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AN OPTIMIZATION PROBLEM RELATED TO GENERALIZED EQUILIBRIUM AND FIXED POINT PROBLEMS WITH APPLICATIONS

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Dedicated to Wataru Takahashi on the occasion of his retirement

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Abstract. We introduce a method for finding a solution of an optimization problem to the set of common solutions of generalized equilibrium problems and the set of common fixed points of a family of infinite nonexpansive mappings. The results presented in this paper improve and extend the corresponding results of many others.
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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of $\mathcal{H}$ and $Q : \mathcal{H} \to 2^{\mathcal{H}}$ be a multi-valued mapping. Let $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function, that is, $\Phi(w, u, v) + \Phi(w, v, u) = 0$ for all $(w, u, v) \in \mathcal{H} \times C \times C$. The *generalized equilibrium problem* is to find $u \in C$ and $w \in Qu$ such that

$$GEP : \Phi(w, u, v) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in C. \tag{1.1}$$

In case of (1.1), we will denote by $u \in GEP(C, Q, \Phi, \varphi)$.

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In particular, if $\varphi \equiv 0$ and $\Phi(w,u,v) = E(u,v)$ where $E : C \times C \to \mathbb{R}$, then problem (1.1) becomes the following equilibrium problem (for short, (EP)), which is to find $x^* \in C$ such that

$$EP : E(x^*, v) \geq 0, \quad \forall v \in C.$$

It is well known that the equilibrium problems (EP) which were introduced by Blum and Oettli [2] and Noor and Oettli [10] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. Hence collectively, equilibrium problems cover a vast range of applications. Consequently, since the problem (1.1) is a generalization of the equilibrium problems (EP), studying the generalized equilibrium problem (GEP) is very useful.

Equally important to the equilibrium problems, we also have the problem of finding the fixed points of the nonlinear mappings. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of nonlinear mappings, for examples, see [12, 16, 17, 20] and the references therein.

On the other hand, the optimization problems are of very interesting and have been studying by many authors. A kind of such problems is the following optimization problem (for shot, OP):

$$OP : \min_{x \in C} \frac{\mu}{2}\langle Ax, x\rangle + \frac{1}{2}\|x - u\|^2 - h(x), \tag{1.2}$$

where $C = \bigcap_{i=1}^{\infty} C_i$, when $C_1, C_2, ...$ are infinitely many closed convex subsets on $\mathcal{H}$ such that $\bigcap_{i=1}^{\infty} C_i \neq \emptyset, u \in \mathcal{H}, \mu \geq 0$ is a real number, A is a strongly positive bounded linear operator on $\mathcal{H}$ and h is a potential function for γf (that is, $h'(x) = \gamma f(x)$ for all $x \in \mathcal{H}$). For more detailed accounts on optimization problems and related problems, we refer to [1, 5, 19, 20].

Inspired by the recent research work going on in this interesting field, we will introduce a general iterative method for finding a solution of the problem (1.2) to the set of common element of the set of solutions for the problem (1.1) and the set of fixed points of an finite family of nonexpansive mappings. Under some suitable conditions, strong convergence theorems are established in the framework of Hilbert space. The results obtained in this paper can be viewed as an important extension of the previously known results.

2. Preliminaries

Now we give some the basic definitions.

Definition 2.1. *A mapping $T : C \to C$ is said to be nonexpansive mapping if*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If T is a mapping, we denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{x \in C : Tx = x\}$.

Definition 2.2. *A mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0, 1)$ such that*

$$\|fx - fy\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

Definition 2.3. *A bounded linear operator A on a Hilbert space $\mathcal{H}$ is said to be strongly positive if there exists a constant $\overline{\gamma} > 0$ such that*

$$\langle Ax, x\rangle \ge \overline{\gamma}\|x\|^2 \quad \forall x, y \in \mathcal{H}.$$

In order to proof our main result, we also need the following concepts.

2.1. **W- mapping.** Let $T_1, T_2, ...$ be infinite mappings of C into itself. In this paper, we consider the mapping W_n defined by

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I,\\
U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I,\\
&\vdots\\
U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I,\\
U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I,\\
&\vdots\\
U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2)I,\\
W_n = U_{n,1} &= \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \qquad (2.1)
\end{aligned}$$

where $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i \in\mathbf{N}$. Note that, the mapping W_n is a nonexpansive mapping provided $T_1, T_2, ...$ are infinite family of non-expansive mappings. Moreover, we have the following lemmas, which are important tools for proving our main results.

Lemma 2.4. [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and $\lambda_1, \lambda_2, \cdots$ be real numbers such that $0 < \lambda_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Using Lemma 2.4, one can define the mapping W of C into itself as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C. \tag{2.2}$$

Such a mapping W is called the *W-mapping* generated by $T_1, T_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$.

Based on Lemma 2.4, throughout this paper, we will assume that $0 < \lambda_n \le b < 1$ for each $n \ge 1$ for some $b \in \mathbb{R}$.

Lemma 2.5. [14] *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \cdots$ be non-expansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $F(W) = \bigcap_{n=1}^{\infty} F(T_n)$.*

Lemma 2.6. ([6]) *Let C be a nonempty closed convex subset of a Hilbert space H, $\{T_i : C \to C\}$ be a family of infinitely non-expansive mappings with $\bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

2.2. **Auxiliary generalized equilibrium problem and the hybrid iterative scheme.** Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $\varphi : C \to \mathbb{R}$ be a real-valued function, $Q : \mathcal{H} \to \mathcal{H}$ a mapping and $\Phi : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function. Let r be a positive number. For a given point $x \in \mathcal{H}$, we consider the problem of finding $y \in C$ such that

$$\Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0 \quad \forall z \in C,$$

which is known as the *auxiliary generalized equilibrium problem.* Related to such a problem, for each $r > 0$, we consider a mapping $S^{(r)} : \mathcal{H} \to C$ which is defined by

$$x \mapsto \{y \in C : \Phi(Qx, y, z) + \varphi(z) - \varphi(y) + \frac{1}{r}\langle y - x, z - y\rangle \geq 0, \quad \forall z \in C\}. \tag{2.3}$$

Now let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be real-valued functions, $Q_1, Q_2 : \mathcal{H} \to \mathcal{H}$ be nonlinear mappings and $\Phi_1, \Phi_2 : \mathcal{H} \times C \times C \to \mathbb{R}$ be equilibrium-like functions. For each $r > 0$, let $S_1^{(r)}$ and $S_2^{(r)}$ be denoted for mappings defined as (2.3) generated by Φ_1, Q_1, φ_1 and Φ_2, Q_2, φ_2, respectively. We will assume the following condition:

Condition (Δ):

(a) $S_i^{(r)}$ is a single-valued mapping, for each $i = 1, 2$;

(b) for each $i = 1, 2$, the mapping $S_i^{(r)}$ is firmly nonexpansive (that is, for any $u, v \in \mathcal{H}$, $\|S_i^{(r)}(u) - S_i^{(r)}(v)\| \leq \langle S_i^{(r)}(u) - S_i^{(r)}(v), u - v\rangle$);

(c) $F(S_i^{(r)}) = GEP(C, \Phi_i, Q_i, \varphi_i)$, for each $i = 1, 2$.

Assuming that the Condition (Δ) is satisfied, we can introduce the following hybrid algorithm:

Algorithm (I). Let $\mu > 0, \gamma > 0, r > 0$ be three constants. Let f be a contraction of $\mathcal{H}$ into itself and let A be a strongly positive bounded linear operator on $\mathcal{H}$. For given $u, x_0 \in \mathcal{H}$, we define the sequence $\{x_n\}$ by

$$\begin{cases} u_n = S_1^{(r)}(x_n), v_n = S_2^{(r)}(x_n) \\ y_n = \delta u_n + (1-\delta)v_n \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + ((1-\beta_n)I - \alpha_n(I + \mu A))\, W_n y_n, \end{cases} \tag{2.4}$$

where $\delta \in (0, 1)$ and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1]$ and W_n is the W-mapping defined by (2.1). Of course, we will use the Algorithm (I) to obtain our main results in this paper. To do this, we also need the following lemmas.

Lemma 2.7. [20] *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. If x^* is a solution to the optimization problem (1.2) then*

$$\langle u + (\gamma f - (I + \mu A))x^*, x - x^*\rangle \leq 0,$$

for all $x \in C$.

Lemma 2.8. [11] *Each Hilbert space* H *satisfies Opial's condition, i.e., for any sequence* $\{x_n\} \subset H$ *with* $x_n \rightharpoonup x$, *the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

hold for any $y \in H$ *such that* $y \neq x$.

Lemma 2.9. [3] *Let* E *be a uniformly convex Banach space,* C *be a nonempty closed convex subset of* E *and* $S : C \to C$ *be a non-expansive mapping. Then* $I - S$ *is demi-closed at zero.*

Lemma 2.10. [4] *Let* C *be a closed convex subset of a strictly convex Banach space* E. *Let* $\{T_n : n \in \mathbb{N}\}$ *be a family of non-expansive mappings on* C *with* $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. *Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{n=1}^{\infty} \lambda_n = 1$. *Then a mapping* S *on* C *defined by*

$$Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$$

for all $x \in C$ *is well defined, non-expansive and* $F(S) = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.11. [15] *Let* $\{x_n\}$ *and* $\{l_n\}$ *be bounded sequences in a Banach space* E *and* b_n *be a sequence in* $[0,1]$ *with*

$$0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty} b_n < 1.$$

Suppose that $x_{n+1} = (1 - b_n)l_n + b_n x_n$ *for all* $n \geq 1$ *and*

$$\limsup_{n\to\infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|l_n - x_n\| = 0$.

Lemma 2.12. [18] *Assume that* $\{\theta_n\}$ *is a sequence of nonnegative real numbers such that*

$$\theta_{n+1} \leq (1 - a_n)\theta_n + \delta_n, \quad \forall n \geq 1,$$

where $\{a_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} a_n = \infty$;

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{a_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \theta_n = 0$.

3. MAIN RESULTS

Now we are in position to prove our main results.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *Let* f *be a contraction of* $\mathcal{H}$ *into itself with coefficient* $\alpha \in (0,1)$ *and let* A *be a strongly positive bounded linear operator on* $\mathcal{H}$ *with coefficient* $\overline{\gamma} > 0$. *Let* $T_1, T_2, \ldots$ *be an* *infinite family of nonexpansive mappings of C into itself. Let $\varphi_1, \varphi_2 : C \to \mathbb{R}$ be real-valued functions, $Q_1, Q_2 : \mathcal{H} \to \mathcal{H}$ be nonlinear mappings and $\Phi_1, \Phi_2 : \mathcal{H} \times C \times C \to \mathbb{R}$ be equilibrium-like functions. Assume that the Condition (Δ) is satisfied and*

$$\Omega = \bigcap_{i=1}^{\infty} (F(T_i) \cap GEP(C, \Phi_1, Q_1, \varphi_1) \cap GEP(C, \Phi_2, Q_2, \varphi_2)) \neq \emptyset.$$

Let $u \in \mathcal{H}$ be fixed, and $\{x_n\}$ be a sequence generated by Algorithm(I). If the following control conditions are satisfied:

(i) $0 < \gamma < (1+\mu)\overline{\gamma}/\alpha$;
(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$,

Then the sequence $\{x_n\}$ converges strongly to $x^* \in \Omega$ which solve the optimization problem (1.2).

Proof. Step 1. We show that $\{x_n\}$ is a bounded sequence.
Let $p \in \Omega$. We see that

$$\begin{aligned} \|y_n - p\| &= \|\delta u_n + (1-\delta)v_n - p\| \\ &= \delta\|S_1^{(r)}x_n - S_1^{(r)}p\| + (1-\delta)\|S_2^{(r)}x_n - S_2^{(r)}p\| \\ &\leq \delta\|x_n - p\| + (1-\delta)\|x_n - p\| = \|x_n - p\|. \end{aligned} \tag{3.1}$$

On the other hand, since A is a linear bounded self-adjoint operator on H, then $\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}$.

Now, without loss of generality we may assume that $\alpha_n \leq (1-\beta_n)(1+\mu\|A\|)^{-1}$. Observe that

$$\begin{aligned} \langle (1-\beta_n)I - \alpha_n(I+\mu A)u, u\rangle &= 1 - \beta_n - \alpha_n - \alpha_n\mu\langle Au, u\rangle \\ &\geq 1 - \beta_n - \alpha_n - \alpha_n\mu\|A\| \\ &\geq 0. \end{aligned}$$

This means, $(1-\beta_n)I - \alpha_n(I+\mu A)$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I - \alpha_n(I+\mu A)\| &= \sup\{\langle((1-\beta_n)I - \alpha_n(I+\mu A))u, u\rangle : u \in \mathcal{H}, \|u\| = 1\} \\ &= \sup\{1 - \beta_n - \alpha_n - \alpha_n\mu\langle Au, u\rangle : u \in \mathcal{H}, \|u\| = 1\} \\ &\leq 1 - \beta_n - \alpha_n - \alpha_n\mu\overline{\gamma}. \end{aligned}$$

Set $\widetilde{A} = (I + \mu A)$. Thus, by (3.1), we have

$$\begin{aligned}
&\|x_{n+1} - p\| \\
&= \|\alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1-\beta_n)I - \alpha_n \widetilde{A}]W_n y_n - p\| \\
&= \|\alpha_n u + \alpha_n(\gamma f(x_n) - \widetilde{A}p) + \beta_n(x_n - p) + [(1-\beta_n)I - \alpha_n \widetilde{A}](W_n y_n - p)\| \\
&\leq (1 - \beta_n - \alpha_n - \alpha_n \mu \overline{\gamma})\|y_n - p\| + \beta_n\|x_n - p\| + \alpha_n\|u\| + \alpha_n\|\gamma f(x_n) - \widetilde{A}p\| \\
&\leq (1 - \alpha_n - \alpha_n \mu \overline{\gamma})\|x_n - p\| + \alpha_n\|u\| + \alpha_n\gamma\|f(x_n) - f(p)\| + \alpha_n\|\gamma f(p) - \widetilde{A}p\| \\
&\leq (1 - \alpha_n - \alpha_n \mu \overline{\gamma})\|x_n - p\| + \alpha_n\|u\| + \alpha_n\gamma\alpha\|x_n - p\| + \alpha_n\|\gamma f(p) - \widetilde{A}p\| \\
&= [1 - ((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n]\|x_n - p\| + ((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n \frac{\|\gamma f(p) - \widetilde{A}p\| + \|u\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}. \qquad (3.2)
\end{aligned}$$

It follows from (3.2) and induction that

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|\gamma f(p) - \widetilde{A}p\| + \|u\|}{(1+\mu)\overline{\gamma} - \gamma\alpha}\right\}, \quad \forall n \geq 0.$$

Therefore, $\{x_n\}$ is bounded and $\{y_n\}, \{W_n x_n\}, \{W_n y_n\}$ and $\{f(x_n)\}$ are also bounded.

Step 2. We show that $\|x_{n+1} - x_n\| \to 0$.

Define a sequence $\{z_n\}$ in $\mathcal{H}$ by $z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}$ for all $n \geq 0$. Observe that, from the definition of z_n, we have

$$\begin{aligned}
z_{n+1} - z_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}(u + \gamma f(x_{n+1})) + ((1-\beta_{n+1})I - \alpha_{n+1}\widetilde{A})W_{n+1}y_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n(u + \gamma f(x_n)) + ((1-\beta_n)I - \alpha_n \widetilde{A})W_n y_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(u + \gamma f(x_{n+1})) - \frac{\alpha_n}{1 - \beta_n}(u + \gamma f(x_n)) \\
&\quad + W_{n+1}y_{n+1} - W_n y_n + \frac{\alpha_n}{1 - \beta_n}\widetilde{A}W_n y_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\widetilde{A}W_{n+1}y_{n+1} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}[u + \gamma f(x_{n+1}) - \widetilde{A}W_{n+1}y_{n+1}] + \frac{\alpha_n}{1 - \beta_n}[\widetilde{A}W_n y_n - u - \gamma f(x_n)] \\
&\quad + W_{n+1}y_{n+1} - W_{n+1}y_n + W_{n+1}y_n - W_n y_n.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(x_{n+1})\|+\|\widetilde{A}W_{n+1}y_{n+1}\|)\\
&\quad+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|\widetilde{A}W_ny_n\|+\|\gamma f(x_n)\|)+\|W_{n+1}y_{n+1}-W_{n+1}y_n\|\\
&\quad+\|W_{n+1}y_n-W_ny_n\|-\|x_{n+1}-x_n\|\\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(x_{n+1})\|+\|\widetilde{A}W_{n+1}y_{n+1}\|)\\
&\quad+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|\widetilde{A}W_ny_n\|+\|\gamma f(x_n)\|)+\|W_{n+1}y_n-W_ny_n\|\\
&\quad+\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|. \qquad (3.3)
\end{aligned}$$

From (2.1), since W_n, T_n and $U_{n,i}$ are all nonexpansive mappings, we have

$$\begin{aligned}
\|W_{n+1}y_n-W_ny_n\| &= \|\lambda_1T_1U_{n+1,2}y_n-\lambda_1T_1U_{n,2}y_n\|\\
&\leq \lambda_1\|U_{n+1,2}y_n-U_{n,2}y_n\|\\
&= \lambda_1\|\lambda_2T_2U_{n+1,3}y_n-\lambda_2T_2U_{n,3}y_n\|\\
&\leq \lambda_1\lambda_2\|U_{n+1,3}y_n-U_{n,3}y_n\|\\
&\leq \cdots\\
&\leq \lambda_1\lambda_2\cdots\lambda_n\|U_{n+1,n+1}y_n-U_{n,n+1}y_n\|\\
&\leq M\prod_{i=1}^n\lambda_i, \qquad (3.4)
\end{aligned}$$

where $M=\sup_n\{\|U_{n+1,n+1}y_n-U_{n,n+1}y_n\|\}$. Meanwhile, by the nonexpansiveness of $S_1^{(r)}$ and $S_2^{(r)}$, we get

$$\begin{aligned}
\|y_{n+1}-y_n\| &= \|\delta u_{n+1}+(1-\delta)v_{n+1}-[\delta u_n+(1-\delta)v_n]\|\\
&\leq \delta\|u_{n+1}-u_n\|+(1-\delta)\|v_{n+1}-v_n\|\\
&= \delta\|S_1^{(r)}x_{n+1}-S_1^{(r)}x_n\|+(1-\delta)\|S_2^{(r)}x_{n+1}-S_2^{(r)}x_n\|\\
&\leq \delta\|x_{n+1}-x_n\|+(1-\delta)\|x_{n+1}-x_n\|\\
&= \|x_{n+1}-x_n\|, \qquad (3.5)
\end{aligned}$$

Substituting (3.4) and (3.5) into (3.3), we obtain

$$\begin{aligned}
\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\| &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|u\|+\|\gamma f(x_{n+1})\|+\|\widetilde{A}W_{n+1}y_{n+1}\|)\\
&\quad+\frac{\alpha_n}{1-\beta_n}(\|u\|+\|\widetilde{A}W_ny_n\|+\|\gamma f(x_n)\|)+M\prod_{i=1}^n\lambda_i,
\end{aligned}$$

which implies that $\limsup_{n\to\infty}(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|)\leq 0$. Hence, by Lemma 2.11, we have $\lim_{n\to\infty}\|z_n-x_n\|=0$. Consequently, it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|z_n-x_n\|=0. \qquad (3.6)$$

Step 3. We show $\|Wy_n-y_n\|\to 0$ as $n\to\infty$.

Note that, from (2.4), we have

$$\begin{aligned}\|x_n - W_n y_n\| \leq & \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\| \\ \leq & \|x_n - x_{n+1}\| + \alpha_n \|u + \gamma f(x_n) - \widetilde{A} W_n y_n\| + \beta_n \|x_n - W_n y_n\|,\end{aligned}$$

that is,

$$\|x_n - W_n y_n\| \leq \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|u + \gamma f(x_n) - A W_n y_n\|. \tag{3.7}$$

It follows from the conditions (ii), (iii) and (3.6) that

$$\lim_{n\to\infty} \|x_n - W_n y_n\| = 0. \tag{3.8}$$

Since $S_1^{(r)}$ and $S_2^{(r)}$ are firmly nonexpansive mappings, we have

$$\begin{aligned}\|y_n - p\|^2 &= \|\delta u_n + (1-\delta)v_n - p\|^2 \\ &= \delta\|S_1^{(r)}x_n - S_1^{(r)}p\|^2 + (1-\delta)\|S_2^{(r)}x_n - S_2^{(r)}p\|^2 \\ &\leq \delta\langle S_1^{(r)}x_n - S_1^{(r)}p, x_n - p\rangle + (1-\delta)\langle S_2^{(r)}x_n - S_2^{(r)}p, x_n - p\rangle \\ &\leq \delta\langle u_n - p, x_n - p\rangle + (1-\delta)\langle v_n - p, x_n - p\rangle \\ &= \langle y_n - p, x_n - p\rangle \\ &= \frac{1}{2}(\|y_n - p\|^2 + \|x_n - p\|^2 - \|x_n - y_n\|^2),\end{aligned}$$

that is, $\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2$. Therefore, we have

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ =&\|\alpha_n(u + \gamma f(x_n) - \widetilde{A}p) + \beta_n(x_n - W_n y_n) + (I - \alpha_n\widetilde{A})(W_n y_n - p)\|^2 \\ \leq&\|(I - \alpha_n\widetilde{A})(W_n y_n - p) + \beta_n(x_n - W_n y_n)\|^2 + 2\alpha_n\langle u + \gamma f(x_n) - \widetilde{A}p, x_{n+1} - p\rangle \\ \leq&[\|(I - \alpha_n\widetilde{A})(W y_n - p)\| + \beta_n\|x_n - W_n y_n\|]^2 + 2\alpha_n\|u + \gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\| \\ \leq&[(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\|y_n - p\| + \beta_n\|x_n - W_n y_n\|]^2 + 2\alpha_n\|\gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\| \\ =&(1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|y_n - p\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 \\ &+ 2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|y_n - p\|\|x_n - W_n y_n\| + 2\alpha_n\|u + \gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\| \\ \leq&(1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\{\|x_n - p\|^2 - \|x_n - y_n\|^2\} + 2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|y_n - p\|\|x_n - W_n y_n\| \\ &+ 2\alpha_n\|u + \gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\| + \beta_n^2\|x_n - W_n y_n\|^2 \\ =&(1 - 2\alpha_n(1+\mu)\overline{\gamma} + \alpha_n(1+\mu)^2\overline{\gamma}^2)\|x_n - p\|^2 - (1 - \alpha_n(1+\mu)\overline{\gamma})^2\|x_n - y_n\|^2 \\ &+ \beta_n^2\|x_n - W_n y_n\|^2 + 2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|y_n - p\|\|x_n - W_n y_n\| \\ &+ 2\alpha_n\|u + \gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\| \\ \leq&\|x_n - p\|^2 + \alpha_n(1+\mu)^2\overline{\gamma}^2\|x_n - p\|^2 + \beta_n^2\|x_n - W_n y_n\|^2 - (1 - \alpha_n - \alpha_n\mu\overline{\gamma})^2\|x_n - y_n\|^2 \\ &+ 2(1 - \alpha_n - \alpha_n\mu\overline{\gamma})\beta_n\|y_n - p\|\|x_n - W_n y_n\| + 2\alpha_n\|u + \gamma f(x_n) - \widetilde{A}p\|\|x_{n+1} - p\|.\end{aligned} \tag{3.9}$$

This implies that

$$
\begin{aligned}
&(1-\alpha_n-\alpha_n\mu\overline{\gamma})^2\|x_n-y_n\|^2\\
\leq&\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(1+\mu)^2\overline{\gamma}^2\|x_n-p\|^2+\beta_n^2\|x_n-W_ny_n\|^2\\
&+2\alpha_n\|u+\gamma f(x_n)-\widetilde{A}p\|\|x_{n+1}-p\|\\
&+2(1-\alpha_n-\alpha_n\mu\overline{\gamma})\beta_n\|y_n-p\|\|x_n-W_ny_n\|\\
\leq&(\|x_n-p\|+\|x_{n+1}-p\|)\|x_{n+1}-x_n\|+\alpha_n(1+\mu)^2\overline{\gamma}^2\|x_n-p\|^2\\
&+\beta_n^2\|x_n-W_ny_n\|^2+2(1-\alpha_n-\alpha_n\mu\overline{\gamma})\beta_n\|y_n-p\|\|x_n-W_nz_n\|\\
&+2\alpha_n\|u+\gamma f(x_n)-\widetilde{A}p\|\|x_{n+1}-p\|.
\end{aligned}\tag{3.10}
$$

Therefore, it follows from (3.8), (3.9) and (3.10) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.\tag{3.11}$$

Since, $\|W_ny_n-y_n\|\leq\|W_ny_n-x_n\|+\|x_n-y_n\|$, it follows that

$$\lim_{n\to\infty}\|W_ny_n-y_n\|=0.\tag{3.12}$$

Now, $\|Wy_n-y_n\|\leq\|Wy_n-W_ny_n\|+\|W_ny_n-y_n\|$. This together with Lemma 2.6, we obtain

$$\lim_{n\to\infty}\|Wy_n-y_n\|=0.\tag{3.13}$$

Step 4. We show $\limsup_{n\to\infty}\langle(u+\gamma f-[I+\mu A]x^*,x_n-x^*\rangle\leq 0$, where $x^*\in\Omega$ is a solution of the optimization problem (1.2).

Since $\{y_n\}$ is a bounded sequence, we can find a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ and $q\in\mathcal{H}$ such that $y_{n_j}\to q$ weakly, and

$$\lim_{j\to\infty}\langle(u+\gamma f-[I+\mu A]x^*,y_{n_j}-x^*\rangle=\limsup_{n\to\infty}\langle(u+\gamma f-[I+\mu A]x^*,y_n-x^*\rangle.\tag{3.14}$$

Moreover, since $\{x_n\}$ is a bounded sequence and $\lim_{n\to\infty}\|x_n-y_n\|=0$, we see that the corresponding subsequence $\{x_{n_j}\}$ of $\{x_n\}$ also converges weakly to $q\in\mathcal{H}$.

Now, let us define a mapping $D:\mathcal{H}\to\mathcal{H}$ by

$$Dx=\delta S_1^{(r)}x+(1-\delta)S_2^{(r)}x,\quad\forall x\in\mathcal{H}.$$

From Lemma 2.10, we know that D is a nonexpansive mapping with

$$F(D)=F(S_1^{(r)})\cap F(S_2^{(r)}).$$

Observe that,

$$\lim_{j\to\infty}\|x_{n_j}-Dx_{n_j}\|=0.\tag{3.15}$$

Indeed, it is easy to see that $\|y_{n_j}-Dx_{n_j}\|=0$ for all $j\in\mathbb{N}$ and it follows that

$$\|x_{n_j}-Dx_{n_j}\|\leq\|x_{n_j}-y_{n_j}\|+\|y_{n_j}-Dx_{n_j}\|=\|x_{n_j}-y_{n_j}\|,$$

and, by (3.11), we obtain (3.15). Consequently, thanks to Lemma 2.9, we have $q\in F(D)$.

Next, we show that $q \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. Assume that $q \notin F(W)$. From Opial's condition (Lemma 2.8), we have

$$\begin{aligned}\liminf_{j\to\infty} \|y_{n_j} - q\| &< \liminf_{j\to\infty} \|y_{n_i} - Wq\| \\ &\leq \liminf_{j\to\infty} (\|y_{n_j} - Wy_{n_j}\| + \|Wy_{n_j} - Wq\|) \\ &\leq \liminf_{j\to\infty} (\|y_{n_j} - Wy_{n_j}\| + \|y_{n_j} - q\|). \end{aligned} \tag{3.16}$$

Thus, it follows from (3.13) that $\liminf_{i\to\infty} \|y_{n_i} - q\| < \liminf_{i\to\infty} \|y_{n_i} - q\|$, which is a contradiction, and so $q \in F(W) = \bigcap_{i=1}^{\infty} F(T_i)$. Hence, $q \in \Omega$. Consequently, by Lemma 2.7 and (3.14), we have

$$\begin{aligned}\limsup_{n\to\infty} \langle (u + [\gamma f - (I + \mu A)]x^*, x_n - x^*\rangle &= \limsup_{n\to\infty} \langle (u + [\gamma f - (I + \mu A)]x^*, y_n - x^*\rangle \\ &= \lim_{j\to\infty} \langle (u + [\gamma f - (I + \mu A)]x^*, y_{n_j} - x^*\rangle \\ &= \langle u + [\gamma f - (I + \mu A)]x^*, q - x^*\rangle \\ &\leq 0. \end{aligned} \tag{3.17}$$

Step 5. We prove $\{x_n\}$ converges strongly to $x^* \in \Omega$, where x^* is a solution of the optimization problem (1.2).

Consider,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 =& \|\alpha_n(u + \gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\widetilde{A})(W_n y_n - x^*)\|^2 \\ \leq& \|\beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n\widetilde{A})(W_n y_n - x^*)\|^2 \\ &+ 2\alpha_n\langle u + \gamma f(x_n) - \widetilde{A}x^*, x_{n+1} - x^*\rangle \\ \leq& [\|((1-\beta_n)I - \alpha_n\widetilde{A})(W_n y_n - x^*)\| + \|\beta_n(x_n - x^*)\|]^2 \\ &+ 2\alpha_n\gamma\langle f(x_n) - f(x^*), x_{n+1} - x^*\rangle \\ &+ 2\alpha_n\langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^*\rangle \\ \leq& [(1 - \beta_n - \alpha_n(1+\mu)\overline{\gamma})\|y_n - x^*\| + \beta_n\|x_n - x^*\|]^2 \\ &+ 2\alpha_n\gamma\alpha\|x_n - x^*\|\|x_{n+1} - x^*\| \\ &+ 2\alpha_n\langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^*\rangle \\ \leq& (1 - \alpha_n(1+\mu)\overline{\gamma})^2\|x_n - x^*\|^2 + \alpha_n\gamma\alpha\{\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\} \\ &+ 2\alpha_n\langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^*\rangle, \end{aligned}$$

this means,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq& \frac{1 - 2\alpha_n(1+\mu)\overline{\gamma} + \alpha_n^2(1+\mu)^2\overline{\gamma}^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^*\rangle \\ =& \left[1 - \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right]\|x_n - x^*\|^2 + \frac{((1+\mu)\alpha_n\overline{\gamma})^2}{1 - \alpha_n\gamma\alpha}\|x_n - x^*\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma u + f(x^*) - \widetilde{A}x^*, x_{n+1} - x^*\rangle \end{aligned}$$

$$\leq \left[1 - \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}\right] \|x_n - x^*\|^2 + \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha}$$
$$\times \left\{ \frac{\alpha_n(1+\mu)\overline{\gamma}^2 M}{2((1+\mu)\overline{\gamma} - \gamma\alpha)} + \frac{1}{(1+\mu)\overline{\gamma} - \gamma\alpha} \langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^* \rangle \right\}$$
$$= (1 - \kappa_n)\|x_n - x^*\|^2 + \kappa_n\sigma_n,$$

where

$$R = \sup\{\|x_n - x^*\|^2 : n \geq 1\}, \quad \kappa_n = \frac{2((1+\mu)\overline{\gamma} - \gamma\alpha)\alpha_n}{1 - \alpha_n\gamma\alpha},$$

$$\sigma_n = \frac{\alpha_n(1+\mu)\overline{\gamma}^2 R}{2((1+\mu)\overline{\gamma} - \gamma\alpha)} + \frac{1}{(1+\mu)\overline{\gamma} - \gamma\alpha} \langle u + \gamma f(x^*) - \widetilde{A}x^*, x_{n+1} - x^* \rangle.$$

It is easy to see that $\sum_{n=1}^{\infty} \kappa_n = \infty$ and $\limsup_{n\to\infty} \sigma_n \leq 0$. Hence, by Lemma 2.12, we conclude that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

On the other hand, as an applications of our main results, we consider the following convex feasibility problem (CFP):

$$\text{find a point } x \in \bigcap_{i=1}^{N} C_i,$$

where $N \geq 1$ is an integer and each C_i is assumed to be the set of solutions of the generalized equilibrium problem with the bi-function Φ_i for $i = 1, 2, \ldots, N$. There is a considerable investigation on CFP in the setting of Hilbert spaces which captures applications in various disciplines such as image restoration ([7, 9]), computer tomography ([13]) and radiation therapy treatment planning ([8]). In fact, using technique as in Theorem 3.1, we can obtain the following result.

Theorem 3.2. *Let C be a nonempty closed and convex subset of a Hilbert space $\mathcal{H}$. Let $f : C \to C$ be a contraction with the coefficient $\alpha \in (0,1)$. Let $\varphi_i : C \to \mathbb{R}$ be real-valued function, $Q_i : C \to \mathcal{H}$ be nonlinear mapping and $\Phi_i : \mathcal{H} \times C \times C \to \mathbb{R}$ be an equilibrium-like function for each $i = 1, 2, \ldots, N$. Assume that the Condition (Δ) is satisfied and*

$$\Omega = \bigcap_{i=1}^{N} GEP(C, \Phi_i, Q_i, \varphi_i) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$\begin{cases} x_1 \in C, \\ \Phi_1(Qx_n, u_{(n,1)}, z) + \varphi(z) - \varphi(u_{(n,1)}) + \frac{1}{r}\langle u_{(n,1)} - x_n, z - u_{(n,1)} \rangle \geq 0, \quad \forall z \in C, \\ \Phi_2(Qx_n, u_{(n,2)}, z) + \varphi(z) - \varphi(u_{(n,2)}) + \frac{1}{r}\langle u_{(n,2)} - x_n, z - u_{(n,2)} \rangle \geq 0, \quad \forall z \in C, \\ \quad \vdots \\ \Phi_N(Qx_n, u_{(n,N)}, z) + \varphi(z) - \varphi(u_{(n,N)}) + \frac{1}{r}\langle u_{(n,N)} - x_n, z - u_{(n,N)} \rangle \geq 0, \quad \forall z \in C, \\ y_n = \sum_{i=1}^{N} \delta_i u_{(n,i)}, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) y_n, \quad \forall n \geq 1, \end{cases}$$

where $r > 0$ be fixed and $\delta_i \in (0,1)$ for each $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^{N} \delta_i = 1$. If $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following control conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ converge strongly to $x^* \in \Omega$.

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