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APPROXIMATE FIXED POINTS AND BEST PROXIMITY PAIRS

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Abstract. In this paper we give existence theorems of ϵ-fixed points for multi-functions from a subset to another in a Banach space. Our result extends previous approximate fixed point theorems. As a consequence we obtain new theorems of existence of best proximity pairs.
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1. Introduction

Let $(E, \|.\|)$ be a real Banach space, and let $d(x, y) = \|x - y\|, x, y \in E$. Let X and Y be subsets of E.
Given a multi-function $F : X \to 2^Y$, and $\epsilon > 0$, we define the set of *ϵ-fixed points* of F by

$$FIX^{\epsilon}(F) := \{x \in X : d(x, F(x)) \leq d(X, Y) + \epsilon\}, \tag{1.1}$$

where $d(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$ and $d(x, F(x) = \inf\{d(x, y) : y \in F(x)\}$. This concept, for $Y = X$, was introduced in [3]. Existence of ϵ-fixed points is known in the literature as existence of approximate fixed points (see [11]).

The pair $(x, F(x))$ is called a *best proximity pair of* F if $d(x, F(x)) = d(X, Y)$. Some times the fixed-point equation $F(x) = x$ does not possess a solution, then the next question that naturally arises is whether it is possible to find an element x in a suitable space such that x is close to $F(x)$ in some sense. Best proximity pair theorems are adequate to be explored in this direction. Under suitable conditions a best proximity theorem boils down to a fixed-point theorem. Thus, best proximity pair theorems also serve as a generalization of fixed-point theorems. Existence theorems of best proximity pairs were given in [4], [7], [9], and [10].

The interest to obtain theorems of existence of fixed points and approximate fixed points is based on its usefulness in applications, as game theory and mathematical economics (see [1]). Existence theorems of best proximity pairs imply existence theorems of equilibrium pairs for constrained generalized games (see [8], [9], and [10]).

In [3] the authors gave new approximate fixed point theorems for multi-functions from a set into itself. Using the technique employed by them, we extend these theorems in two ways. In first place we consider multi-functions from a set to another and in second place we substitute the assumption that X has nonempty interior by a more general condition over a certain set X^0. Our approach allow us to obtain, in Section 2, new existence theorems of approximate fixed points which extend previous results proved in [3](see Corollary 2.5, 2.8, and Theorem 2.9). In Section 3, we also obtain new existence theorems of best proximity pairs.

Let A and B be subsets of E, and let $F : A \to 2^B$ be a multi-function. F is called *closed* (*weak closed*) on A if for all net $x_\alpha \in A$, converging (weakly converging) to $x \in X$, and $y_\alpha \in F(x_\alpha)$, converging (weakly converging) to $y \in E$, imply $y \in F(x)$. The multi-function F is called *upper semi-continuous* (*weakly upper semi-continuous*) if for all $x \in A$ and for all open set U (weak open) in B such that $F(x) \subset U$, there is an open neighborhood (weak open neighborhood) V of x in A, verifying $F(y) \subset U$, for all $y \in V$. .

In this work we consider the following subsets of E,

$$X^0 := \{x \in X : \exists y \in \overline{Y} \quad \text{such that} \quad d(x, y) = d(X, Y)\}, \tag{1.2}$$

and

$$Y^0 := \{y \in Y : \exists x \in \overline{X} \quad \text{such that} \quad d(x, y) = d(X, Y)\}. \tag{1.3}$$

As usual $\overline{Y}$ denote the closure of Y with the norm topology. If X and Y are convex sets, it is easy to prove that X^0 and Y^0 are convex sets.

Definition 1.1. *We say that the set X has the property (**I**) if for all $0 < \delta < 1$, there exists $x_0 \in X$ such that $\delta \overline{X} + (1 - \delta)x_0 \subset X$.*

Definition 1.2. *We say that the pair (X, Y) has the property (**II**) if $u \in \overline{X}$ and $d(u, v) = d(X, Y)$ for some $v \in \overline{Y}$ imply $u \in \overline{X^0}$.*

Remark 1.3. The property (**I**) plays an important role in this paper. The paper [3] was the inspiration source to define it.
Suppose $A \subset V$, where V is a subspace of E, and let $INT_V(A)$ be the V-relative interior of A i.e., $\{x \in A : \exists r > 0, \text{ with } B(x, r) \cap V \subset A\}$. Here $B(x, r)$ is the open ball in E, of center x and radius r. If $INT_V(A) \neq \emptyset$ and A is convex, then A has the property (**I**). Every convex closed nonempty set has the property (**I**).
For all nonempty set X, the pair (X, X) has the property (**II**). The property (**II**), in general, is not an extremely restrictive condition.

2. Existence of approximate fixed points

In this Section we give new theorems of existence of approximate fixed points.
We begin establishing several lemmas concerning to the metric projection.

Lemma 2.1. *Let E be a Banach space. If X is a bounded set then there exists $r > 0$ such that $Y^0 \subset B(0, r) \cap Y$.*

Proof. There is $M > 0$ such that $\|x\| \leq M$ for all $x \in X$. Let $r = 2M + \beta$, where $\beta = d(X, Y)$. Let $y \in Y$ be such that $\|y\| > r$. Then for all $x \in X$ we have $d(x, y) \geq r - \|x\| \geq M + \beta$. So, $y \notin Y^0$. In consequence, $Y^0 \subset B(0, r) \cap Y$. $\square$

Lemma 2.2. *Let E be a reflexive Banach space. Suppose that X and Y are bounded and convex sets, $X^0 \neq \emptyset$ and $Y^0 \neq \emptyset$. If the pair (X, Y) has the property **(II)**, then the metric projection $P_{\overline{X^0}} : Y^0 \to \overline{X^0}$ is closed (weakly closed) on Y^0.*

Proof. Since E is reflexive, it is known that the metric projection on the convex closed set $\overline{X^0}$ is well defined, i.e.,

$$P_{\overline{X^0}}(y) := \{x \in \overline{X^0} : d(y, x) = d(y, \overline{X^0})\} \neq \emptyset,$$

for all $y \in E$. Let $z_\alpha \in Y^0$ be converging (weakly converging) to $z \in Y^0$ and let $w_\alpha \in P_{\overline{X^0}}(z_\alpha)$ be with w_α converging (weakly converging) to w. There exists a net $u_\alpha \in \overline{X}$ with $d(u_\alpha, z_\alpha) = d(X, Y)$. Since the pair (X, Y) has the property (**II**), then $u_\alpha \in \overline{X^0}$. Therefore we obtain $d(z_\alpha, w_\alpha) = d(z_\alpha, \overline{X^0}) \leq d(z_\alpha, u_\alpha) = d(X, Y)$. As $w_\alpha - z_\alpha$ weakly converges to $w - z$, using the Lemma 27([5],p. 68), in either case we have $d(w, z) \leq \underline{\lim} d(w_\alpha, z_\alpha) = d(X, Y)$. In consequence $w \in P_{\overline{X^0}}(z)$. The proof is complete. □

Lemma 2.3. *Let E be a Banach space. Suppose that $X^0 \neq \emptyset$, $\overline{X^0}$ compact, and $Y^0 \neq \emptyset$. Let $P_{\overline{X^0}} : E \to \overline{X^0}$ be the metric projection. We have*

(1) *If the pair (X, Y) has the property **(II)**, then $P_{\overline{X^0}}$ is closed on Y^0.*
(2) *$P_{\overline{X^0}}$ is upper-semi continuous on E.*

Proof. Since $\overline{X^0}$ is compact it is known that the metric projection on the compact set $\overline{X^0}$ is well defined and with compact values. Now, the proof of that it is closed follows as in the proof of Lemma 2.2.
Next, we prove that the projection is upper semi-continuous. Let $y \in E$ and let U be an open set such that $P_{\overline{X^0}}(y) \subset U$. Since $P_{\overline{X^0}}(y)$ is compact, there is $r > 0$ such that

$$P_{\overline{X^0}}(y) + r := \{z \in E : d(z, P_{\overline{X^0}}(y)) < r\} \subset U.$$

Then it will be sufficient to prove that there exists $s > 0$, such that $P_{\overline{X^0}}(B(y, s)) \subset P_{\overline{X^0}}(y)) + r$. Suppose that it is not true, so there are two sequences $z_n \in E, y_n \in P_{\overline{X^0}}(z_n)$ such that $d(z_n, y) < \frac{1}{n}$ and $d(y_n, P_{\overline{X^0}}(y) \geq r$. From the compactness of $\overline{X^0}$, there exists a subsequence y_{n_k} of y_n with y_{n_k} converging to a point $y_0 \in \overline{X^0}$. Clearly, we have $d(y_{n_k}, \overline{X^0}) = d(y_{n_k}, z_{n_k}) \to d(y_0, y)$, so $d(y_0, y) = d(y_0, \overline{X^0}) = 0$. It is a contradiction. □

Theorem 2.4. *Let E be a reflexive Banach space. Let X and Y be convex subsets where either X or Y is bounded. Assume $X^0 \neq \emptyset$, with the property **(I)**, and that the pair (X, Y) has the property **(II)**. Let $F : X \to 2^Y$ be a weakly closed multi-function with convex set values such that $F(x) \cap Y^0 \neq \emptyset, x \in X^0$. Then $FIX^\epsilon(F) \neq \emptyset$, for each $\epsilon > 0$.*

Proof. Let $F' : X^0 \to 2^{Y^0}$ defined by $F'(x) = F(x) \cap Y^0, x \in X^0$.
If $P_{\overline{X^0}} : E \to \overline{X^0}$ is the metric projection, we also write $P_{\overline{X^0}} : 2^E \to \overline{X^0}$ for the multi-function given by

$$P_{\overline{X^0}}(A) = \bigcup_{a \in A} P_{\overline{X^0}}(a), A \subset E.$$

Since X^0 satisfies the property (**I**), given $\lambda \in (0,1)$ there exists $x_0 \in X^0$ such that $Z := \lambda\overline{X^0} + (1-\lambda)x_0 \subset X^0$.
Now, we consider the multi-function $H : Z \to 2^Z$ defined by

$$H(x) = \lambda P_{\overline{X^0}} \circ F'(x) + (1-\lambda)x_0,\ x \in Z.$$

We will prove that H is weakly closed on Z. By Lemma 2.1, without loss of generality, we can assume X^0 and Y^0 bounded. Let $x_\alpha \in Z$ be weakly converging $x \in Z$, and let $u_\alpha \in H(x_\alpha)$ be weakly converging to $u \in \overline{X^0}$. From definition of H, there are $y_\alpha \in F'(x_\alpha)$ and $t_\alpha \in P_{\overline{X^0}}(y_\alpha)$ such that $u_\alpha = \lambda t_\alpha + (1-\lambda)x_0$. Since $\overline{X^0}$ is bounded and E is reflexive there exists a sub-net t_{α_k} weakly converging to a point $t \in \overline{X^0}$. Therefore, u_{α_k} weakly converges to $\lambda t + (1-\lambda)x_0 \in \overline{X^0}$. In consequence, $u = \lambda t + (1-\lambda)x_0$. Since $y_{\alpha_k} \in F(x_{\alpha_k}) \cap Y^0$ and Y^0 is bounded, there is a sub-net, which we again denote by y_{α_k}, weakly converging to a point $y \in \overline{Y}$. As F is weakly closed, $y \in F(x)$, so $y \in Y$. In addition, there exists $v_{\alpha_k} \in \overline{X}$ such that $d(v_{\alpha_k}, y_{\alpha_k}) = d(X,Y)$. The sub-net v_{α_k} has a sub-net, that we again denote in the same way, weakly converging to a point $v \in \overline{X}$. Thus we get

$$d(v,y) \le \underline{\lim} d(v_{\alpha_k}, y_{\alpha_k}) = d(X,Y),$$

which implies that $y \in Y^0$. We have proved that $y \in F'(x)$. By Lemma 2.2, $t \in P_{\overline{X^0}} \circ F'(x)$, in consequence $u \in H(x)$.
Next, we show that H has convex set values. In fact, let $\gamma \in [0,1]$ and let $t_i \in P_{\overline{X^0}} \circ F'(x), i = 1,2, x \in X^0$, then there are $r_i \in F'(x)$ such that $t_i \in P_{\overline{X^0}}(r_i)$. Since $F'(x)$ is a convex set, $\gamma r_1 + (1-\gamma)r_2 \in F'(x)$. Further, $r_i \in Y^0$ implies that there is $b_i \in \overline{X}$, with $d(r_i, b_i) = d(X,Y)$. Thus, $d(r_i,t_i) = d(r_i, \overline{X^0}) \le d(r_i, b_i)$, which implies that $d(r_i,t_i) = d(X,Y)$. It follows that

$$d(\gamma r_1 + (1-\gamma)r_2, \gamma t_1 + (1-\gamma)t_2) = d(X,Y).$$

In consequence, $\gamma t_1 + (1-\gamma)t_2 \in P_{\overline{X^0}}(\gamma r_1 + (1-\gamma)r_2)$.
Since H is a weakly closed multi-function, from a weakly compact set with convex nonempty values into itself and E with the weak topology is a convex locally Hausdorff vectorial space, by Glicksberg Theorem [6], H has a fixed point in Z, say x^*.
Clearly, there are $y \in F'(x^*)$ and $x \in P_{\overline{X^0}}(y)$ such that $x^* = \lambda x + (1-\lambda)x_0$. We recall that $\overline{X^0}$ is bounded, so there is $M > 0$ such that $\|u\| \le M$, for all $u \in \overline{X^0}$. Given $\epsilon > 0$, we chose $\lambda \in (0,1)$ satisfying $1 - \lambda < \frac{\epsilon}{2M}$. Then

$$d(x^*, x) = (1-\lambda)d(x_0, x) \le 2(1-\lambda)M < \epsilon. \tag{2.1}$$

As $y \in Y^0$ then $d(y, \overline{X^0}) = d(X,Y)$. Finally, from (2.1) we get

$$d(x^*, F(x^*) \le d(x^*, y) \le d(x^*, x) + d(x,y) = d(x^*, x) + d(y, \overline{X^0}) \le \epsilon + d(X,Y).$$

□

The next Corollary was proved in [3].

Corollary 2.5. *Let E be a reflexive Banach space. Let X be a convex bounded subset of E. Assume that X has nonempty interior. Let $F : X \to 2^X$ be a weakly closed multi-function with convex nonempty set values for each $x \in X$. Then $FIX^\epsilon(F) \neq \emptyset$, for each $\epsilon > 0$.*

Proof. Here we have $X = Y = X^0 = Y^0$. As we have observed the property (**II**) for the pair (X, Y) immediately follows. If $x_0 \in IntX$, it is easy see that $\lambda\overline{X} + (1-\lambda)x_0 \subset X$ for all $\lambda \in [0, 1]$, so X^0 has the property (**I**). In addition, we have $F(x) \cap Y^0 \neq \emptyset$. Then the corollary immediately follows from Theorem 2.4. □

Corollary 2.6. *Let E be a reflexive Banach space. Let X and Y be convex nonempty subsets where either X or Y is bounded. Assume that X is closed. Let $F : X \to 2^Y$ be a weakly closed multi-function with convex set values such that $F(x) \cap Y^0 \neq \emptyset, x \in X^0$. Then $FIX^\epsilon(F) \neq \emptyset$, for each $\epsilon > 0$.*

Proof. Clearly the pair (X, Y) has the property (**II**). Let $x_n \in X$ be a sequence fulfilling $\lim d(x_n, Y) = d(X, Y)$. Since x_n is bounded, there exists a subsequence, denoted also by x_n, and $x \in X$, such that x_n weakly converges to x, because X is a convex and closed set. Let $\delta > 0$. For each $n \in \mathbb{N}$ there is $y_n \in Y$ such that $d(x_n, y_n) \leq \frac{1}{n} + d(x_n, Y)$. For sufficiently big n we have $d(x_n, y_n) \leq \frac{1}{n} + \delta + d(X, Y)$. Since y_n is bounded, we can get a subsequence again denoted by y_n, weakly converging to $y \in \overline{Y}$. Thus

$$d(x, Y) = d(x, \overline{Y}) \leq d(x, y) \leq \underline{\lim} d(x_n, y_n) \leq \delta + d(X, Y). \tag{2.2}$$

As $\delta > 0$ is arbitrary, from (2.2) follows that $d(x, y) = d(X, Y)$. Therefore, $x \in X^0$. On the other hand, X^0 is closed. In fact, let $z_n \in X^0$ be with $d(z_n, x) \to 0$, as $n \to \infty$. Clearly $z \in X$. For each $n \in \mathbb{N}$ let $y_n \in \overline{Y}$ be such that $d(z_n, y_n) = d(X, Y)$. As before we can assume that y_n weakly converges to $y \in \overline{Y}$. Since $z_n - y_n$ weakly converges to $z - y$ we have $d(z, y) \leq \underline{\lim} d(z_n, y_n) = d(X, Y)$. It follows that $z \in X^0$. Finally, since X^0 is nonempty, convex and closed, it has the property (**I**), then we can apply the Theorem 2.4. □

Theorem 2.7. *Let E be a Banach space. Let X and Y be convex subsets where either X or Y is bounded. Assume $X^0 \neq \emptyset$, with the property (**I**), and totally bounded. Let $F : X \to 2^Y$ be a multi-function with convex set values such that $F(x) \cap Y^0 \neq \emptyset, x \in X^0$. We have*

(1) *If the pair (X, Y) has the property (**II**) and F is closed, or*
(2) *F is upper semi-continuous,*

then $FIX^\epsilon(F) \neq \emptyset$, for each $\epsilon > 0$.

Proof. Let $\epsilon > 0$. Let $F' : X^0 \to 2^{Y^0}$, $P_{\overline{X^0}} : E \to \overline{X^0}, M$, and $\lambda \in (0, 1)$ as in the proof of Theorem 2.4. There exists $x_0 \in X^0$ such that $Z := \lambda\overline{X^0} + (1-\lambda)x_0 \subset X^0$. Let the multi-function $H : Z \to 2^Z$ be defined by $H(x) = \lambda P_{\overline{X^0}} \circ F'(x) + (1-\lambda)x_0$. First, we suppose that the pair (X, Y) has the property (**II**) and F is closed. By Lemma 2.3, $P_{\overline{X^0}}$ is closed on Y^0. Using that $\overline{X^0}$ is compact and that it has the property (**I**), we can analogously prove to the proof of Theorem 2.4 that H is closed. Suppose now F upper semi-continuous. We now consider the multi-function $G : Z \to Z$ defined by

$$G(z) = \lambda P_{\overline{X^0}}(\overline{F'(x)}) + (1-\lambda)x_0,\ x \in Z.$$

We will see that G is upper semi-continuous on Z. Let $x \in Z$ and let U be an open set in E such that $H(x) \subset U$. By Lemma 2.3, $P_{\overline{X^0}}$ is upper semi-continuous on E, then for each $y \in \overline{F'(x)}$ there exists an open neighborhood V_y of y such that

$$P_{\overline{X^0}}(z) \subset \frac{1}{\lambda}(U - (1-\lambda)x_0) =: C, \text{ for all } z \in V_y.$$

We consider the following open set

$$Q = \bigcup_{y \in \overline{F'(x)}} V_y.$$

Since E is a normal topological space, there is an open set D such that

$$\overline{F'(x)} \subset D \subset \overline{D} \subset Q.$$

F is upper semi-continuous, then there is an open neighborhood J of x such that $F'(t) \subset D$ for all $t \in J$. So, $\overline{F'(t)} \subset \overline{D} \subset Q$ for all $t \in J$. In consequence, $P_{\overline{X^0}} \circ F'(t) \subset C$ for all $t \in J$, or i.e., $G(t) \subset U$ for all $t \in J$. Then, G is upper semi-continuous.
On the other hand, $G(x)$ is closed for all $x \in Z$, it follows that G is closed (see [1]). We have proved that both multi-functions H and G are closed. We can apply the Glicksberg's Theorem to H and G ([6]). Thus, G has a fixed point in Z, say x^*. Clearly, there are $y \in \overline{F'(x^*)}$ and $x \in P_{\overline{X^0}}(y)$ such that $x^* = \lambda x + (1-\lambda)x_0$. Then

$$d(x^*, x) = (1-\lambda)d(x_0, x) \leq 2(1-\lambda)M < \epsilon. \tag{2.3}$$

In addition, for each $n \in \mathbb{N}$ we can get $z \in F(x^*)$ such that $d(y, z) < \frac{1}{n}$. As $z \in Y^0$ then $d(z, \overline{X^0}) = d(X, Y)$. Finally, from (2.3) we get

$$\begin{aligned} d(x^*, F(x^*)) \leq d(x^*, z) \leq d(x^*, x) + d(z, y) + d(y, x) \leq \\ \leq d(x^*, x) + \frac{1}{n} + d(y, \overline{X^0}) \leq \epsilon + \frac{2}{n} + d(z, \overline{X^0}) = \epsilon + \frac{2}{n} + d(X, Y). \end{aligned} \tag{2.4}$$

As n is arbitrary we obtain $d(x^*, F(x^*)) \leq \epsilon + d(X, Y)$.
The proof of $FIX^\epsilon(F) \neq \emptyset$, under the hypothesis (1), follows the same patterns that the proof of Theorem 2.4, using that the multi-function H is closed. □

The next Corollary was proved in [3].

Corollary 2.8. *Let E be a Banach space and let X be a convex totally bounded set with non-empty interior. Assume that $F : X \to 2^X$ is a closed or upper semi-continuous multi-function such that $F(x)$ is a non-empty and convex set for each $x \in X$. Then $FIX^\epsilon(F) \neq \emptyset$ for each $\epsilon > 0$.*

Proof. In the Theorem 2.7 we consider $X = Y = X^0 = Y^0$. Then $X^0 \neq \emptyset$ with the property (**I**), and X^0 is a totally bounded set. Further the pair (X, Y) has the property (**II**). Therefore the Corollary immediately follows from Theorem 2.7. □

The following Theorem was established in [3], with the assumptions $IntX \neq \emptyset$ and E separable. As we show the last hypothesis is not necessary.

Theorem 2.9. *Let E be a reflexive Banach space. Let X be a non-empty convex subset and bounded of E fulfilling the property (**I**). Assume that $F : X \to 2^X$ is a weakly upper semi-continuous multi-function such that $F(x)$ is a non-empty and convex subset of X for each $x \in X$. Then $FIX^\epsilon(F) \neq \emptyset$ for each $\epsilon > 0$.*

Proof. Let $\epsilon > 0$ and let $M > 0$ be such that $\|x\| \leq M$ for all $x \in X$. We choose $\lambda \in (0, 1)$ with $1 - \lambda < \frac{\epsilon}{2M}$, and $x_0 \in X$ satisfying

$$Z := \lambda\overline{X} + (1 - \lambda)x_0 \subset X.$$

Here, we consider the multi-function $G : Z \to 2^Z$, defined by $G(x) = \lambda\overline{F(x)}+(1-\lambda)x_0$. We will prove that G is weakly upper semi-continuous. Let $x \in G$ and let U be a weak open set containing to $G(x)$. Then

$$\overline{F(x)} \subset \frac{1}{\lambda}(U - (1 - \lambda)x_0) =: C.$$

Since $\overline{X}$ is weakly compact, it is normal with the weak topology. In addition, $F(x)$ convex implies $\overline{F(x)} = \overline{F(x)}^w$. Thus, there is a weak open V in E such that

$$\overline{F(x)} \subset V \cap \overline{X} \subset \overline{V}^w \cap \overline{X} \subset C \cup \overline{X}.$$

On the other hand, as F is weakly upper semi-continuous and $F(x) \subset V$, there is a weak open neighborhood of x, say D, such that $F(z) \subset V \cap X$ for all $z \in D \cap X$. Therefore,

$$\overline{F(z)} = \overline{F(z)}^w \subset \overline{V}^w \cap \overline{X} \subset C,$$

for all $z \in D \cap X$. In consequence, $G(z) \subset U$ for all $z \in D$, so G is weakly upper semi-continuous. Since $G(x)$ is a weakly closed set for all x$\in Z$ we have G is weakly closed. Then, G has a fixed point. Now, the proof follows the same patterns that the proof of Theorem 2.4. □

Remark 2.10. To establish the Theorem 2.9 for a weakly upper semi-continuous multi-function $F : X \to Y$ following our arguments, we need that $P_{\overline{X^0}}$ be weakly upper semi-continuous. However, as far as we know, the metric projection weakly upper semi-continuous essentially implies $\overline{X^0}$ compact (see [2], Remark 1, p. 798). In consequence, we establish the next theorem with this requirement.

Theorem 2.11. *Let E be a Banach space. Let X and Y be convex subsets where either X or Y is bounded. Assume that $X^0 \neq \emptyset$ with the property (**I**), and X^0 is a totally bounded set. Let $F : X \to 2^Y$ be a weakly upper semi-continuous multi-function with convex set values such that $F(x) \cap Y^0 \neq \emptyset, x \in X^0$. Then $FIX^\epsilon(F) \neq \emptyset$, for each $\epsilon > 0$.*

Proof. The proof is analogous to the proof of Theorem 2.7. □

3. Existence of best proximity pairs

In this section we will give two theorems of existence of best proximity pairs and a theorem of existence of fixed points. They will be consequence of the existence theorems of ϵ-fixed points proved in Section 2.

Theorem 3.1. *Assume the hypothesis of Theorem 2.4. In addition, we suppose that X is a closed set. Then F has a best proximity pair.*

Proof. By Lemma 2.1 we can suppose that X and Y are bounded sets. The Theorem 2.4 implies that for each $n \in \mathbb{N}$, there is a $1/n$-fixed point, say x_n. On the other hand, there exists $y_n \in F(x_n)$ such that

$$d(x_n, y_n) \leq d(x_n, F(x_n)) + 1/n,$$

for all $n \in \mathbb{N}$. Then $d(x_n, y_n) \leq d(X, Y) + 2/n$, for all $n \in \mathbb{N}$.
Since E is reflexive we can find subsequences x_{n_k} and y_{n_k}, weakly converging to $x \in X = \overline{X}$ and $y \in \overline{Y}$, respectively. Now, F is weakly closed, so $y \in F(x)$. Further

$$d(x, y) \leq \underline{\lim} d(x_{n_k}, y_{n_k}) = d(X, Y).$$

In consequence, $d(x, F(x)) = d(X, Y)$. The proof is complete. □

The proof of the following theorems are analogous to the proof of Theorem 3.1 and we omit it.

Theorem 3.2. *Assume the hypothesis of Theorem 2.9. In addition, we suppose that X is a closed set. Then F has a fixed point, i.e., there is $x \in X$ such that $x \in F(x)$.*

Theorem 3.3. *Assume the same hypothesis of Theorem 2.11. In addition, we suppose that X is a closed set. Then F has a best proximity pair.*

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