\|x-y\|^2.$$

On the other hand, it is easy to see that

$$\langle x-y, \gamma fx - \gamma fy\rangle \leq \gamma\alpha\|x-y\|^2.$$

Therefore, for all $x, y \in C$, we have
$\langle x-y, (B-\gamma f)x - (B-\gamma f)y\rangle = \langle x-y, B(x-y)\rangle - \langle x-y, \gamma fx - \gamma fy\rangle \geq (\bar{\gamma}-\gamma\alpha)\|x-y\|^2$. □

2. Main results

Now, we are ready to give our main results in this paper.

Theorem 2.1. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H *such that* $C \pm C \subset C$ *and* $A : C \to H$ *be a* μ*-inverse-strongly monotone mapping. Let* $f : C \to C$ *be a contraction with the coefficient* α $(0 < \alpha < 1)$ *and* $T_1, T_2, \cdots$ *be a sequence of nonexpansive self-mappings on* C*. Let* B *be a strongly positive linear bounded self-adjoint operator of* C *into itself with the coefficient* $\bar{\gamma} > 0$*. Assume that* $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$*. Let the sequence* $\{x_n\}$ *be generated by* (1.8)*, where the mapping* W_n *is defined by* (1.6)*,* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\lambda_n\}$ *is a sequence in* $(0, 2\mu)$*. If* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C,A) \neq \emptyset$ *and* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$ *and* $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$;

(C4) $\{\lambda_n\}\subset[u,v]$ *for some* u,v *with* $0<u<v<2\mu$,

then the sequence $\{x_n\}$ *converges strongly to some* $x^*\in F$*, which uniquely solves the following variation inequality:*

$$\langle Bx^*-\gamma f(x^*),x^*-p\rangle\le 0,\quad \forall p\in F. \tag{2.1}$$

Equivalently, we have $x^*=P_F(\gamma f+I-B)x^*$.

Proof. First, we show that the mapping $I-\lambda_nA$ is nonexpansive for each $n\ge 1$. Indeed, from the condition (C4), for $x,y\in C$, we have

$$\begin{aligned}&\|(I-\lambda_nA)x-(I-\lambda_nA)y\|^2\\&=\|(x-y)-\lambda_n(Ax-Ay)\|^2\\&\le\|x-y\|^2-2\lambda_n\langle Ax-Ay,x-y\rangle+\lambda_n^2\|Ax-Ay\|^2\\&\le\|x-y\|^2-2\lambda_n\mu\|Ax-Ay\|^2+\lambda_n^2\|Ax-Ay\|^2\\&=\|x-y\|^2+\lambda_n(\lambda_n-2\mu)\|Ax-Ay\|^2\\&\le\|x-y\|^2.\end{aligned}$$

This shows that $I-\lambda_nA$ is a nonexpansive mapping for each $n\ge 1$. Noticing that condition (C1), we may assume, with no loss of generality, that $\alpha_n\le\|B\|^{-1}$ for all $n\ge 1$. From Lemma 1.5, we know that, if $0<\alpha_n\le\|B\|^{-1}$ for all $n\ge 1$, then $\|I-\alpha_nB\|\le 1-\alpha_n\bar{\gamma}$.

Now, we are in a position to show that the sequence $\{x_n\}$ is bounded. Letting $p\in F$, we have

$$\begin{aligned}&\|x_{n+1}-p\|\\&=\|\alpha_n(\gamma f(x_n)-Bp)+(I-\alpha_nB)(W_nP_C(I-\lambda_nA)x_n-p)\|\\&\le\alpha_n\|\gamma f(x_n)-Bp\|+(1-\alpha_n\bar{\gamma})\|W_nP_C(I-\lambda_nA)x_n-p\|\\&\le\alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Bp\|+(1-\alpha_n\bar{\gamma})\|x_n-p\|\\&\le[1-\alpha_n(\bar{\gamma}-\gamma\alpha)]\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\|.\end{aligned}$$

By simple inductions, we obtain

$$\|x_n-p\|\le\max\{\|x_1-p\|,\frac{\|Bp-\gamma f(p)\|}{\bar{\gamma}-\gamma\alpha}\}\quad\forall n\ge 1,$$

which yields that the sequence $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. Putting $\rho_n=P_C(I-\lambda_nA)x_n$, we have

$$\begin{aligned}&\|\rho_{n+1}-\rho_n\|\\&=\|P_C(I-\lambda_{n+1}A)x_{n+1}-P_C(I-\lambda_nA)x_n\|\\&\le\|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_nA)x_n\|\\&=\|(I-\lambda_{n+1}A)x_{n+1}-(I-\lambda_{n+1}A)x_n+(I-\lambda_{n+1}A)x_n-(I-\lambda_nA)x_n\|\\&=\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.\end{aligned}\tag{2.2}$$

From (1.8), we see that

$$\begin{aligned}
&\|x_{n+2}-x_{n+1}\| \\
&=\|(I-\alpha_{n+1}B)(W_{n+1}\rho_{n+1}-W_n\rho_n)-(\alpha_{n+1}-\alpha_n)BW_n\rho_n \\
&\quad+\gamma[\alpha_{n+1}(f(x_{n+1})-f(x_n))+f(x_n)(\alpha_{n+1}-\alpha_n)]\| \\
&\le(1-\alpha_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|)+|\alpha_{n+1}-\alpha_n|\|BW_n\rho_n\| \\
&\quad+\gamma[\alpha_{n+1}\alpha\|x_{n+1}-x_n\|+\|f(x_n)\||\alpha_{n+1}-\alpha_n|].
\end{aligned}\tag{2.3}$$

Since T_i and $U_{n,i}$ are nonexpansive, it follows from (1.6) that

$$\begin{aligned}
\|W_{n+1}\rho_n-W_n\rho_n\| &= \|\gamma_1T_1U_{n+1,2}\rho_n-\gamma_1T_1U_{n,2}\rho_n\| \\
&\le\gamma_1\|U_{n+1,2}\rho_n-U_{n,2}\rho_n\| \\
&=\gamma_1\|\gamma_2T_2U_{n+1,3}\rho_n-\gamma_2T_2U_{n,3}\rho_n\| \\
&\le\gamma_1\gamma_2\|U_{n+1,3}\rho_n-U_{n,3}\rho_n\| \\
&\le\cdots \\
&\le\gamma_1\gamma_2\cdots\gamma_n\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\| \\
&\le M_1\prod_{i=1}^n\gamma_i,
\end{aligned}\tag{2.4}$$

where $M_1\ge 0$ is an appropriate constant such that $\|U_{n+1,n+1}\rho_n-U_{n,n+1}\rho_n\|\le M_1$ for all $n\ge 1$. Substituting (2.2) and (2.4) into (2.3), we arrive at

$$\begin{aligned}
\|x_{n+2}-x_{n+1}\| &\le[1-\alpha_{n+1}(\bar{\gamma}-\alpha\gamma)]\|x_{n+1}-x_n\| \\
&\quad+M_2(\prod_{i=1}^n\gamma_i+2|\alpha_{n+1}-\alpha_n|+|\lambda_{n+1}-\lambda_n|),
\end{aligned}\tag{2.5}$$

where M_2 is an appropriate constant such that

$$M_2=\max\{M_1,\sup_{n\ge1}\{\|Ax_n\|\},\gamma\sup_{n\ge1}\{\|f(x_n)\|\},\sup_{n\ge1}\{\|BW_n\rho_n\|\}\}.$$

Observing the conditions (C1)-(C3) and applying Lemma 1.7 to (2.5), we obtain that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{2.6}$$

Next, we prove that $\lim_{n\to\infty}\|W\rho_n-\rho_n\|=0$. For all $p\in F$, we have

$$\begin{aligned}
&\|\rho_n-p\|^2 \\
&=\|P_C(I-\lambda_nA)x_n-P_C(I-\lambda_nA)p\|^2 \\
&\le\|(x_n-p)-\lambda_n(Ax_n-Ap)\|^2 \\
&=\|x_n-p\|^2-2\lambda_n\langle x_n-p,Ax_n-Ap\rangle+\lambda_n^2\|Ax_n-Ap\|^2 \\
&\le\|x_n-p\|^2-2\lambda_n\mu\|Ax_n-Ap\|^2+\lambda_n^2\|Ax_n-Ap\|^2 \\
&=\|x_n-p\|^2+\lambda_n(\lambda_n-2\mu)\|Ax_n-Ap\|^2.
\end{aligned}\tag{2.7}$$

On the other hand, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&=\|\alpha_n\gamma f(x_n)+(I-\alpha_nB)W_n\rho_n-p\|^2\\
&\le(\alpha_n\|\gamma f(x_n)-Bp\|+(1-\alpha_n\bar{\gamma})\|W_n\rho_n-p\|)^2\\
&\le(\alpha_n\|\gamma f(x_n)-Bp\|+(1-\alpha_n\bar{\gamma})\|\rho_n-p\|)^2\\
&\le\alpha_n\|\gamma f(x_n)-Bp\|^2+\|\rho_n-p\|^2+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.
\end{aligned}\tag{2.8}$$

Substituting (2.7) into (2.8), we obtain

$$\begin{aligned}
\|x_{n+1}-p\|^2\le\ &\alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2+\lambda_n(\lambda_n-2\mu)\|Ax_n-Ap\|^2\\
&+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.
\end{aligned}$$

It follows from the condition (C4) that

$$\begin{aligned}
&u(2\mu-v)\|Ax_n-Ap\|^2\\
&\le\alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|\\
&\le\alpha_n\|\gamma f(x_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.
\end{aligned}$$

From the condition (C1) and (2.6), it follows that

$$\lim_{n\to\infty}\|Ax_n-Ap\|=0.\tag{2.9}$$

On the other hand, from the firm nonexpansivity of P_C, we have

$$\begin{aligned}
\|\rho_n-p\|^2=&\|P_C(I-\lambda_nA)x_n-P_C(I-\lambda_nA)p\|^2\\
\le&\langle(I-\lambda_nA)x_n-(I-\lambda_nA)p,\rho_n-p\rangle\\
=&\frac{1}{2}\{\|(I-\lambda_nA)x_n-(I-\lambda_nA)p\|^2+\|\rho_n-p\|^2\\
&-\|(I-\lambda_nA)x_n-(I-\lambda_nA)p-(\rho_n-p)\|^2\}\\
\le&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|(x_n-\rho_n)-\lambda_n(Ax_n-Ap)\|^2\}\\
=&\frac{1}{2}\{\|x_n-p\|^2+\|\rho_n-p\|^2-\|x_n-\rho_n\|^2-\lambda_n^2\|Ax_n-Ap\|^2\\
&+2\lambda_n\langle x_n-\rho_n,Ax_n-Ap\rangle\},
\end{aligned}$$

which yields that

$$\|\rho_n-p\|^2\le\|x_n-p\|^2+2\lambda_n\|x_n-\rho_n\|\|Ax_n-Ap\|-\|x_n-\rho_n\|^2.\tag{2.10}$$

Substitute (2.10) into (2.8) yields that

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le\alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2+2\lambda_n\|x_n-\rho_n\|\|Ax_n-Ap\|\\
&\quad+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|-\|x_n-\rho_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_n-\rho_n\|^2\\&\le \alpha_n\|\gamma f(x_n)-Bp\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\&\quad+2\lambda_n\|x_n-\rho_n\|\|Ax_n-Ap\|+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|\\&\le \alpha_n\|\gamma f(x_n)-Bp\|^2+(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|\\&\quad+2\lambda_n\|x_n-\rho_n\|\|Ax_n-Ap\|+2\alpha_n\|\gamma f(x_n)-Bp\|\|\rho_n-p\|.\end{aligned}$$

From the condition (C1), (2.6) and (2.9), we have

$$\lim_{n\to\infty}\|x_n-\rho_n\|=0. \tag{2.11}$$

Notice that

$$\begin{aligned}\|\rho_n-W_n\rho_n\|&\le \|x_{n+1}-W_n\rho_n\|+\|x_n-x_{n+1}\|+\|x_n-\rho_n\|\\&\le \alpha_n\|\gamma f(x_n)-BW_n\rho_n\|+\|x_n-x_{n+1}\|+\|x_n-\rho_n\|.\end{aligned}$$

It follows from the condition (C1), (2.5) and (2.10) that

$$\lim_{n\to\infty}\|\rho_n-W_n\rho_n\|=0. \tag{2.12}$$

Since the sequence $\{x_n\}$ is bounded, we see that $\{\rho_n\}$ is also a bounded sequence in C. Without loss of generality, we can assume that there exists a bounded set $K\subset C$ such that $\rho_n\in K$ for all $n\ge 1$. On the other hand, we have

$$\begin{aligned}\|W\rho_n-\rho_n\|&\le \|W\rho_n-W_n\rho_n\|+\|W_n\rho_n-\rho_n\|\\&\le \sup_{\rho\in K}\|W\rho-W_n\rho\|+\|W_n\rho_n-\rho_n\|.\end{aligned}$$

From Lemma 1.3 and (2.12), we obtain

$$\lim_{n\to\infty}\|W\rho_n-\rho_n\|=0. \tag{2.13}$$

Finally, we show that $x_n\to x^*$ as $n\to\infty$. First, we prove that the uniqueness of the solution of the variational inequality (2.1), which is indeed a consequence of the strong monotonicity of $B-\gamma f$. Suppose that $x^*\in F$ and $x^{**}\in F$ both are solutions to (2.1). Then we have

$$\langle (B-\gamma f)x^*, x^*-x^{**}\rangle\le 0$$

and

$$\langle (B-\gamma f)x^{**}, x^{**}-x^*\rangle\le 0.$$

Adding up the two inequalities, we see that

$$\langle (B-\gamma f)x^*-(B-\gamma f)x^{**}, x^*-x^{**}\rangle\le 0.$$

The strong monotonicity of $B-\gamma f$ (see Lemma 1.8) implies that $x^*=x^{**}$ and the uniqueness is proved. Let x^* be the unique solution of (2.1). That is, $x^*=P_F(\gamma f+(I-B))x^*$.

Next, we show that

$$\limsup_{n\to\infty}\langle Bx^*-\gamma f(x^*), x^*-x_n\rangle\le 0. \tag{2.14}$$

To show it, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_n\rangle = \lim_{i\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_{n_i}\rangle.$$

Since $\{x_{n_i}\}$ is bounded, it follows that there is a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ converges weakly to p. We may assume, without loss of generality, that $x_{n_i} \rightharpoonup p$. Therefore, we have $p \in F$. Indeed, let us first show that $p \in VI(C, A)$. Put

$$Tw = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $\rho_n \in C$, we have

$$\langle v - \rho_n, w - Av\rangle \geq 0.$$

On the other hand, from $\rho_n = P_C(I - \lambda_n A)x_n$, we have

$$\langle v - \rho_n, \rho_n - (I - \lambda_n A)x_n\rangle \geq 0$$

and hence

$$\langle v - \rho_n, \frac{\rho_n - x_n}{\lambda_n} + Ax_n\rangle \geq 0.$$

It follows that

$$\begin{aligned}
&\langle v - \rho_{n_i}, w\rangle \\
&\geq \langle v - \rho_{n_i}, Av\rangle \geq \langle v - \rho_{n_i}, Av\rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i}\rangle \\
&\geq \langle v - \rho_{n_i}, Av - \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}} - Ax_{n_i}\rangle \\
&= \langle v - \rho_{n_i}, Av - A\rho_{n_i}\rangle + \langle v - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i}\rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\
&\geq \langle v - \rho_{n_i}, A\rho_{n_i} - Ax_{n_i}\rangle - \langle v - \rho_{n_i}, \frac{\rho_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle,
\end{aligned}$$

which implies that $\langle v - p, w\rangle \geq 0$. We have $p \in A^{-1}0$ and hence $p \in VI(C, A)$.

Next, let us show $p \in \bigcap_{i=1}^{\infty} F(T_i)$. Since Hilbert spaces are Opial's spaces, it follows from (2.11) that

$$\begin{aligned}
\liminf_{i\to\infty}\|\rho_{n_i} - p\| &< \liminf_{i\to\infty}\|\rho_{n_i} - Wp\| \\
&= \liminf_{i\to\infty}\|\rho_{n_i} - W\rho_{n_i} + W\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty}\|W\rho_{n_i} - Wp\| \\
&\leq \liminf_{i\to\infty}\|\rho_{n_i} - p\|,
\end{aligned}$$

which is a contradiction. Thus we have $p \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. On the other hand, we have

$$\limsup_{n\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_n\rangle = \lim_{i\to\infty}\langle Bx^* - \gamma f(x^*), x^* - x_{n_i}\rangle$$
$$= \langle Bx^* - \gamma f(x^*), x^* - p\rangle \le 0.$$

That is, (2.14) holds. It follows from Lemma 1.4 that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Bx^*) + (I - \alpha_n B)(W_n\rho_n - x^*)\|^2 \\
&\le (1 - \alpha_n\bar{\gamma})^2\|W_n\rho_n - x^*\|^2 + 2\alpha_n\langle\gamma f(x_n) - Bx^*, x_{n+1} - x^*\rangle \\
&\le (1 - \alpha_n\bar{\gamma})^2\|x_n - x^*\|^2 + \alpha\gamma\alpha_n(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2) \\
&\quad + 2\alpha_n\langle\gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\le \frac{(1 - \alpha_n\bar{\gamma})^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle\gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle \\
&= \frac{(1 - 2\alpha_n\bar{\gamma} + \alpha_n\alpha\gamma)}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 + \frac{\alpha_n^2\bar{\gamma}^2}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle\gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle \\
&\le [1 - \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}]\|x_n - x^*\|^2 \\
&\quad + \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}[\frac{1}{\bar{\gamma} - \alpha\gamma}\langle\gamma f(x^*) - Bx^*, x_{n+1} - x^*\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_3],
\end{aligned} \tag{2.15}$$

where M_3 is an appropriate constant such that $M_3 \ge \sup_{n\ge1}\|x_n - x^*\|^2$. Put

$$b_n = \frac{2\alpha_n(\bar{\gamma} - \alpha\gamma)}{1 - \alpha_n\alpha\gamma},$$

$$c_n = \frac{1}{\bar{\gamma} - \alpha\gamma}\langle\gamma f(x^*) - Bx^*, x_{n+1} - q\rangle + \frac{\alpha_n\bar{\gamma}^2}{2(\bar{\gamma} - \alpha\gamma)}M_3.$$

Then, from (2.15), we have

$$\|x_{n+1} - x^*\|^2 \le (1 - b_n)\|x_n - x^*\| + b_n c_n. \tag{2.16}$$

It follows from the conditions (C1), (C2) and (2.14) that

$$\lim_{n\to\infty} b_n = 0,\ \sum_{n=1}^{\infty} b_n = \infty,\ \limsup_{n\to\infty} c_n \le 0.$$

Therefore, applying Lemma 1.7, we have $x_n \to x^*$ as $n \to \infty$. This completes the proof.

Taking $\gamma = 1$ and $B = I$ (the identity mapping) in Theorem 2.1, we have the following results.

Theorem 2.2. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be a μ-inverse-strongly monotone mapping. Let $f : C \to C$ be a contraction with the coefficient α $(0 < \alpha < 1)$ and $T_1, T_2, \cdots$ be a sequence of nonexpansive self-mappings on C. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n P_C(I - \lambda_n A)x_n, \quad \forall n \geq 1,$$

where the mapping W_n is defined by (1.6), $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\mu]$. If $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$ and $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen such that

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$;

(C4) $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < 2\mu$,

then the sequence $\{x_n\}$ converges strongly to $x^* \in F$, where $x^* = P_F f(x^*)$, which solves the following variation inequality:

$$\langle f(x^*) - x^*, p - x^* \rangle \leq 0, \quad \forall p \in F.$$

Remark 2.3. Theorem 2.2 mainly improves the corresponding results in Chen et al. [8] which just involved a single nonexpansive mapping.

Further, if $f(x) = x_1$ for all $x \in C$ in Theorem 2.2, we have the following theorem.

Theorem 2.4. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H and $A : C \to H$ be a μ-inverse-strongly monotone mapping. Let $T_1, T_2, \cdots$ be a sequence of nonexpansive self-mappings on C. Let the sequence $\{x_n\}$ be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) W_n P_C(I - \lambda_n A)x_n, \quad \forall n \geq 1,$$

where the mapping W_n is defined by (1.6), $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\mu]$. If $F = \bigcap_{i=1}^{\infty} F(T_i) \cap VI(C, A) \neq \emptyset$ and $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen such that

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$;

(C4) $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < 2\mu$, *then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in F$.

Remark 2.5. Theorem 2.4 includes Theorem 3.1 of Iiduka and Takahashi [12] as a special case.

If we take $A = 0$ in Theorem 2.4, then we have the following results.

Theorem 2.6. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H. *Let* $T_1, T_2, \cdots$ *be a sequence of nonexpansive self-mappings on* C. *Let the sequence* $\{x_n\}$ *be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) W_n x_n, \quad \forall n \geq 1,$$

where the mapping W_n *is defined by* (1.6), $\{\alpha_n\}$ *is a sequence in* $(0, 1)$. *If* $F = \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$ *and* $\{\alpha_n\}$ *is chosen such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$,

then the sequence $\{x_n\}$ *converges strongly to* $x^* \in F$.

Remark 2.7. Theorem 2.6 mainly improves the results of Wittmann [30] from a single mapping to a family of mappings.

3. Applications

As some applications of our main results, we consider another class of important nonlinear operator: strict pseudo-contractions.

Recall that a mapping $S : C \to C$ is said to be a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Note that the class of k-strict pseudo-contractions strictly includes the class of nonexpansive mappings. Put $A = I - S$, where $S : C \to C$ is a k-strict pseudo-contraction. Then A is $\frac{1-k}{2}$-inverse-strongly monotone (see [2, 6, 12]).

Theorem 3.1. *Let* H *be a real Hilbert space,* C *be a nonempty closed convex subset of* H *and* $S : C \to C$ *be a* k*-strict pseudo-contraction. Let* $f : C \to C$ *be a contraction with the coefficient* α $(0 < \alpha < 1)$ *and* $T_1, T_2, \cdots$ *be a sequence of nonexpansive self-mappings on* C. *Let the sequence* $\{x_n\}$ *be generated by*

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) W_n((1 - \lambda_n)x_n + \lambda_n S x_n), \quad \forall n \geq 1,$$

where W_n *is defined by* (1.6), $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\lambda_n\}$ *is a sequence in* $[0, 2(1 - k)]$. *If* $F = \bigcap_{i=1}^{\infty} F(T_i) \cap F(S) \neq \emptyset$ *and* $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are chosen such that*

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| \leq \infty$;

(C4) $\{\lambda_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < 2(1 - k)$,

then the sequence $\{x_n\}$ *converges strongly to* $x^* \in F$.

Proof. Put $\gamma = 1$, $B = I$ and $A = I - S$. Then A is $\frac{1-k}{2}$-inverse-strongly monotone. We have

$$F(S) = VI(C, A),\ P_C(I - \lambda_n A)x_n = (1 - \lambda_n)x_n + \lambda_n S x_n.$$

It is easy to conclude the desired conclusion from Theorem 2.1.

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References

[1] F. E. Browder, *Nonexpansive nonlinear operators in Banach spaces*, Proc. Nat. Acad. Sci. USA, **54**(1965), 1041-1044.

[2] F.E. Browder, W.V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl., **20**(1967), 197-228.

[3] H.H. Bauschke, J.M. Borwein, *On projection algorithms for solving convex feasibility problems*, SIAM Rev., **38**(1996), 367-426.

[4] H.H. Bauschke, *The approximation of fixed points of compositions of nonexpansive mappings in Hilbert space*, J. Math. Anal. Appl., **202** 1996), 150-159.

[5] P.L. Combettes, *The foundations of set theoretic estimation*, Proc. IEEE, **81**(1993), 182-208.

[6] S.S. Chang, H.W.J. Lee, C.K. Chan, *A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization*, Nonlinear Anal., **70**(2009), 3307-3319.

[7] L.C. Ceng, J.C. Yao, *Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, Appl. Math. Comput., **198**(2008), 729-741.

[8] J.M. Chen, L.J. Zhang, T.G. Fan, *Viscosity approximation methods for nonexpansive mappings and monotone mappings*, J. Math. Anal. Appl., **334**(2007), 1450-1461.

[9] Y.J. Cho, S.M. Kang, X. Qin, *Approximation of common fixed points of an infinite family of nonexpansive mappings in Banach spaces*, Comput. Math. Appl., **56**(2008), 2058-2064.

[10] F. Deutsch, H. Hundal, *The rate of convergence of Dykstra's cyclic projections algorithm: the polyhedral case*, Numer. Funct. Anal. Optim., **15**(1994), 537-565.

[11] F. Deutsch, I. Yamada, *Minimizing certain convex functions over the intersection of the fixed point set of nonexpansive mappings*, Numer. Funct. Anal. Optim., **19**(1998), 33-56.

[12] H. Iiduka, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Anal., **61**(2005), 341-350.

[13] H. Iiduka, W. Takahashi, M. Toyoda, *Approximation of solutions ofvariational inequalities for monotone mappings*, Pan Amer. Math. J., **14** (2004), 49-61.

[14] A.N. Iusem, A.R. De Pierro, *On the convergence of Han's method for convex programming with quadratic objective*, Math. Program, Ser. B, **52**(1991), 265-284.

[15] W.A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly, **72**(1965), 1004-1006.
[16] G. Marino, H.K. Xu, *A general iterative method for nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **318**(2006), 43-52.
[17] A. Moudafi, *Viscosity approximation methods for fixed points problems*, J. Math. Anal Appl., **241**(2000), 46-55.
[18] P.E. Maingé, *A hybrid extragradient-viscosity method for monotone operators and fixed point problems*, SIAM J. Control Optim., **47**(2008), 1499-1515.
[19] X. Qin, M. Shang, H. Zhou, *Strong convergence of a general iterative method for variational inequality problems and fixed point problems in Hilbert spaces*, Appl. Math. Comput., **200**(2008), 242-253.
[20] X. Qin, M. Shang, Y. Su, *Strong convergence of a general iterative algorithm for equilibrium problems and variational inequality problems*, Math. Comput. Modelling, **48** (2008), 1033-1046.
[21] X. Qin, Y.J. Cho, S. . Kang, H. Zhou, *Convergence theorems of common fixed points for a family of Lipschitz quasi-pseudocontractions*, Nonlinear Anal., **71**(2009), 685-690.
[22] X. Qin, Y.J. Cho, S.M. Kang, H. Zhou, *Convergence of a modified Halpern-type iteration algorithm for quasi-ϕ-nonexpansive mappings*, Appl. Math. Lett., **22**(2009), 1051-1055.
[23] X. Qin, Y.J. Cho, J.I. Kang, S.M. Kang, *Strong convergence theorems for an infinite family of nonexpansive mappings in Banach spaces*, J. Comput. Appl. Math., **230**(2009), 121-127.
[24] X. Qin, Y.J. Cho, S.M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math., **225**(2009), 20-30.
[25] X. Qin, S.Y. Cho, S.M. Kang, *Convergence of an iterative algorithm for systems of variational inequalities and nonexpansive mappings with applications*, J. Comput. Appl. Math., **233**(2009), 231-240.
[26] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149**(1970), 75-88.
[27] K. Shimoji, W. Takahashi, *Strong convergence to common fixed points of infinite nonexpansive mappings and applications*, Taiwanese J. Math., **5**(2001), 387-404.
[28] W. Takahashi, M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118**(2003), 417-428.
[29] W. Takahashi, K. Shimoji, *Convergence theorems for nonexpansive mappings and feasibility problems*, Math. Comput. Modelling, **32**(2000), 1463-1471.
[30] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., **58**(1992), 486-491.
[31] H.K. Xu, *Iterative algorithms for nonlinear operators*, J. London Math. Soc., **66**(2002), 240-256.
[32] H.K. Xu, *An iterative approach to quadratic optimization*, J. Optim. Theory Appl., **116**(2003), 659-678.
[33] I. Yamada, *The hybrid steepest descent method for the variational inequality problem of the intersection of fixed point sets of nonexpansive mappings*, in: D. Butnariu, Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization, Elsevier, 2001, 473-504.
[34] D.C. Youla, *Mathematical theory of image restoration by the method of convex projections*, in: H. Stark (Ed.), Image Recovery: Theory and Applications, Academic Press, Florida, 1987, 29-77.

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