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THE EQUIVALENCE OF CONE METRIC SPACES AND METRIC SPACES

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Abstract. In this note, we introduce a metric on the cone metric space and then prove that a complete cone metric space is always a complete metric space and verify that a contractive mapping on the cone metric space is a contractive mapping on the metric space. Hence, fixed point theorems on cone metric space are, essentially, fixed point theorems on metric space.

Key Words and Phrases: Cone metric space, metric, contractive mapping.
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1. Introduction

Banach contraction principle plays an important role in several branches of mathematics and applied mathematics. For this reason, it has been extended in many directions, for example, see [3, 11, 2, 5, 4] and references therein.

Cone metric spaces were researched by Huang and Zhang in [6]. They defined cone metric and cone metric spaces, which generalize metric and metric spaces, and proved some fixed point theorems for contractive mappings on these spaces. Then in [10, 9, 7, 12, 8, 1], the authors extend some fixed point theorems on metric spaces to cone metric spaces.

In this note, without the assumption that the cone is normal, we introduce a metric D on the cone metric space (X, d) and then we point out that a complete cone metric space is always a complete metric space and show that contractive mappings on a cone metric space (X, d) are contractive on the metric space (X, D).

Consistent with Huang and Zhang [6], the following definitions and results will be needed in the sequel.

Let E always be a real Banach space and P a subset of E. P is called a cone if:

(i) P is closed, nonempty and $P \neq \{0\}$;

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(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b;

(iii) $P \bigcap (-P) = \{0\}$.

Given a cone $P \subset E$, we can define a partial order $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ indicates that $y - x \in intP$, where $intP$ denotes the interior of P.

In the rest of the paper, we always suppose that E is a Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is a partial order with respect to P.

Lemma 1.1 ([10]) *Let $\{x_n\}, \{y_n\}$ are two sequences in E . If $x_n \leq y_n$ for any $n \in N$, $x_n \to x, y_n \to y$, $(n \to \infty)$, then $x \leq y$.*

Definition 1.2 ([6]) *Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies*

(d1) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.3 ([6]) *Let (X, d) be a cone metric space, $x \in X$ and $\{x_n\}$ a sequence in X. Then*

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \geq N$. we denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_m, x_n) << c$ for all $m, n \geq N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Definition 1.4 ([6]) *Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X, then X is called a sequentially compact cone metric space.*

2. Main results

We start this section with an auxiliary result.

Lemma 2.1 *Let (X, d) be a cone metric space, then*

$$D(x, y) = \inf_{\{u \in P | u \geq d(x,y)\}} \|u\|, \quad x, y \in X$$

is a metric on X.

Proof. (1) It is obvious that $D(x, y) \geq 0$.

If $D(x, y) = 0$, i.e., $\inf_{\{u \in P | u \geq d(x,y)\}} \|u\| = 0$, then for arbitrary $n \in N$, there exists $u_n \in P, u_n \geq d(x, y)$ such that $\|u_n\| < \frac{1}{n}$.

Since $u_n \geq d(x, y)$ and $u_n \to 0 (n \to \infty)$, by Lemma 1.1, we have $0 \geq d(x, y)$, which implies $d(x, y) \in P \bigcap (-P)$. Hence $d(x, y) = 0$ and $x = y$.

(2)$d(x, y) = d(y, x)$ implies $D(x, y) = D(y, x), \quad x, y \in X$.

(3) Let $x, y, z \in X$, then $D(x, z) \leq D(x, y) + D(y, z)$.

In fact, since

$$D(x, z) = \inf_{\{u_1 \in P | u_1 \geq d(x,z)\}} \|u_1\|,$$

$$D(x, y) = \inf_{\{u_2 \in P | u_2 \geq d(x,y)\}} \|u_2\|,$$

$$D(y,z) = \inf_{\{u_3 \in P | u_3 \geq d(y,z)\}} \|u_3\|,$$

for arbitrary $u_2, u_3 \in P, u_2 \geq d(x,y), u_3 \geq d(y,z)$,

$$u_2 + u_3 \geq d(x,y) + d(y,z) \geq d(x,z),$$

then

$$\{u_1 \in P | u_1 \geq d(x,z)\} \supset \{u_2 + u_3 \in P | u_2 \geq d(x,y), u_3 \geq d(y,z)\},$$

which implies

$$\inf_{\{u_2,u_3 \in P | u_2 \geq d(x,y), u_3 \geq d(y,z)\}} \|u_2 + u_3\| \geq \inf_{\{u_1 \in P | u_1 \geq d(x,z)\}} \|u_1\|.$$

Note that

$$\begin{aligned} &\inf_{\{u_2,u_3 \in P | u_2 \geq d(x,y), u_3 \geq d(y,z)\}} \|u_2 + u_3\| \\ &\leq \inf_{\{u_2,u_3 \in P | u_2 \geq d(x,y), u_3 \geq d(y,z)\}} \|u_2\| + \|u_3\| \\ &= \inf_{\{u_2 \in P | u_2 \geq d(x,y)\}} \|u_2\| + \inf_{\{u_3 \in P | u_3 \geq d(y,z)\}} \|u_3\| \end{aligned},$$

thus

$$\inf_{\{u_2 \in P | u_2 \geq d(x,y)\}} \|u_2\| + \inf_{\{u_3 \in P | u_3 \geq d(y,z)\}} \|u_3\| \geq \inf_{\{u_1 \in P | u_1 \geq d(x,z)\}} \|u_1\|,$$

i.e.

$$D(x,y) + D(y,z) \geq D(x,z).$$

(1-3) show that D is a metric on X, (X,D) is a metric space. □

Theorem 2.2 *The metric space (X,D) is complete if and only if the cone metric space (X,d) is complete .*

Proof. (1) If the cone metric space (X,d) is complete.

Let $\{x_n\}$ be a Cauchy sequence of the metric space (X,D).

For any $c >> 0$, there exists $\delta > 0$, such that $c + B(0,\delta) \subset P$. Note that $\{x_n\}$ is a Cauchy sequence, there is N such that $D(x_n, x_m) \leq \frac{\delta}{4}$ for $m,n > N$,i.e.,

$$\inf_{\{u \in P | u \geq d(x_n,x_m)\}} \|u\| \leq \frac{\delta}{4}.$$

Hence there exists $v \in P$, $\| v \| \leq \frac{\delta}{2}$ such that $d(x_n, x_m) \leq v$.

Note that $c - v \in intP$, thus $d(x_n, x_m) \leq v << c$ for $m,n > N$, which implies $\{x_n\}$ is a Cauchy sequence of the cone metric space (X,d).

Since (X,d) is complete, there is $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

Given $c >> 0$, note that $\frac{c}{k} >> 0$ for $k \geq 1$, there exists N_k such that for all $n > N_k$, $d(x, x_n) << \frac{c}{k}$. Hence

$$D(x_n, x) = \inf_{\{u \in P | u \geq d(x_n,x)\}} \|u\| \leq \frac{\|c\|}{k} \; for\ all\ n > N_k$$

Since $\frac{\|c\|}{k} \to 0 (k \to \infty)$, then

$$D(x_n, x) \to 0, (n \to \infty).$$

Hence the metric space (X,D) is complete.

(2) Assume the metric space (X, D) is complete.

Let $\{x_n\}$ be a Cauchy sequence of the cone metric space (X, d).

Given $c >> 0$ and a positive number $\varepsilon > 0$, there is $k \geq 1$, such that $\|\frac{c}{k}\| < \varepsilon$.

Noting that $\frac{c}{k} >> 0$ and $\{x_n\}$ be a Cauchy sequence of the cone metric space (X, d), then there exists N such that for all $m, n > N$, $d(x_m, x_n) << \frac{c}{k}$. Hence

$$D(x_n, x_m) = \inf_{\{u \in P | u \geq d(x_m, x_n)\}} \|u\| \leq \frac{\|c\|}{k} < \varepsilon \ for\ all\ m, n > N.$$

which implies $\{x_n\}$ is a Cauchy sequence of the cone metric space (X, D).

Since (X, D) is complete, there is $x \in X$ such that $\lim_{n\to\infty} D(x_n, x) = 0$.

For any $c >> 0$, there exists $\delta > 0$, such that $c + B(0, \delta) \subset P$. For this $\delta > 0$, there is N such that $D(x_n, x) \leq \frac{\delta}{4}$ for $n > N$,i.e.,

$$\inf_{\{u \in P | u \geq d(x_n, x)\}} \|u\| \leq \frac{\delta}{4}.$$

Hence there exists $v \in P$, $\| v \| \leq \frac{\delta}{2}$ such that $d(x_n, x) \leq v$.

Note that $c - v \in intP$, thus $d(x_n, x) \leq v << c$ for $n > N$, which implies $\{x_n\}$ convergent to x in the cone metric space (X, d).

Hence the cone metric space (X, d) is complete. □

As a consequence of Theorem 2.2, we easily get the following:

Theorem 2.3 *If (X, d) is a sequentially compact cone metric space, then (X, D) is a compact metric space.*

Another result of this paper says that a contractive mapping on cone metric space is always contractive on the metric space. More precisely, we have:

Theorem 2.4 *Let (X, d) be a complete cone metric space. If $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq kd(x, y), \quad for\ all\ x, y \in X,$$

where $k \in [0, 1)$ is a constant, then T is a contractive mapping on (X, D),i.e.,

$$D(Tx, Ty) \leq kD(x, y), \quad for\ all\ x, y \in X.$$

Hence T has a unique fixed point in X.

Proof. In fact, let $v \in P, v \geq d(x, y)$, then $kv \geq d(Tx, Ty)$, which implies

$$\{kv \mid v \in P, v \geq d(x, y)\} \subset \{u \mid u \in P, u \geq d(Tx, Ty)\},$$

thus

$$\inf_{\{kv | v \in P, v \geq d(x,y)\}} \|kv\| \geq \inf_{\{u | u \in P, u \geq d(Tx,Ty)\}} \|u\|,$$

or equivalent,

$$k \inf_{\{v \in P | v \geq d(x,y)\}} \|v\| \geq \inf_{\{u \in P | u \geq d(Tx,Ty)\}} \|u\|,$$

that is

$$kD(x, y) \geq D(Tx, Ty), \quad for\ all\ x, y \in X,$$

then T has a unique fixed point in X. □

Remark. In a similar way, we can show that the fixed point theorems established in [1]-[6] are still true without the assumption that cone P is normal and they are, in essence, fixed point theorems on metric spaces.

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