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THE STABILITY OF A QUARTIC TYPE FUNCTIONAL EQUATION WITH THE FIXED POINT ALTERNATIVE

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Abstract. Cădariu and Radu [Fixed points and the stability of Jensen's functional equation, *J. Inequal. Pure Appl. Math.* **4** (2003), Art. ID 4.] applied the fixed point alternative to the investigation of Cauchy and Jensen functional equations. In this paper, we adopt the the fixed point alternative method of Cădariu and Radu to prove the generalized Hyers-Ulam stability for the quartic functional equation

$$f(kx+y)+f(kx-y)=k^2[f(x+y)+f(x-y)]+2k^2(k^2-1)f(x)-2(k^2-1)f(y)$$

for each $k \in \mathbb{N} \setminus \{1\}$.
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1. Introduction

In 1940, Ulam [24] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive functions was solved by Hyers [13] under the assumption that G_1 and G_2 are Banach spaces. Indeed, he proved that each solution of the inequality $\|f(x+y) - f(x) - f(y)\| \le \epsilon$, for all x and y, can be approximated by an exact solution, say, an additive function. Th. M. Rassias [22] attempted to weaken the condition for the bound of the norm of the Cauchy difference as follows

$$\|f(x+y)-f(x)-f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

and generalized the result of Hyers. In 1950, Aoki [1] generalized Hyers' theorem for approximately additive mappings. Since then, the stability of several functional equations has been extensively investigated.

The terminology Hyers-Ulam stability originates from these historical backgrounds. The terminology can also be applied to the case of other functional equations. For more detailed definitions of such terminologies, we can refer to [2, 4], [9]-[17] and [23].

Lee, Im and Hwang [18] introduced the following functional equation

$$f(2x+y)+f(2x-y) = 4[f(x+y)+f(x-y)]+24f(x)-6f(y), \tag{1.1}$$

and they established the general solution of the functional equation (1.1). It is easy to see that the function $f(x) = cx^4$ is a solution of the functional equation (1.1). Thus, it is natural that (1.1) is called a quartic functional equation and every solution of the quartic functional equation is said to be a quartic mapping. For more detailed definitions of such terminologies, we can refer to [7] and [21].

Now we introduce the following quartic functional equation

$$f(kx+y)+f(kx-y) = k^2[f(x+y)+f(x-y)]+2k^2(k^2-1)f(x)-2(k^2-1)f(y) \tag{1.2}$$

for each $k \in \mathbb{N} \setminus \{1\}$. Recently, Cădariu and Radu [5] applied the fixed point method to the investigation of the Cauchy additive functional equation ([3, 4, 19]).

In this paper, we will adopt the fixed point alternative of Cădariu and Radu to prove the generalized Hyers-Ulam stability of the functional equation (1.2).

2. Stability of Eq. (1.2)

For completeness, we will first present solution of the functional equation (1.2).

Lemma 2.1. *Let X and Y be real vector spaces. A function $f : X \to Y$ satisfies the functional equation* (1.2) *if and only if f is quartic.*

Proof. ($\Rightarrow$) Letting $k = 2$ in (1.2), we get (1.1) which implies that f is quartic.

($\Leftarrow$) Suppose that f is quartic. We prove (1.2) for $k = j$ by induction on $j \in \mathbb{N}$. For the case $j = 1$, (1.2) holds obviously.
Letting $j = 2$ in (1.2), we have (1.1). For $j = 3$, we have

$$\begin{aligned} f(3x+y)+f(3x-y) &= [f(2x+(x+y))+f(2x-(x+y))] \\ &\quad + [f(2x+(x-y))+f(2x-(x-y))] \\ &\quad - [f(x+y)+f(x-y)] \end{aligned} \tag{2.1}$$

for all $x, y \in X$. On the other hand, we have

$$f(2x+(x+y))+f(2x-(x+y)) = 4[f(x+(x+y))+f(x-(x+y))]+24f(x)-6f(x+y)$$

so, by evenness of f we obtain

$$f(2x+(x+y))+f(2x-(x+y)) = 4[f(2x+y)+f(y)]+24f(x)-6f(x+y), \tag{2.2}$$

for all $x, y \in X$. So we have

$$f(2x+(x-y))+f(2x-(x-y)) = 4[f(x+(x-y))+f(x-(x-y))]+24f(x)-6f(x-y)$$

again, by evenness of f, we obtain

$$f(2x+(x-y))+f(2x-(x-y)) = 4[f(2x-y)+f(y)]+24f(x)-6f(x-y), \tag{2.3}$$

for all $x, y \in X$. By (2.1), (2.2) and (2.3), we get

$$f(3x+y)+f(3x-y) = 4[f(2x+y)+f(2x-y)]+48f(x)+8f(y)-7f(x-y)-7f(x+y) \tag{2.4}$$

for all $x, y \in X$. By (1.1) and (2.4) we have

$$f(3x+y)+f(3x-y) = 9[f(x+y)+f(x-y)]+144f(x)-16f(y)$$

for all $x, y \in X$. Hence (1.2) holds for $j = 3$.
Suppose (1.2) holds for $j = n-1$ and $j = n$, in which ($2 \le n \le k$). It is easy to see that

$$\begin{aligned} f((k+1)x+y)+f((k+1)x-y) &= f(kx+(x+y))+f(kx-(x+y)) \\ &\quad - [f((k-1)x+y)+f((k-1)x-y)] \\ &\quad + f(kx-(x-y))+f(kx+(x-y)) \end{aligned} \tag{2.5}$$

for all $x, y \in X$. On the other hand, we have

$$\begin{aligned} &[f(kx+(x+y))+f(kx-(x+y))]+[f(kx+(x-y))+f(kx-(x-y))] \\ &= k^2[f(2x+y)+f(y)]+2k^2(k^2-1)f(x)-2(k^2-1)f(x+y) \\ &+ k^2[f(2x-y)+f(y)]+2k^2(k^2-1)f(x)-2(k^2-1)f(x-y) \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} f((k-1)x+y)+f((k-1)x-y) &= (k-1)^2[f(x+y)+f(x-y)] \\ &\quad + 2(k-1)^2[(k-1)^2-1]f(x) \\ &\quad - 2[(k-1)^2-1]f(y) \end{aligned} \tag{2.7}$$

for all $x, y \in X$. By (2.5), (2.6) and (2.7), we get

$$\begin{aligned} f((k+1)x+y)+f((k+1)x-y) &= (k+1)^2[f(x+y)+f(x-y)] \\ &\quad + 2(k+1)^2[(k+1)^2-1]f(x) \\ &\quad - 2[(k+1)^2-1]f(y), \end{aligned}$$

for all $x, y \in X$, this means that (1.2) holds for $j = k+1$, then, by using (2.5) , one inductively obtains (1.2). $\square$

Definition 2.2. *Let X be a non-empty set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies*
(1) $d(x, y) = 0$ *if and only if* $x = y$;
(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity. We now introduce one of fundamental results of fixed point theory. For the proof, refer to [16]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [14, 15].

Theorem 2.3. [3, 19] *(The alternative of fixed point.) Let* (X, d) *be a generalized complete metric space. Assume that* $T : X \to X$ *is a strictly contractive operator with the Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* k *such that* $d(T^k x, T^{k+1}x) < \infty$ *for some* $x \in X$, *then the followings are true:*
(1) *The sequence* $\{T^n x\}$ *converges to a fixed point* x^* *of* T;
(2) x^* *is the unique fixed point of* T *in*

$$X^* = \{y \in X \mid d(T^k x, y) < \infty\};$$

(3) *If* $y \in X^*$, *then* $d(y, x^*) \leq \frac{1}{1-L}\, d(y, Ty)$.
See also [4, 8].

Utilizing the above-mentioned fixed point alternative, we now obtain our main result, i.e., the generalized Hyers-Ulam stability of the functional equation (1.2). Let X be a real vector space and Y a real Banach space. Given a mapping $f : X \to Y$, we set

$$Df(x, y) := f(kx+y)+f(kx-y)-k^2[f(x+y)+f(x-y)]-2k^2(k^2-1)f(x)+2(k^2-1)f(y)$$

for all $x, y \in X$.

From now on, assume that k is a fixed integer greater than 1. Let $\varphi : X \times X \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\varphi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{4n}} = 0 \tag{2.8}$$

for all $x, y \in X$, where $\lambda_i = k$ if $i = 0$ and $\lambda_i = \frac{1}{k}$ if $i = 1$.

Theorem 2.4. *Suppose that a function* $f : X \to Y$ *satisfies the functional inequality*

$$\|Df(x, y)\| \leq \varphi(x, y) \tag{2.9}$$

for all $x, y \in X$ *and* $f(0) = 0$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \longmapsto \psi(x) = \varphi(\frac{x}{k}, 0)$$

has the property

$$\psi(x) \leq L.\lambda_i^4.\psi(\frac{x}{\lambda_i}) \tag{2.10}$$

for all $x \in X$, *then there exists a quartic function* $Q : X \to Y$ *such that the inequality*

$$\|f(x) - Q(x)\| \leq \begin{cases} \frac{L}{2(1-L)}, & i=0; \\ \frac{1}{2(1-L)}, & i=1; \end{cases} \tag{2.11}$$

holds for all $x \in X$.

Proof. Consider

$$X := \{g : \ g : X \to Y, \ g(0) = 0\}$$

and introduce the generalized metric on X,

$$d(g,h) = inf\{M \in (0,\infty) : \ \|g(x) - h(x)\| \le M\psi(x), \ x \in X\}.$$

Let $\{g_n\}$ be a Cauchy sequence in (X,d). According to the definition of Cauchy sequences, there exists, for any given $\epsilon > 0$, a positive integer N_ϵ such that $d(g_m, g_n) \le \epsilon$ for all $m, n \ge N_\epsilon$. By considering the definition of the generalized metric d, we see that

$$\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N} \ \forall m, n \ge N_\epsilon \ \forall x \in X \ : \ \|g_m(x) - g_n(x)\| \le M\psi(x) \tag{2.12}$$

If x is any given point of X, (2.12) implies that $\{g_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, $\{g_n(x)\}$ converges in Y for each $x \in X$. Hence we can define a function $g : X \to Y$

$$g(x) = \lim_{n\to\infty} g_n(x) \tag{2.13}$$

for any $x \in X$. We have $g \in X$ and $\lim_n g_n = g$.

Now we define a function $T : X \to X$ by

$$Tg(x) = \frac{1}{\lambda_i^4} g(\lambda_i x)$$

for all $x \in X$. Note that for all $g, h \in X$,

$$\begin{aligned} d(g,h) < M \implies & \|g(x) - h(x)\| \le M\psi(x), \ x \in X \\ \implies & \|\frac{1}{\lambda_i^4} g(\lambda_i x) - \frac{1}{\lambda_i^4} h(\lambda_i x)\| \le \frac{1}{\lambda_i^4} M\psi(\lambda_i x), \ x \in X, \\ \implies & \|\frac{1}{\lambda_i^4} g(\lambda_i x) - \frac{1}{\lambda_i^4} h(\lambda_i x)\| \le LM\psi(x), x \in X, \\ \implies & d(Tg, Th) \le LM. \end{aligned}$$

Hence we see that

$$d(Tg, Th) \le Ld(g,h)$$

for all $g, h \in X$, that is, T is a strictly self-mapping of X with the Lipschitz constant L.

If we put $y = 0$ in (2.9) and use (2.10) with the case $i = 0$, then we see that

$$\|2f(kx) - 2k^4 f(x)\| \le \varphi(x, 0) \tag{2.14}$$

which is reduced to

$$\|f(x) - \frac{1}{k^4} f(kx)\| \le \frac{1}{2k^4}\psi(kx) \le \frac{L}{2}\psi(x)$$

for all $x \in x$, that is, $d(f, Tf) \le \frac{L}{2} = \frac{L^1}{2} < \infty$. If we substitute $x := \frac{x}{k}$ in (2.14) and use (2.10) with the case $i = 1$, then we see that

$$\|2f(x) - 2k^4 f(\frac{x}{k})\| \le \varphi(\frac{x}{k}, 0)$$

which is reduced to

$$\|f(x) - k^4 f(\frac{x}{k})\| \le \frac{1}{2}\psi(x)$$

for all $x \in X$, that is, $d(f, Tf) \leq \frac{1}{2} = \frac{L^0}{2} < \infty$.
Now, from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in X such that

$$Q(x) = \lim_{n\to\infty} \frac{f(\lambda_i^n x)}{\lambda_i^{4n}} \tag{2.15}$$

for all $x \in X$ since $\lim_{n\to\infty} d(T^n f, Q) = 0$.
To show that the function $Q : X \to Y$ is quartic, let us replace x and y by $\lambda_i^n x$ and $\lambda_i^n y$ in (2.9), respectively, and divide by λ_i^{4n}. Then it follows from (2.8) and (2.15) that

$$\|DQ(x,y)\| = \lim_{n\to\infty} \frac{\|Df(\lambda_i^n x, \lambda_i^n y)\|}{\lambda_i^{4n}} \leq \frac{\varphi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{4n}} = 0$$

for all $x, y \in X$, that is, Q satisfies the functional equation (1.2). Thus Lemma 2.1 guarantees that Q is quartic.
According to the fixed point alternative, Since Q is the unique fixed point of T in the set $X^* = \{g \in X : \ d(f,g) < \infty\}$, Q is the unique mapping such that

$$\|f(x) - Q(x)\| \leq M\psi(x)$$

for all $x \in X$ and some $M > 0$. Again using the fixed point alternative, we have

$$d(f, Q) \leq \frac{1}{1-L} d(f, Tf),$$

and so obtain the inequality

$$d(f, Q) \leq \frac{L^{1-i}}{2(1-L)}$$

which yields the inequality (2.11). This completes the proof of the theorem. $\square$

From Theorem 2.4, we obtain the following corollary concerning the generalized Hyers-Ulam stability of the functional equation (1.2).

Corollary 2.5. *Let X and Y be a normed space and a Banach space, respectively. Let $p \geq 0$ be given with $p \neq 4$. Assume that $\delta \geq 0$ and $\epsilon \geq 0$ are fixed. Suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y)\| \leq \delta + \epsilon(\|x\|^p + \|y\|^p) \tag{2.16}$$

for all $x, y \in X$. Furthermore, assume that $f(0) = 0$ and $\delta = 0$ in (2.16) for the case $p > 4$. Then there exists a unique quartic mapping $Q : X \to Y$ such that the inequality

$$\|f(x) - Q(x)\| \leq \frac{\delta}{2(k^{4-p} - 1)} + \frac{\epsilon}{2(k^4 - k^p)} \|x\|^p \tag{2.17}$$

holds for all $x \in X$, where $p < 4$, or the inequality

$$\|f(x) - Q(x)\| \leq \frac{\epsilon}{2(k^p - k^4)} \|x\|^p \tag{2.18}$$

holds for all $x \in x$, where $p > 4$.

Proof. Let $\varphi(x,y) := \delta + \epsilon(\|x\|^p + \|y\|^p)$ for all $x, y \in x$. then it follows that

$$\frac{\varphi(\lambda_i^n x, \lambda_i^n y)}{\lambda_i^{4n}} = \frac{\delta}{\lambda_i^{4n}} + (\lambda_i^n)^{p-4}\epsilon(\|x\|^p + \|y\|^p) \longrightarrow 0$$

as $n \longrightarrow \infty$, where $p < 4$ if $i = 0$ and $p > 4$ if $i = 1$, we see that the inequality (2.10) holds with either $L = k^{p-4}$ or $L = \frac{1}{k^{p-4}}$. Now the inequality (2.11) yields the inequalities (2.17) and (2.18) which complete the proof of the corollary. □

The following corollary is the generalized Hyers-Ulam stability of the functional equation (1.2).

Corollary 2.6. *Let X and Y be a normed space and a Banach space, respectively. Assume that $\theta \geq 0$ is fixed. suppose that a mapping $f : X \to Y$ satisfies the functional inequality*

$$\|Df(x,y)\| \leq \theta \tag{2.19}$$

for all $x, y \in X$. then there exists a unique quartic mapping $Q : X \to Y$ such that the inequality

$$\|f(x) - Q(x)\| \leq \frac{\theta}{4(k^4 - 1)} \tag{2.20}$$

for all $x \in X$.

Proof. In Corollary 2.5, putting $\delta := 0$, $p := 0$ and $\epsilon := \frac{\theta}{2}$, we obtain the conclusion of the corollary. □

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